Canonical equivalence of gravity and acceleration — two-page-tutorial

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(Dated: August 21, 2014)

Homogeneous Newtonian gravitational acceleration g in inertial frame is equivalent with zero gravity in free-falling g-accelerated frame. We present the corresponding canonical transformation of the free particle dynamics both classical and quantum.

I. FREE PARTICLE

In inertial (IN) frame, with IN canonical variables (Q, P), the Hamiltonian of free motion in gravity g reads:

$$H_g = \frac{P^2}{2m} + mgQ \ . \tag{1}$$

The solution of the canonical equations is

$$\dot{P} = -mg \ , \quad \dot{Q} = \frac{P}{m} \ , \tag{2}$$

leading to the acceleration $\ddot{Q} = -g$ of the particle.

Let us look at the same motion from the free-falling (FF) frame, in FF canonical variables (q, p). The Hamiltonian must be just

$$H_0 = \frac{p^2}{2m} \ . \tag{3}$$

The solution of the canonical equations is

$$\dot{p} = 0 \ , \quad \dot{q} = \frac{p}{m} \ , \tag{4}$$

leading to $\ddot{q} = 0$. The motion is completely forceless as it should be due to the equivalence principle.

Suppose that the IN and FF frames coincide instantaneously at $t = 0$. Then the solutions (2,4) are related by

$$Q = q - \frac{1}{2}gt^2 \ , \quad P = p - mgt \ , \tag{5}$$

which is a canonical transformation since it preserves the classical Poisson bracket (the quantum commutator) of the canonical variables. The Hamiltonians (1,3) are canonical transforms of each other but the naive expectation $H_G(Q, P) = H_0(q, p)$ is false, the correct transformation is different.

Let us consider the canonical transformation, e.g., from the FF to the IN frames:

$$q, \ p, \ H_0 \ \longrightarrow \ Q, \ P, \ H_g \ . \tag{6}$$

We construct it for the classical and for both the quantum Heisenberg as well as Schrödinger dynamics.

II. CANONICAL TRANSFORMATION OF CLASSICAL DYNAMICS

As to the general form, we invoke Landau-Lifschitz [1]:

$$p = \frac{\partial\phi}{\partial q} \ , \quad Q = \frac{\partial\phi}{\partial P} \ , \tag{7}$$

$$H_g = H_0 + \frac{\partial\phi}{\partial t} \ , \tag{8}$$

where $\phi(q, P, t)$ is a suitably chosen generator function.

In our case, we have to choose

$$\phi(q, P, t) = mgtq - \frac{1}{2}gt^2P - \frac{1}{3}mg^2t^3 + qP \ , \tag{9}$$

upto an irrelevant constant. This leads to the coordinate and momentum transformations (5) and to the desired transformation of the Hamiltonian:

$$H_g = \frac{p^2}{2m} - gtP + mgq - \frac{1}{2}mg^2t^2 = \frac{P^2}{2m} + mgQ. \tag{10}$$

Recall that $q = Q$ and $p = P$ at $t = 0$, the IN and FF frames coincide instantaneously.

III. CANONICAL TRANSFORMATION OF HEISENBERG DYNAMICS

Again, we construct the transformation of the canonical operators and Hamiltonians from the FF to the IN frames. The general form that preserves the structure of Heisenberg dynamics reads

$$Q = UqU^\dagger \ , \quad P = UpU^\dagger \ , \tag{11}$$

$$H_g = UH_0U^\dagger - i\dot{U}U^\dagger \ , \tag{12}$$

where the unitary U generates the transformation (we set $\hbar = 1$). The role of $-i\dot{U}U^\dagger$ is clear: the Heisenberg equation $\dot{q} = i[H_0, q]$ must be transformed into $\dot{Q} = i[H_g, Q]$, and similarly for the momenta.

We can choose U in function of q and p, we write it as $U = \exp(i\chi)$ where, as we see below, χ must be linear in q and p. To get (12) in a form closer to the classical version (8), we insert the identity $\dot{U} = i[H_0, U] + (\partial U/\partial t)$, yielding $H_g = H_0 - i(\partial U/\partial t)U^\dagger$. Using the Baker-Haussdorf identity for $U = \exp(i\chi)$ we get the ultimate form

$$H_g = H_0 + \frac{\partial\chi}{\partial t} - \frac{i}{2}\left[\frac{\partial\chi}{\partial t}, \chi\right] \ . \tag{13}$$

