NON-MARKOVIAN CONTINUOUS QUANTUM MEASUREMENT OF RETARDED OVSERVABLES

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Contrary to longstanding doubts, diffusive non-Markovian quantum trajectories are single system trajectories and correspond to the true continuous measurement of a certain retarded potential.

CONTENTS:

PEOPLE:

- **Gisin 1984, Belavkin, D. (1988)**
- **Strunz 1996**
- **Gambetta+Wiseman (2002-3)**
- **Chou+Su+Hao+Yu (1985)**

Continuous Quantum Measurement

Markovian continuous measurement of $\hat{x}_t$: stochastic Schrödinger equation (SSE) of the collapsing state vector depending on the history of the read-outs:

$$\psi_t[x]$$

where $x = \{x_\tau; \tau \in [0,t]\}$.

Formal extension for the non-Markovian (even relativistic) case (1990). Only $\psi_\infty[x]$ and $p_\infty[x]$ were given. The concept of continuous read-out was missing.

Smart non-Markovian quantum trajectories (1996) and their non-Markovian SSE (1997).

Doubts: non-Markov quantum trajectory is mathematical fiction (2002).

The present work comes to the positive conclusion: the non-Markovian trajectories are measurable single system trajectories. The equations concerning the measurement of $\hat{x}_t$ must be reparametrized in terms of a given $\hat{z}_t$ which is a retarded function of $\hat{x}_t$. Then we are continuously reading out $\hat{z}_t$ instead of $\hat{x}_t$.

Stochastic Unravelling

Suppose openness is caused by continuous measurement:

$$\hat{\rho}_t = \mathcal{M}_t \hat{\rho}_0$$

where $\hat{\rho}_t$ is density matrix, $\mathcal{M}_t$ is completely positive map (superoperator). Simplest non-Markovian:

$$\mathcal{M}_t = \mathcal{T} \exp\left(-\frac{1}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\sigma \hat{x}_{\tau,\Delta}\alpha(\tau-\sigma)\hat{x}_{\sigma,\Delta}\right)$$

where $\alpha(\tau-\sigma)$ is real positive kernel. Superoperator notation: $\hat{x}_{\tau,\Delta}\hat{O} = [\hat{x}_\tau, \hat{O}]$ for any $\hat{O}$; $\mathcal{T}$ is time-ordering for all *Heisenberg (super)operators.*

The decohered quantity is $\hat{x}_t$ but the measured quantity may be different, say $\hat{z}_t$.

Meausured state: either $\psi_t[x]$, or, e.g.: $\psi_t[z]$ where x or z are two different read-outs of the detectors. They must equally unravel the ensemble evolution:

$$\hat{\rho}_t \ = \ \mathrm{M}\psi_t[x]\psi_t^\dagger[x] \ = \ \mathrm{M}\psi_t[z]\psi_t^\dagger[z] \ .$$

It was hard to find a non-Markovian unravelling. Yet, in 1990 i got some $\psi_t[x]$ and in 1996 Strunz got some $\psi_t[z]$, in 1997 we found a linear SSE for the latter:

$$\frac{\mathrm{d}\Psi_t[z]}{\mathrm{d}t} = z_t\hat{x}_t\Psi_t[z] - 2\hat{x}_t\int_0^t \alpha(t-\tau)\frac{\delta\Psi_t[z]}{\delta z_\tau}\mathrm{d}\tau \ ,$$

z_τ is a real random variable for $\tau \in [0,t]$. True state is obtained via normalization: $\psi_t[z] = \Psi_t[z]/\|\Psi_t[z]\|$. Probability distribution of z:

$$p_t[z] = \tilde{G}_{[0,t]}[z] \ \|\Psi_t[z]\|^2 \ ,$$

$\tilde{G}_{[0,t]}[z]$ is a Gaussian distribution defined through $\alpha(\tau-\sigma)$. We showed (1997):

$$\mathrm{M}z_t = 2\int_0^t \alpha(t-\sigma)\langle\hat{x}_\sigma\rangle_t \mathrm{d}\sigma \ ,$$

$\langle\hat{x}_\sigma\rangle_t$ is $\hat{x}_\sigma$'s quantum expectation value at time t in state $\psi_t[z]$.

The SSE "measures" the retarded "potential" of $\hat{x}_t$ rather than $\hat{x}_t$ itself.

