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Localized Objects Formed by Self-Trapped Gravitational Waves*♯

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Abstract—Geons are localized horizonless objects formed by gravitational waves, held together by the gravitational attraction of their own field energy. In many respects they are similar to scalar field pulson/oscillon configurations, which were found numerically in 1976 by Kudryavtsev, Bogolyubskii, and Makhankov. If there is a negative cosmological constant, the spacetime of geons asymptotically approaches the anti-de Sitter (AdS) metric. AdS geons are time-periodic regular localized vacuum solutions without any radiation loss at infinity. A higher order perturbative construction in terms of an amplitude parameter shows that there are one-parameter families of AdS geon solutions emerging from combinations of identical-frequency linear modes of the system.

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1. INTRODUCTION

The search for long-living localized finite energy solutions is a very important longstanding problem in theoretical physics. For the profound understanding, it is crucial to identify the simplest physical models in which such states may exist. It is also instructive to explore what are the simplest matter fields that can form such structures, and to see whether vacuum gravitational waves are capable to make long-living localized objects. A general feature that can be observed is that nonlinearity is essential in these systems. The size of the objects may cover a huge range, from the size of particles to stars or even galaxies, depending on the value of the parameters in the theories.

In many important systems there are no time-independent localized configurations, but still, there are long-living solutions oscillating in time. Probably the simplest such physical system is a single real scalar field on a flat Minkowski background. The object formed in this case is named pulson, or more recently oscillon. When the real scalar field is coupled to Einstein's gravity the accepted name is oscillaton. The localized configurations formed by gravitational or electromagnetic waves are known under the name geons. We give some more information about these object in the next few paragraphs.

The equation describing a spherically symmetric real scalar field ϕ, with self-interaction potential $U(\phi)$, in case of d spatial dimensions can be written as

$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial r^2}+\frac{d-1}{r}\frac{\partial\phi}{\partial r}=U'(\phi), \qquad (1)$$

where t is the time coordinate, r is the radial coordinate, and the prime denotes derivative with respect to ϕ. For $d\geq 2$ dimensions we require a regular center, and for $d=1$ we ask for mirror symmetry at $r=0$. Exactly time-periodic, localized, regular, finite energy solution for ϕ only exist for $d=1$ spatial dimension, and even in that case only for $U(\phi)=1-\cos\phi$, the sine-Gordon potential. The solution is called sine-Gordon breather, and it is an exact solution that can be written in a well known simple form.

However, this does not mean that there are no regular long living stable finite energy solutions in higher dimensions or for other potentials. The most thoroughly studied case is the $U(\phi)=\frac{1}{4}(\phi^2-1)^2$ symmetric double well potential. For a large class of potentials there are oscillating "almost-breather" solutions, which are weakly emitting energy by scalar field radiation. Because of this, they have a slowly changing amplitude and frequency. The likelihood of the existence of these configurations for not exactly solvable one dimensional systems were first studied in 1975 by Kosevich and Kovalev [1], and also by Dashen, Hasslacher, and Neveu [2]. In the same year Kudryavtsev have shown by numerical methods that long living localized states form by the collision

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of two kinks [3]. Spherically symmetric states in $3+1$ spacetime dimensions were first found in 1977 by Bogolyubskii and Makhan'kov in Dubna, by a numerical time evolution code [4, 5]. The original name for these solutions were *pulsons*. The interest about these configurations began to grow after 1994, when Marcelo Gleiser studied them in more details [6]. Next year, in their paper Copeland, Gleiser, and Müller started to call these objects *oscillons* [7], which is still the most widely used name for them in the literature. Oscillons in case of one or two space dimensions live forever, with more and more slowly decreasing amplitude, but oscillons in three or more dimensions suddenly decay after a few hundred, or possibly few thousand oscillations. The reason for this quick decay is that when the amplitude falls below a certain value they enter into an unstable domain.

There are localized configurations similar to oscillons even when the scalar field is coupled to gravity. Seidel and Suen in 1991 discovered spherically symmetric oscillating solutions for a real scalar field in the theory of general relativity [8]. These objects became known under the name *oscillatons* [9]. In this case gravity already provides the necessary nonlinearity, hence a simple $U(\phi) = \frac{1}{2}m^2\phi^2$ Klein–Gordon potential can already support these structures. Even if one chooses a more general self-interaction potential, for small amplitude oscillatons the leading order behavior will be same as for the Klein–Gordon potential. For large distances gravitation dominates over the self-interaction of the scalar field. There is also a difference in the small amplitude scaling behavior between oscillons and oscillatons. As a result of that, for three spatial dimensions, small amplitude oscillatons are stable. This means that oscillatons live forever, there is no moment of time when they suddenly decay.