As to χ, we have to choose

$$\chi = mgtq - \frac{1}{2}gt^2p - \frac{1}{12}mg^2t^3 \ , \tag{14}$$

upto an irrelevant real constant. The Baker-Haussdorf identity can factorize $U = \exp(i\chi)$ into

$$U = \exp\left(-igt^2p/2\right)\exp(imgtq)\exp(img^2t^3/6)\ , \quad (15)$$

useful to confirm that the unitary transform (11) of the canonical variables leads to (5). As to the transformed Hamiltonian (13), we insert H_0 from (3) and χ from (14), obtaining the desired result:

$$H_g = \frac{p^2}{2m} + mgq - gtp = \frac{P^2}{2m} + mgQ\ . \quad (16)$$

Observe that $q = Q$ and $p = P$ at $t = 0$, the IN and FF frames coincide instantaneously. This also means that the canonical transformation between the (Heisenberg) states ρ_0 and ρ_g is trivial:

$$\rho_g = \rho_0\ . \quad (17)$$

IV. CANONICAL TRANSFORMATION OF SCHRÖDINGER DYNAMICS

Since at time t the FF Heisenberg operators (q, p) coincide with the IN Heisenberg operators (Q, P), therefore from now on we use notation q, p to denote the time-independent canonical Schrödinger operators in both frames. The dynamics in either frames is fully described by the corresponding Hamiltonians:

$$H_g = \frac{p^2}{2m} + mgq\ , \quad H_0 = \frac{p^2}{2m}\ . \quad (18)$$

The state vector $|\psi_g\rangle$ in the IN frame satisfies the Schrödinger equation $|\dot{\psi}_g\rangle = -iH_g|\psi_g\rangle$ and similarly does $|\psi_0\rangle$ in the FF frame. Since $|\psi_g\rangle$ and $|\psi_0\rangle$ coincide at $t = 0$, it is easy to find their relationship at any other time t:

$$|\psi_g\rangle = e^{-iH_gt}e^{iH_0t}|\psi_0\rangle \equiv U^\dagger|\psi_0\rangle\ , \quad (19)$$

where we expect that U coincides with the unitary operator of the canonical transformation of Heisenberg dynamics. We read out U from the equation above:

$$U = e^{-iH_0t}e^{iH_gt}\ . \quad (20)$$

This closed equation is the convenient one to get U, instead of the implicit method used previously in the Heisenberg dynamics. The r.h.s. really coincides with (15). An easy proof is if we write

$$\dot{U} = -ie^{-iH_0t}(H_0 - H_g)e^{iH_gt} = i(mgq - gpt)U\ , \quad (21)$$

and show the time-derivative of (15) coincides with it.

How do wave functions transform? The wave function is the projection amplitude of the state vector on the eigenstate $|q\rangle$ of the coordinate operator q. Remember, q is time-independent and common for both frames in question. Hence the IN wave function is $\psi_g(q) = \langle q|\psi_g\rangle$ and the FF wave function is $\psi_0(q) = \langle q|\psi_0\rangle$. Observe that

$$\langle q|\psi_g\rangle = \langle q|U^\dagger|\psi_0\rangle\ . \quad (22)$$

Using (15), we can write $U|q\rangle$ as

$$U|q\rangle = \exp(imgtq + img^2t^3/6)|q - gt^2/2\rangle\ . \quad (23)$$

Insert this into (22) and recognize the wave functions on both sides, yielding the relationship we have been looking for:

$$\psi_0(q - gt^2/2) = \exp(imgtq + img^2t^3/6)\psi_g(q)\ . \quad (24)$$

We can verify this result directly from elementary considerations. At time t, the FF frame is shifted by a distance $gt^2/2$ and moving at velocity gt with respect to the IN frame, therefore the corresponding wave functions would be related simply by $\psi_g(q) = \exp(-imgtq)\psi_0(q - gt^2/2)$. The presence of the additional phase is not a surprize. The acceleration g imposes an additional kinetic energy $mg^2t^2/2$ upon the IN wave function, and during time t this accumulates an addional phase, leading to $\psi_g(q) = \exp(-imgtq - img^2t^3/6)\psi_0(q - gt^2/2)$. For an elementary derivation, see reference in [2].

V. LESSONS

Transformation between inertial (IN) and free-falling (FF) canonical variables (5) is easy to write up—do it for yourself—, they are what we expect naively. The systematic canonical transformation, including that of the Hamiltonian, is less trivial, it has been our main goal. In classical dynamics, we just quoted the generating function method from Landau-Lifshitz, without teaching about the underlying math—you may learn it from there. In quantum mechanics, the central object is the unitary operator U of canonical transformation. In Heisenberg dynamics, we showed an indirect construction of U— a but involved but instructive. Schrödinger dynamics is more convenient, the canonical transformation U can be directly constructed (20)—like it! Finally, one can easily relate the IN and FF wave functions, the strange relative phase $mg^2t^3/6$ gets clear physical interpretation.

[1] L.D. Landau and E.M. Lifshitz: Mechanics (Pergamon Press, 1960).
[2] R. Colella, A.W. Overhauser, S.A. Werner, Phys. Rev. Lett. **34** 1472 (1975).