Non-Markovian Measurement Device

Example: single vonN detector of initial density matrix $D_0(x;x')$:

$$D_0(x;x)\hat{\rho}_0 \longrightarrow D_0(x-\hat{x}_{\tau,L};x-\hat{x}_{\tau,R})\hat{\rho}_0\ .$$

Superoperator notation: $\hat{x}_{\tau,L}\hat{O}=\hat{x}_\tau\hat{O}$ and $\hat{x}_{\tau,R}\hat{O}=\hat{O}\hat{x}_\tau$. After *read-out* of the pointer x: total state goes into the system's conditional state, depending on the read-out:

$$\hat{\rho}(x)=\frac{1}{p(x)}D_0(x-\hat{x}_{\tau,L};\ x-\hat{x}_{\tau,R})\hat{\rho}_0\ ,$$

$$p(x)=\mathrm{tr}D_0(x-\hat{x}_{\tau,L};\ x-\hat{x}_{\tau,R})\hat{\rho}_0\ .$$

Choose discrete time $\tau=n\epsilon$, $n=0,\pm1,\pm2,\dots$. Install an infinite sequence of vonN detectors, labelled by $\tau=n\epsilon$. Pointer coordinates of the detectors: x_τ.

The detector of *label* $\tau=n\epsilon$ measures the Heisenberg operator $\hat{x}_\tau$ of the system via the above mechanism. We switch on the detectors for $\tau\geq0$.

Assume *initially correlated detectors*, of intitial wave function:

$$\phi_0[x]=\sqrt{\mathcal{N}}\exp\left(-\epsilon^2\sum_{\tau,\sigma}x_\tau\alpha(\tau-\sigma)x_\sigma\right)\ .$$

In continuous (or weak measurement) limit $\epsilon\rightarrow0$:

$$\phi_0[x]=\sqrt{G[x]}\ .$$

Introduce the characteristic function $\theta_{[0,t]}$ of the period $[0,t]$. The total density matrix reads:

$$\hat{\rho}_t[x;x]=\mathcal{T}G[x-\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\hat{\rho}_0\ ,$$

Superoperator notation $\hat{x}_c\hat{O}=\frac{1}{2}\{\hat{x},\hat{O}\}$. This form guarantees the unravelling of the open system dynamics $\hat{\rho}_t=\mathcal{M}_t\hat{\rho}_t$.

Continuous Read-Out

It is crucial to realize that the true time-evolution of the system's conditional state depends on our chosen schedule to reading out the pointers x_τ. In fact, *we can read out any x_τ at any time* since all detectors are alaways available. Of course, we better read out the value x_τ at a *time* which is later than the *label τ* of the detector because the detector will only have coupled to the system at time τ. Hence, a natural schedule is that we read out x_τ immediately, i.e., at time τ. As a result, until any given time $t > 0$ we would read out all pointers x_τ for the period $[0, t]$ and no others. To calculate the conditional post-measurement state $\hat{\rho}_t[x]$ of the system at time t, we trace (integrate) the total density matrix $\hat{\rho}[x; x]$ over all x_τ with $\tau \notin [0, t]$:

$$\hat{\rho}_t[x] = \frac{1}{p_t[x]} \int \hat{\rho}_t[x; x] \prod_{\tau \notin [0,t]} \mathrm{d}x_\tau \ .$$

This post-measurement density matrix $\hat{\rho}_t[x]$ of the system depends on the read-outs x_τ of τ from $[0, t]$ only:

$$\hat{\rho}_t[x] = \frac{1}{p_t[x]} \mathcal{T} G_{[0,t]}[x - \hat{x}_c] \mathcal{M}_t \hat{\rho}_0 \ ,$$

$G_{[0,t]}[x]$ is the marginal distribution of $G[x]$.

Instead of reading out the coordinates $\{x_\tau; \tau \in [0, t]\}$, read out

$$z_\tau = 2 \int_{-\infty}^{\infty} \alpha(\tau - \sigma) x_\sigma \mathrm{d}\sigma \ ,$$

the postmeasurement density matrix becomes:

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]} \mathcal{T} \tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]} \hat{x}_c] \mathcal{M}_t \hat{\rho}_0 \ ,$$

where $\tilde{G}_{[0,t]}[z]$ is the marginal distribution of $\tilde{G}[z]$. This is our ultimate equation for the non-Markovian continuous measurement of the observable

$$\hat{z}_t = 2 \int_0^t \alpha(t - \sigma) \hat{x}_\sigma \mathrm{d}\sigma \ ,$$

which is a sort of *retarded potential* generated by the Heisenberg variable $\hat{x}_\tau$.

