Pauli-GKLS Hybrid Master Equations

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Prelude: before I learned about CP & GKLS

Master Eq. for positive dynamical semigroup: $d\hat\rho/dt = \mathcal{L}\hat\rho$.
For pure initial state $\hat\rho = |\Psi\rangle\langle\Psi| \equiv \hat P$:

$$d\hat P/dt = \mathcal{L}\hat P = -i\hat H_\Psi \hat P + i\hat P \hat H_\Psi^\dagger + \hat W_\Psi$$

Effective Hamiltonian: $-i\hat H_\Psi = \mathcal{L}\hat P - \langle\mathcal{L}\hat P\rangle_\Psi$
Transition rate operator: $\hat W_\Psi = \mathcal{L}\hat P - \{\mathcal{L}\hat P, \hat P\} + \langle\mathcal{L}\hat P\rangle_\psi \hat P \ \geq \ 0$
Ito-Stochastic Schrodinger Eq. for all P-preserving ME's (D. 1986):

$$d\Psi = -i\hat H_\Psi |\Psi\rangle\, dt + d|\Phi\rangle \qquad d|\Phi\rangle d\langle\Phi| = \hat W_\Psi dt$$

Example of non-CP but P ME: $d\hat\rho/dt = \mathcal{L}\hat\rho = \sum_k c_k(\hat\sigma_k\hat\rho\hat\sigma_k - \hat\rho)$
$c_1 = 1,\ c_2 = 1,\ c_3 = -1$ (Benatti & al. 2002)
$\hat H_\Psi = -\frac{1}{2}\sum c_k(\hat\sigma_k - \langle\hat\sigma_k\rangle)^2; \qquad \hat W_\Psi = \sum c_k(\hat\sigma_k - \langle\hat\sigma_k\rangle)\hat P(\hat\sigma_k - \langle\hat\sigma_k\rangle)$

Hybrid

- Dynamical coupling between Classical and Quantum (Aleksandrov 1981)
- Coupling between measurement device and Quantum (Sherry & Sudarshan 1978)
- Fundational coexistence between Classical and Quantum (D. 1998)

Mathematics:

	Classical	Quantum	Hybrid
State:	$\rho(x)$	$\hat{\rho}$	$\hat{\rho}(x)$
Dynamics:	Pauli	GKLS	?

Hybrid density

Simplest construction: $\{\rho(x)\ and\ \hat{\rho}\} \Rightarrow \rho(x)\hat{\rho} \equiv \hat{\rho}(x)$

Generically:

$$\text{any } \hat{\rho}(x) \geq 0 \quad \forall x; \quad \mathrm{tr} \sum_x \hat{\rho}(x) = 1.$$

Reduced Q	Reduced C	Conditional Q	Conditional C
$\hat{\rho} = \sum_x \hat{\rho}(x)$	$\rho(x) = \mathrm{tr}\hat{\rho}(x)$	$\hat{\rho}_x = \hat{\rho}(x)/\rho(x)$	$\nexists$

E.g.:

- $\hat{\rho}(r,p)$ where $\hat{\rho}$: electrons, (r,p) : nuclei
- $\hat{\rho}[A]$ where $\hat{\rho}$: e^+e^-, A : e.m.field
- $\hat{\rho}(n)$ where $\hat{\rho}$: Q-dot, n : charge count
- $\hat{\rho}(k)$ where $\hat{\rho}$: Q-system, k : measurement outcome

Measurement

What happens to $\hat\rho$ under measurement of c.s.o.p. $\{\hat P_x\}$?

- *Text-book formalism:* $\hat\rho$ jumps randomly to the conditional $\hat\rho_x$,

$$\hat\rho \longrightarrow \hat\rho_x = \frac{1}{\rho(x)}\hat P_x \hat\rho \hat P_x \text{ with probability } \rho(x) = \mathrm{tr}(\hat P_x \hat\rho \hat P_x).$$

- *Hybrid formalism:* $\hat\rho$ jumps to the hybrid density $\hat\rho(x)$,

$$\hat\rho \longrightarrow \hat\rho(x) = \hat P_x \hat\rho \hat P_x.$$

$$\text{Generically:} \quad \hat\rho \longrightarrow \hat\rho(x) = \hat M_x \hat\rho \hat M_x^\dagger \text{ where } \sum_x \hat M_x^\dagger \hat M_x = \hat I.$$

E.g.: unsharp measurement of $\hat q$

$$\hat M_x = \hat M_x^\dagger = (2\pi\sigma^2)^{-1/4}\exp[-(\hat q - x)^2/4\sigma^2]$$

$$\hat\rho \longrightarrow \hat\rho(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{[-(\hat q - x)^2/4\sigma^2]} \hat\rho e^{[-(\hat q - x)^2/4\sigma^2]}$$

Canonical hybrid dynamics?

- *Hybrid canonical bracket*, with hybrid Hamiltonian $\hat{H}(q,p)$:
  $$\frac{d\hat{\rho}(q,p)}{dt} = -i[\hat{H}(q,p),\hat{\rho}(q,p)] + \mathbf{Herm}\{H(q,p),\hat{\rho}(q,p)\}_{\text{Poisson}}$$
  Fatally wrong: $\hat{\rho}(q,p) \geq 0$ is not preserved!
  Positivity is preserved if we add noise to the r.h.s.

- *Partial Husimi projection*, from unitary dynamics of Q-system + Q-harmonic-oscillator: $d\widehat{\rho}/dt = -i[\widehat{H},\widehat{\rho}]$.
  Generate hybrid state and ME by partial Husimi projection:
  $$\hat{\rho}(q,p) = \mathrm{tr}_{\mathrm{osc}}\left((\hat{1}\otimes|q,p\rangle\langle q,p|)\widehat{\rho}\right)$$
  $$d\hat{\rho}/dt = -i\mathrm{tr}_{\mathrm{osc}}\left((\hat{1}\otimes|q,p\rangle\langle q,p|)[\widehat{H},\widehat{\rho}]\right)$$
  Yields canonical hybrid bracket plus "noisy" terms.
  $\hat{\rho}(q,p) \geq 0$ preserved by construction.

