Feynman path integral and Weyl ordering — one-page-tutorial

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For those who know already what standard Feynman path integral is, the operator ordering ambiguity is discussed and resolved.

I. FEYNMAN PATH INTEGRAL

Matrix-elements of time-ordered operator functionals, defined on a given Hilbert space, can formally be expressed by Feynman path integrals over classical fields:

$$\langle x_f | e^{-iT\hat{H}} \mathrm{T}\Phi[\hat{x}] | x_i \rangle = \int \Phi[x] D_F x \ ,$$

for an arbitrary functional $\Phi[x]$ of the time-dependent canonical coordinate $\{x_t; t \in [0,T]\}$. Here $\hat{H} = \hat{p}^2/2$ with the canonical momentum $\hat{p}$, the symbol T stands for time-ordering, and $\hat{x}_t$ is the time-dependent Heisenberg coordinate operator $\hat{x}_t = e^{it\hat{H}} \hat{x}_0 e^{-it\hat{H}}$. The Feynman measure is the standard one:

$$D_F x \equiv exp\left(\frac{i}{2}\int_0^T \dot{x}_t^2 dt\right) \delta(x_T - x_f)\delta(x_0 - x_i) \prod_{t\in[0,T]} dx_t \ .$$

The formal proof of our starting equation is usually achieved by substituting the Fourier form:

$$\Phi[x] = \int \tilde{\Phi}[k] \exp(i\int_0^T k_t x_t dt) \prod_{t\in[0,T]} dk_t \ ,$$

then by introducing the standard discretized time-slicing procedure.

II. ORDERING AMBIGUITIES

Consider the particular case when the functional Φ depends *explicitly* on the velocities $\dot{x}$ as well:

$$\Phi[x] = \Phi[x, \dot{x}] \ .$$

It is well known that the interpretation of the above functional as the integrand of a Feynman path integral becomes ill-defined. We have to fix the problem. The simplest innocent ansatz is that, when defining $D_F x$ in the limit of a time-sliced discretization, the discrete *symmetric* time-derivative is adopted for $\dot{x}$. It is less common that the interpretation of the corresponding time-ordered functional $\Phi[\hat{x}, \dot{\hat{x}}] = \Phi[\hat{x}, \hat{p}]$, too, becomes ill-defined. We fix this ordering issue in accordance with the previous ansatz of symmetric discrete time-derivatives for $\dot{\hat{x}}$ on a time-sliced basis. With these additional conventions the formal equivalence of time-ordered operator functionals and Feynman path integrals becomes correct.

III. WEYL ORDERING

It turns out that the chosen ordering is just the Weyl ordering, we denote it by W. Weyl ordering prescribes complete symmetrization between equal time operators $\hat{x}_t$ and $\dot{\hat{x}}_t = \hat{p}_t$. In general:

$$\begin{aligned} \mathrm{W}\hat{x}\mathrm{W}\phi(\hat{x},\hat{p}) &= \frac{1}{2}\{\hat{x}, \mathrm{W}\phi(\hat{x},\hat{p})\} \ , \\ \mathrm{W}\hat{p}\mathrm{W}\phi(\hat{x},\hat{p}) &= \frac{1}{2}\{\hat{p}, \mathrm{W}\phi(\hat{x},\hat{p})\} \ . \end{aligned}$$

In particular: $\mathrm{W}\hat{x}\hat{p} = \frac{1}{2}\{\hat{x},\hat{p}\}$. By using Weyl ordering on the l.h.s. and symmetric time-derivatives

$$\dot{x}_{t,\mathrm{sym}} = \frac{1}{2}\left(\dot{x}_{t-0} + \dot{x}_{t+0}\right)$$

on the r.h.s. of our starting identity, it becomes well-defined:

$$\langle x_f | e^{-iT\hat{H}} \mathrm{WT}\Phi[\hat{x},\hat{p}] | x_i \rangle = \int \Phi[x, \dot{x}_{\mathrm{sym}}] D_F x \ .$$