The general structure of oscillons and oscillatons is shown on Fig. 1. In both cases there is a large amplitude core with slow spatial variations in the scalar field, and a very small amplitude outgoing wave tail. If the central amplitude is ε, then the tail amplitude is proportional to $\exp(-a/\varepsilon)$, where a is a positive constant. Detailed discussion about the radiation rate can be found for oscillons in the papers [10–12], and for oscillatons in [13–15]. As time passes, the energy of the core decreases slowly, together with its amplitude. However, the amplitude of the radiating tail decreases quickly because of the exponential dependence, and the radiation becomes extremely small in a relatively short time. Self-gravitating oscillatons never decay, and a typical oscillaton loses less than half of its energy during the lifetime of our universe.

A further type of localized object formed by oscilating fields are called *geons*. The concept of geon was introduced in 1955 by John Archibald Wheeler,

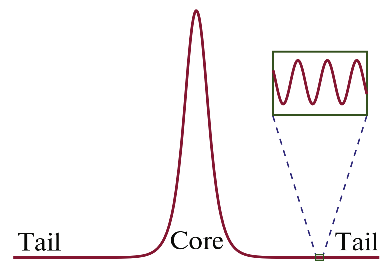

Fig. 1. Oscillons and oscillatons consist of a core and an outgoing wave tail with exponentially small amplitude.

considering high frequency self-gravitating electromagnetic fields [16]. The name comes from the phrase "gravitational-electromagnetic entity". Wheeler first considered a toroidal geon, where the electromagnetic radiation goes around in a circle, inside a toroidal region. The structure is held together by the gravitational attraction of the mass associated with the electromagnetic field energy. Wheeler studied very thin tori, where the minor radius is much smaller than the major radius. In this case small amplitude high frequency waves are going around in a circle. Wheeler had the idea to consider a system which includes a large number of identical-size thin toroidal geons with different orientations. In this way, the metric on the large scale becomes spherically symmetric and static. There is a thin sphere active region, where the high frequency waves are concentrated. The metric of the inner region is flat, outside of the sphere it is Schwarzschild. Brill and Hartle demonstrated in 1964, that instead of electromagnetic waves, geons can also form from vacuum gravitational waves, and these geons have the same large scale metric structure [17]. There are no exact or numerical solutions for geons, we can only infer their existence using spacetime averaging methods, considering small amplitude high frequency perturbations. The averaging procedure was made more precise in 1997 by Anderson and Brill [18]. The stability and lifetime of these geon structures is still under debate.

2. NEGATIVE COSMOLOGICAL CONSTANT

For $\Lambda < 0$ the spacetime of localized objects must tend asymptotically to the anti-de Sitter (AdS) metric. Negative cosmological constant provides an effective attractive force, and because of this, the formation of localized structures becomes easier.

If $\Lambda = 0$, when the amplitude of oscillons or oscillatons decreases, their spatial size grows without limit. If there is a negative Λ, oscillons or oscillatons have finite sized small amplitude limits, corresponding to the linear modes of the theory. There are one-parameter families of solutions emerging from each linear mode [19–21]. A further important difference from the $\Lambda = 0$ case, is that for negative Λ there are exactly periodic oscillons and oscillatons, without any energy loss by scalar field radiation at infinity. Being similar to the sine-Gordon breather, it is more appropriate to call these scalar field objects AdS breathers.

A negative cosmological constant also makes the formation of vacuum gravitational wave geons easier. For $\Lambda < 0$ there are smooth time-periodic vacuum geon solutions without any high frequency low wavelength components, and without energy loss [22–28]. These solutions are called AdS geons, which naming we will use in the following. Conceptually, the AdS breather name might have been more appropriate, since these configurations are quite different in many respects from the $\Lambda = 0$ geons. We will discuss AdS geons in detail in the next section. Their study is technically more involved than that of the scalar field case because of the absence of spherically symmetric solutions.