Hybrid dynamics

- Markovian Classical ME (Pauli):
$$\dot\rho(x) = -\partial_x v(x)\rho(x) + \sum_y \left(T(x,y)\rho(y) - T(y,x)\rho(x)\right), \quad T(x,y) \geq 0$$
diffusion: $\dot\rho(x) = D\partial_x^2\rho(x), \qquad T(x,y) = \lim_{\tau\to 0} \frac{1/\tau}{\sqrt{4\pi D\tau}} e^{-[(x-y)^2/4D\tau]}$

- Markovian Quantum ME (GKLS):
$$\dot{\hat\rho} = -i[\hat H, \hat\rho] + \sum_\alpha \left(\hat F_\alpha \hat\rho \hat F_\alpha^\dagger - \tfrac{1}{2}\{\hat F_\alpha^\dagger \hat F_\alpha, \hat\rho\}\right)$$
decoherence: $\dot{\hat\rho} = -i[\hat H, \hat\rho] - D[\hat q, [\hat q, \hat\rho]], \qquad \hat F = \hat F^\dagger = \sqrt{2D}\hat q$

- Generic Markovian Hybrid ME (‘Pauli-GKLS’):
$$\dot{\hat\rho}(x) = -i[\hat H(x), \hat\rho(x)] +$$
$$+ \sum_{y,\alpha} \left(\hat F_\alpha(x,y)\hat\rho(y)\hat F_\alpha^\dagger(x,y) - \tfrac{1}{2}\{\hat F_\alpha^\dagger(y,x)\hat F_\alpha(y,x), \hat\rho(x)\}\right)$$

(Alicki & Kryszewski 2004, D. 2014)

Hybrid ME: Derivation

- Embed Hybrid into a bigger Quantum:

$$\hat{\rho}(x) \rightarrow \widehat{\rho} = \sum_x \hat{\rho}(x) \otimes |x\rangle\langle x| , \quad \hat{H}(x) \rightarrow \widehat{H} = \sum_x \hat{H}(x) \otimes |x\rangle\langle x|$$

- Assume GKLS ME:

$$\dot{\widehat{\rho}} = -i[\widehat{H}, \widehat{\rho}] + \widehat{F}\widehat{\rho}\widehat{F}^\dagger - \tfrac{1}{2}\{\widehat{F}^\dagger\widehat{F}, \widehat{\rho}\}]$$

- Project back by $\hat{I} \otimes |x\rangle\langle x|$, introduce $\hat{F}(x,y) = \mathrm{tr}'[(\hat{I} \otimes |y\rangle\langle x|)\widehat{F}]$

$$\dot{\hat{\rho}}(x) = -i[\hat{H}(x), \hat{\rho}(x)] + \\ + \sum_y \left[\hat{F}(x,y)\hat{\rho}(y)\hat{F}^\dagger(x,y) - \tfrac{1}{2}\{\hat{F}^\dagger(y,x)\hat{F}(y,x), \hat{\rho}(x)\}\right]$$

Tha's general (Markovian) hybrid ME if $\hat{F}(x,y)$ is regular.
But, e.g., $\hat{F}(x,y) \sim \delta'(x-y)$, yields different forms.
Coming example: $\widehat{F}$ contains $(\partial_x|x\rangle)\langle x|$.

Case study: Q-monitoring

$\hat{\rho}$: Q-particle, $\dot{x}$: monitored value of $\hat{q}$

- Naive hybrid ME:
  $$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \tfrac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\}$$
  Feature: $d\langle x\rangle/dt = \langle\hat{q}\rangle$
  Problem: term $-\frac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\}$ violates positivity of $\hat{\rho}(x)$

- Try GKLS ME for $\widehat{\rho}$ in 'big' space, choose
  $$\widehat{F} = \frac{\hat{q}}{\sqrt{2D}}\otimes\hat{I} + \sqrt{2D}\,\hat{I}\otimes\int\frac{\partial|x\rangle}{\partial x}\langle x|dx \quad \Rightarrow$$
  $$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \tfrac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\} - \frac{1}{16D}[\hat{q}, [\hat{q}, \hat{\rho}(x)]] + D\partial_x^2\hat{\rho}(x)$$
  That's Hybrid ME of Q-monitoring $\hat{q}$ (D. 2014).
  Equivalent with the Ito-formalism (Belavkin 1988, D. 1988) of Q-monitoring of $\hat{q}$.
  (Similar HMEs: Bauer & Bernard & Tilloy 2014)

Summary

- Q-measurements ⇒ fundational hybrids:

$$\hat{\rho} \longrightarrow \hat{P}_x \hat{\rho} \hat{P}_x \equiv \hat{\rho}(x)$$

- Pauli-GKLS theorem is missing for HME. Instead:

$$\text{dilate: } \hat{\rho}(x) \Rightarrow \widehat{\rho}, \quad \text{project: } \dot{\widehat{\rho}} = \mathcal{L}_{GKLS}\widehat{\rho} \;\Rightarrow\; \dot{\hat{\rho}}(x) = \mathcal{L}_{hybr}\hat{\rho}(x)$$

- Time-continuous Q-measurement ⇒ fundational HME.

$$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \tfrac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\} - \frac{1}{16D}[\hat{q}, [\hat{q}, \hat{\rho}(x)]] + D\partial_x^2\hat{\rho}(x)$$