Hybrid Master Equations

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Irreversible evolution rules of classical as well as quantum systems are respectively described by classical and quantum master (kinetic) equations. Their standard structure is well-known. We discuss hybrid master equations of irreversible composite systems which are hybrids of classical and quantum subsystems.

Hybrid

- Phenomenology: dynamical coupling between Classical and Quantum
- Measurement: coupling between Device and Quantum
- Foundations: whatever coexistence between Classical and Quantum

Mathematics:

	Classical	Quantum	Hybrid
Density:	$\rho(x)$	$\hat{\rho}$	$\hat{\rho}(x)$
Master Eq.:	Pauli	Lindblad	?

Hybrid density

Simplest construction: $\{\rho(x)\ \textit{and}\ \hat{\rho}\} \Rightarrow \rho(x)\hat{\rho} \equiv \hat{\rho}(x)$

Generically:

$$\text{any } \hat{\rho}(x) \geq 0 \ \ \forall x; \quad \mathrm{Tr} \sum_x \hat{\rho}(x) = 1.$$

Reduced Q	Reduced C	Conditional Q	Conditional C
$\hat{\rho} = \sum_x \hat{\rho}(x)$	$\rho(x) = \mathrm{Tr}\hat{\rho}(x)$	$\hat{\rho}_x = \hat{\rho}(x)/\rho(x)$	$\nexists$

E.g.:

- $\hat{\rho}(r, p)$ where $\hat{\rho}$: electrons, (r, p) : nuclei
- $\hat{\rho}[A]$ where $\hat{\rho}$: e^+e^-, A : e.m.field
- $\hat{\rho}(n)$ where $\hat{\rho}$: Q-dot, n : charge count
- $\hat{\rho}(k)$ where $\hat{\rho}$: Q-system, k : measurement outcome

Measurement

What happens to $\hat{\rho}$ under measurement of c.s.o.p. $\{\hat{P}_x\}$?

- Text-book formalism: $\hat{\rho}$ jumps randomly to the conditional $\hat{\rho}_x$,

$$\hat{\rho} \longrightarrow \hat{\rho}_x = \frac{1}{\rho(x)}\hat{P}_x\hat{\rho}\hat{P}_x \text{ with probability } \rho(x) = \mathrm{Tr}(\hat{P}_x\hat{\rho}\hat{P}_x).$$

- Hybrid formalism: $\hat{\rho}$ jumps to the hybrid density $\hat{\rho}(x)$,

$$\hat{\rho} \longrightarrow \hat{\rho}(x) = \hat{P}_x\hat{\rho}\hat{P}_x.$$

$$\text{Generically:} \quad \hat{\rho} \longrightarrow \hat{\rho}(x) = \hat{M}_x\hat{\rho}\hat{M}_x^\dagger \text{ where } \sum_x \hat{M}_x^\dagger\hat{M}_x = \hat{I}.$$

E.g.: $\hat{M}_x = \hat{M}_x^\dagger = (2\pi\sigma^2)^{-1/4}\exp[-(\hat{q}-x)^2/4\sigma^2]$,

$$\hat{\rho} \longrightarrow \hat{\rho}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{[-(\hat{q}-x)^2/4\sigma^2]}\hat{\rho}e^{[-(\hat{q}-x)^2/4\sigma^2]}$$

Hybrid dynamics

- All Markovian Classical ME (Pauli):
  $\dot{\rho}(x)=-\partial_x v(x)\rho(x)+\sum_y [T(x,y)\rho(y)-T(y,x)\rho(x)],\quad T(x,y)\geq 0$
  Diffusion: $T(x,y)=\lim_{\tau\to 0}\frac{1/\tau}{\sqrt{4\pi D\tau}}e^{-[(x-y)^2/4D\tau]} \Rightarrow \dot{\rho}(x)=D\partial_x^2\rho(x)$

- All Markovian Quantum ME (Lindblad):
  $$\dot{\hat{\rho}}=-i[\hat{H},\hat{\rho}]+\sum_\alpha[\hat{L}_\alpha\hat{\rho}\hat{L}_\alpha^\dagger-\frac{1}{2}\{\hat{L}_\alpha^\dagger\hat{L}_\alpha,\hat{\rho}\}]$$
  Decoherence: $\hat{L}=\hat{L}^\dagger=\sqrt{2D}\hat{q} \Rightarrow \dot{\hat{\rho}}=-i\left[\hat{H},\hat{\rho}\right]-D[\hat{q},[\hat{q},\hat{\rho}]]$