Since the AdS geons that we consider can be constructed as higher order perturbations of anti-de Sitter spacetime, we first discuss some of its important properties. The AdS metric in global spatially compactified coordinates can be written as

$$\mathrm{d}s^2 = \frac{L^2}{\cos^2 x}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + \sin^2 x\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\Big], \tag{2}$$

where $L^2 = -\dfrac{3}{\Lambda}$. We can represent the spacetime as an infinitely long strip, where each point corresponds to a 2-sphere with radius $L\tan x$. The center of symmetry is at $x = 0$, while infinity is represented by $x = \dfrac{\pi}{2}$. The range of the time coordinate is $-\infty < t < \infty$. The metric is static in these coordinates, but the radial outwards acceleration of constant x observers is $\dfrac{\sin x}{L}$. All timelike geodesics emanating from a point meet again at another point later. An excellent review of various AdS coordinate systems can be found in the book of Griffiths and Podolský [29].

Using the t, x coordinate system, it is obvious, that a light ray can travel to infinity and arrive back in a finite time, when time is measured by a central observer. This is related to the observed instability of the AdS spacetime [30]. Since a wave packet can bounce back many times to the center, it can become more and more concentrated, and after many reflections it still can collapse to a black hole. For smaller amplitude initial packets more bounces needed, as it has been demonstrated numerically by Bizoń and Rostworowski in 2011 for a spherically symmetric massless scalar field coupled to gravity [30]. For this instability the assumption of reflective boundary conditions is essential. Although other type of boundary conditions can be considered, for our purposes the most natural one appears to be the reflective boundary condition.

Since the metric of geons will approach asymptotically the AdS metric, we have to discuss asymptotically AdS spacetimes. The most natural definition is by conformal compactification, based on the original ideas of Roger Penrose [31–34]. This procedure also gives definition for conserved quantities, the total mass and the three components of the angular momentum. The main ingredient of the method is the introduction of a conformally rescaled unphysical metric, $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. This rescaling makes $\tilde{g}_{\mu\nu}$ regular at the surface corresponding to infinity, which is the surface $x = \dfrac{\pi}{2}$ in our case. For the AdS metric (2) the choice $\Omega = \dfrac{\cos x}{L}$ is the most natural.

3. AdS GEONS

AdS geons are localized time-periodic vacuum solutions for $\Lambda < 0$, with regular center and no horizon. They are bound states formed by vacuum gravitational waves. Their typical size is given by the length-scale determined by the cosmological constant, $L = \sqrt{-\dfrac{3}{\Lambda}}$. AdS geon solutions were first constructed by a higher order perturbative expansion [22], and later also by numerical methods [25, 26].

In this paper we will concentrate on the results that can be obtained by the small-amplitude expansion method [35]. Let us consider a one-parameter family of solutions depending on a parameter ε, and expand the metric $g_{\mu\nu}$ of the AdS geon spacetime as

$$g_{\mu\nu} = \sum_{k=0}^{\infty} \varepsilon^k g^{(k)}_{\mu\nu}. \tag{3}$$

The background, $g^{(0)}_{\mu\nu}$, is the anti-de Sitter metric in the form given in (2). We can keep the conformal factor $\Omega = \dfrac{\cos x}{L}$, independently of ε. In this way, $g^{(0)}_{\mu\nu}$ has components that diverge as Ω^{-2} at infinity $x = \dfrac{\pi}{2}$. A natural choice would be to require that for $k \geq 1$ all $g^{(k)}_{\mu\nu}$ diverge at most as Ω^{-1}. That way the metric induced at the surface corresponding to infinity would not change, and $g_{\mu\nu}$ would be asymptotically AdS according to the conformal definition.

However, since the oscillation frequency of AdS geons is amplitude dependent, it is advantageous to make a slight modification on the above asymptotic behavior of $g^{(k)}_{\mu\nu}$. The $\bar{\omega}$ physical frequency depends on the parameter ε, and as we will see, in the small ε limit it always approaches an integer value. For technical simplicity we require, that in terms of the time coordinate t that we use, the coordinate frequency ω should remain an ε independent integer. Because of this, the asymptotically AdS time coordinate $\hat{t}$ will be different from the t we use, the two connected by a factor depending only on ε. This setting can be achieved by requiring the boundary conditions for $k \geq 1$,

$$\lim_{x\to\frac{\pi}{2}} \left(\Omega^2 g^{(k)}_{tt}\right) = -\nu_k, \tag{4}$$

$$\lim_{x\to\frac{\pi}{2}} \left(\Omega^2 g^{(k)}_{\mu\nu}\right) = 0 \quad \text{for} \quad \mu \neq t \quad \text{or} \quad \nu \neq t, \tag{5}$$

where ν_k are constants. Then the asymptotic behavior of g_{tt} is

$$\lim_{x\to\frac{\pi}{2}} \left(\Omega^2 g_{tt}\right) = -\nu, \quad \nu = 1 + \sum_{k=1}^{\infty} \varepsilon^k \nu_k. \tag{6}$$