Stochastic Schrödinger Equation

Let us find the postmeasurement conditional state

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]}\mathcal{T}\tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\hat{\rho}_0$$

in the form:

$$\hat{\rho}_t[z] = \frac{1}{p_t[z]}\tilde{G}_{[0,t]}[z]\Psi_t[z]\Psi_t^\dagger[z] \ ,$$

where $\Psi_t[z]$ is the unnormalized conditional state vector of the system. Trace over both sides, norm condition yields:

$$p_t[z] = \tilde{G}_{[0,t]}[z] \ \|\Psi_t[z]\|^2 \ ,$$

just like for the SSE. Comparing our eqs., they reduce to:

$$\Psi_t[z]\Psi_t^\dagger[z] = \frac{1}{\tilde{G}_{[0,t]}[z]}\mathcal{T}\tilde{G}_{[0,t]}[z - 2\alpha\theta_{[0,t]}\hat{x}_c]\mathcal{M}_t\psi_0\psi_0^\dagger \ .$$

The r.h.s. factorizes and we can write equivalently:

$$\Psi_t[z] = \mathcal{T}\exp\left(\int_0^t z_\tau\hat{x}_\tau \mathrm{d}\tau - \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\sigma \hat{x}_\tau\alpha(\tau-\sigma)\hat{x}_\sigma\right)\psi_0 \ .$$

This $\Psi_t[z]$ is the solution of the SSE.

Conclusion

We proved for the first time that both the formalism of non-Markovian measurement theory (1990) and the non-Markovian SSE (1997) are equivalent with using of correlated von Neumann detectors in the weak-measurement continuous limit, i.e., with the continuous read-out of the values of a given retarded potential of a Heisenberg variable on a singe quantum system.

Hint of efficient simulation?

Immediate generalizations: complex $\alpha(\tau - \sigma)$, indirect measurement on the reservoir.

Appendix.– Assume a random time-dependent real variable x_τ defined for all time τ and consider the following Gaussian distribution functional of $\{x_\tau; \tau \in (-\infty, \infty)\}$:

$$G[x] = \mathcal{N} \exp\left(-2\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\sigma x_\tau \alpha(\tau-\sigma) x_\sigma\right) , \tag{1}$$

where $\alpha(\tau-\sigma)$ is a real positive definite kernel. We define its inverse through:

$$\int_{-\infty}^{\infty} \alpha^{-1}(\tau-s)\alpha(s-\sigma)\mathrm{d}s = \delta(\tau-\sigma) . \tag{2}$$

We also introduce the normalized functional Fourier transform of $G[x]$:

$$\tilde{G}[z] = \tilde{\mathcal{N}} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\sigma z_\tau \alpha^{-1}(\tau-\sigma) z_\sigma\right) . \tag{3}$$

Both distributions are normalized: $\int G[x]\Pi_\tau \mathrm{d}x_\tau = \int \tilde{G}[z]\Pi_\tau \mathrm{d}z_\tau = 1$. Instead of their functional distributions $G[x], \tilde{G}[z]$, the statistics of x_τ, z_τ can equivalently be characterized by their vanishing means $\mathrm{M}x_\tau = \mathrm{M}z_\tau = 0$ and correlation functions, respectively:

$$\mathrm{M}x_\tau x_\sigma = \tfrac{1}{4}\alpha^{-1}(\tau-\sigma) , \qquad \mathrm{M}z_\tau z_\sigma = \alpha(\tau-\sigma) . \tag{4}$$

We need certain marginal distributions as well, e.g.:

$$\tilde{G}_{[0,t]}[z] = \int \tilde{G}[z] \prod_{\tau \notin [0,t]} \mathrm{d}z_\tau , \tag{5}$$

and similarly for $G_{[0,t]}[x]$. These marginal distributions are still Gaussian, e.g.:

$$\tilde{G}_{[0,t]}[z] = \tilde{\mathcal{N}}_{[0,t]} \exp\left(-\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}\sigma z_\tau \alpha_{[0,t]}^{-1}(\tau,\sigma) z_\sigma\right) , \tag{6}$$

where the new kernel is defined by:

$$\int_0^t \alpha_{[0,t]}^{-1}(\tau,s)\alpha(s-\sigma)\mathrm{d}s = \delta(\tau-\sigma) , \qquad \tau,\sigma \in [0,t] . \tag{7}$$

In most cases, $\alpha_{[0,t]}^{-1}(\tau,\sigma)$ is a hard nut to calculate explicitly.

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