- Generic Markovian Hybrid ME ('Pauli-Lindblad'):
  $\dot{\hat{\rho}}(x)=-i[\hat{H}(x),\hat{\rho}(x)]+$
  $$+\sum_{y,\alpha}\left[\hat{L}_\alpha(x,y)\hat{\rho}(y)\hat{L}_\alpha^\dagger(x,y)-\tfrac{1}{2}\{\hat{L}_\alpha^\dagger(y,x)\hat{L}_\alpha(y,x),\hat{\rho}(x)\}\right]$$

Hybrid ME: Derivation

- Embed Hybrid into a bigger Quantum:
  $$\hat\rho(x)\rightarrow\widehat{\rho}=\sum_x\hat\rho(x)\otimes|x\rangle\langle x|\,,\qquad \hat H(x)\rightarrow\widehat{H}=\sum_x\hat H(x)\otimes|x\rangle\langle x|$$
- Assume Lindblad ME:
  $$\dot{\widehat{\rho}}=-i[\widehat{H},\widehat{\rho}]+\widehat{L}\widehat{\rho}\widehat{L}^\dagger-\frac{1}{2}\{\widehat{L}^\dagger\widehat{L},\widehat{\rho}\}]$$
- Project back by $\hat I\otimes|x\rangle\langle x|$, introduce $\hat L(x,y)=\mathrm{Tr}'[(\hat I\otimes|y\rangle\langle x|)\widehat{L}]$
  $$\dot{\hat\rho}(x)=-i[\hat H(x),\hat\rho(x)]+ \\ +\sum_y\left[\hat L(x,y)\hat\rho(y)\hat L^\dagger(x,y)-\tfrac{1}{2}\{\hat L^\dagger(y,x)\hat L(y,x),\hat\rho(x)\}\right]$$

Quite general (Markovian) hybrid ME if $\hat L(x,y)$ is regular.
But, e.g., $\hat L(x,y)\sim\delta'(x-y)$, yields different forms.
Coming example: $\widehat{L}$ contains $(\partial_x|x\rangle)\langle x|$.

Case study: Q-monitoring

$\hat{\rho}$: Q-particle, $\dot{x}$: monitored value of $\hat{q}$

Naive hybrid ME:

$$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \tfrac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\}$$

Check: $d\langle x\rangle/dt = \mathrm{Tr}\int x\dot{\hat{\rho}}(x)dx = \mathrm{Tr}\int \hat{q}\hat{\rho}(x)dx = \mathrm{Tr}(\hat{q}\hat{\rho}) = \langle\hat{q}\rangle$

Problem: term $-\tfrac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\}$ may violate positivity of $\hat{\rho}(x)$

Try Lindblad ME for $\widehat{\rho}$ in big space, choose

$\widehat{L} = \hat{q}/\sqrt{2D} \otimes \hat{I} + \sqrt{2D}\,\hat{I} \otimes \int (\partial_x|x\rangle)\langle x|\,dx \quad \Rightarrow$

$$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \frac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\} - \frac{1}{16D}[\hat{q}, [\hat{q}, \hat{\rho}(x)]] + D\partial_x^2\hat{\rho}(x)$$

Hybrid ME of Q-monitoring.

In particular: $\dot{\rho}(x) = -\partial_x\langle\hat{q}\rangle_x\rho(x) + D\partial^2\rho(x)$

Equivalent with the Ito-formalism of Q-monitoring.

Summary

- Q-measurements $\Rightarrow$ natural hybrids:

$$\hat{\rho} \longrightarrow \hat{P}_k \hat{\rho} \hat{P}_k \equiv \hat{\rho}(k)$$

- Pauli-Lindblad theorem is missing for HME. Instead:

$$\hat{\rho}(x) \Rightarrow \widehat{\rho} \quad \Rightarrow \quad \dot{\widehat{\rho}} = \mathcal{L}\widehat{\rho} \quad \Rightarrow \quad \dot{\hat{\rho}} = \mathcal{L}_{hybr}\hat{\rho}(x)$$

- Time-continuous Q-measurement $\Rightarrow$ natural HME.

$$\dot{\hat{\rho}}(x) = -i[\hat{H}, \hat{\rho}(x)] - \frac{1}{2}\partial_x\{\hat{q}, \hat{\rho}(x)\} - \frac{1}{16D}[\hat{q}, [\hat{q}, \hat{\rho}(x)]] + D\partial_x^2\hat{\rho}(x)$$

- Q-measurement-generated hybrids are consistent ones: x is 'tangible', hybrid ME satisfies 'free will' test (D. 2012). Elze (2012): There are 7 further consistency tests at least.