It follows that the asymptotically AdS time coordinate is $\hat{t} = t\sqrt{\nu}$. The physical frequency $\bar{\omega}$ has to be calculated with respect to a time coordinate that asymptotically agrees with the Schwarzschild time coordinate $\bar{t} = L\hat{t} = tL\sqrt{\nu}$. The relation between the physical and the coordinate frequencies is

$$\bar{\omega} = \frac{\omega}{L\sqrt{\nu}}. \tag{7}$$

We $2+2$ decompose the metric along the symmetry spheres of the background AdS. We use indices that take the values $a, b, c \ldots = 1, 2$, and $i, j, k \ldots = 3, 4$. The coordinates along the time–radius plane are $x^a = (x^1, x^2) = (t, x)$, and the coordinates along the symmetry spheres are $x^i = (x^3, x^4) = (\theta, \varphi)$. We decompose the background AdS spacetime as

$$\mathrm{d}s^2_{(0)} = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + r^2\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{8}$$

where $r = L\tan x$, and

$$g_{ab} = \frac{L^2}{\cos^2 x}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad \gamma_{ij} = \begin{pmatrix}1 & 0\\ 0 & \sin^2\theta\end{pmatrix}. \tag{9}$$

We expand the solutions in terms of real spherical harmonics $\mathbb{S}_{lm}$, where $l \geq 0$ and $-l \leq m \leq l$ integers. The definition of real harmonics is the same as the complex ones, except that the φ dependence is $\cos(m\varphi)$ for $m \geq 0$, and $\sin(|m|\varphi)$ for $m < 0$. The use of complex harmonics would be less practical when going to nonlinear orders in the expansion. Tensors on arbitrary dimensional sphere can be decomposed into three types of parts. The first is the scalar-type part (also called polar or even parity), the second is the vector-type part (axial or odd parity), and the third is the tensor-type part. The tensor-type only exists for more than two dimensional spheres, so we do have to consider that now. In case of four spacetime dimensions, the spheres are two dimensional, and the components of vector spherical harmonics $\mathbb{V}_{(lm)i}$ can be expressed in terms of the scalar harmonics,

$$\mathbb{V}_{(lm)\theta} = \frac{1}{\sqrt{l(l+1)}}\frac{1}{\sin\theta}\frac{\partial\mathbb{S}_{lm}}{\partial\varphi},$$
$$\mathbb{V}_{(lm)\varphi} = \frac{-1}{\sqrt{l(l+1)}}\sin\theta\frac{\partial\mathbb{S}_{lm}}{\partial\theta}. \tag{10}$$

When constructing AdS geons by the small-amplitude expansion procedure, we start with the linear order contribution in ε, and then proceed order by order. At each order in ε scalar- and vector-type perturbations for each l, m can be considered separately. There are separate systems of linear equations determining the chosen type contribution to the metric components for each l, m, independently for scalar and vector perturbations. These equations contain inhomogeneous source terms which are already fixed by the results obtained at lower orders in the ε expansion.

3.1. Vector-Type Perturbations of the Metric

We consider vector-type first, because it is technically simpler than the scalar-type. Even if we start with scalar-type perturbations at linear order in ε, vector-type components appear at ε^2 order. In the Regge–Wheeler gauge [36], the $\mathbb{V}_{(lm)i}$ vector-type contributions to the metric at ε^k order in the expansion are

$$g^{(k)}_{ab} = 0, \quad g^{(k)}_{ai} = Z_a\mathbb{V}_i, \quad g^{(k)}_{ij} = 0, \tag{11}$$

where $a, b = 1, 2$ and $i, j = 3, 4$. There are only two unknown functions, $Z_a \equiv (Z_t, Z_x)$, depending on the coordinates $x^a = (t, x)$. The $l = 1$ spherical harmonics have to be considered separately. For the linear order in ε the $l = 1$ case only has the trivial solution $Z_a = 0$, while for higher orders in ε the $l = 1$ contribution determines the angular momentum. For $l \geq 2$, from the (i, j) components of Einstein's equations follows that there exists a scalar function ϕ such that

$$Z_t = \frac{\partial\phi}{\partial x} + \bar{Z}_t, \quad Z_x = \frac{\partial\phi}{\partial t} + \bar{Z}_x, \tag{12}$$

where $\bar{Z}_t$ and $\bar{Z}_x$ are already known functions fixed by lower order perturbations. Obviously, $\bar{Z}_t$ and $\bar{Z}_x$ are zero at linear order. Defining a rescaled scalar function by $\phi = r\Phi$, where $r = L\tan x$, from the (a, i) components of Einstein's equations follows that

$$-\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial^2\Phi}{\partial x^2} - \frac{l(l+1)}{\sin^2 x}\Phi = \frac{\bar{\Phi}}{\sin^2 x}, \tag{13}$$

where $\bar{\Phi}$ is a known function of t, x determined at lower order in ε. Having the solution for the scalar function Φ, it gives Z_a using (12), and then it determines all vector-type metric perturbation components according to (11). The function Φ is related to the Regge–Wheeler function in black hole perturbation theory [36]. The generated metric in the vector-type case will be asymptotically AdS if

$$\lim_{x\to\frac{\pi}{2}} \Phi = 0. \tag{14}$$

3.2. Scalar-Type Perturbations of the Metric

For each $l \geq 2$ and m indices, scalar-type perturbations are also governed by a function Φ, which satisfies the same equation as the vector-type Φ, equation (13). Only the boundary conditions at infinity are different in the two cases. In the scalar-type case the generated metric will be asymptotically AdS if

$$\lim_{x\to\frac{\pi}{2}} \frac{\mathrm{d}\Phi}{\mathrm{d}x} = 0. \tag{15}$$

The function Φ in the scalar-type case is related to the Zerilli function in black hole pertubation theory [37, 38]. Defining again a rescaled function by $\phi = r\Phi$, we can calculate the quantities Z_{ab} and Z,

$$Z_{tt} = \partial_t^2\phi - \tan x\partial_x\phi + \frac{\phi}{\cos^2 x} + \bar{Z}_{tt}, \tag{16}$$

$$Z_{tx} = \partial_t\partial_x\phi - \tan x\partial_t\phi + \bar{Z}_{tx}, \tag{17}$$

$$Z_{xx} = \partial_x^2\phi - \tan x\partial_x\phi - \frac{\phi}{\cos^2 x} + \bar{Z}_{xx}, \tag{18}$$

$$Z = \frac{\cos^2 x}{L^2}\left(Z_{xx} - Z_{tt}\right) + \bar{Z}, \tag{19}$$

where $\bar{Z}_{ab}$ and $\bar{Z}$ are determined from the lower order results in ε. Then, from these we can define

$$H_L = \frac{r^2}{2}Z, \quad H_{ab} = Z_{ab} - \frac{1}{2}Zg_{ab}. \tag{20}$$

In terms of these quantities, the general $\mathbb{S}_{lm}$ scalar-type metric perturbations in the Regge–Wheeler gauge [36] at ε^k order can be written as ($a, b = 1, 2$; $i, j = 3, 4$)

$$g_{ab}^{(k)} = H_{ab}\mathbb{S}, \quad g_{ai}^{(k)} = 0, \quad g_{ij}^{(k)} = H_L\gamma_{ij}\mathbb{S}. \tag{21}$$

The $l = 0, 1$ scalar-type perturbations have to be treated separately. In these cases there is no generating scalar function. They only give gauge modes at linear order in ε. The $l = 0$ mode determines the contribution to the mass of the AdS geon at higher orders. The method that we have applied is a higher order generalization of the gauge invariant formalism of Mukohyama [39], Kodama, Ishibashi, Seto [40, 41], and Wald [42].

3.3. Periodic Solutions at Linear Order

At order ε^1 there are no inhomogeneous source terms, so $\bar{\Phi} = 0$ in (13). We search solutions in the form $\Phi = p\cos(\omega t)$, where p depends only on the radial coordinate x. For scalar-type perturbations, when the boundary condition is given by (15), centrally regular and asymptotically AdS solutions only exist with frequencies $\omega = l + 1 + 2n$, where $n \geq 0$ is an integer, and the function p is

$$p = \sin^{l+1} x \frac{n!}{(l + \frac{3}{2})_n} P_n^{(l+\frac{1}{2}, -\frac{1}{2})}(\cos(2x)). \tag{22}$$

Here $(c)_n = \Gamma(c + n)/\Gamma(c)$ is the Pochhammer's symbol, and $P_n^{\alpha,\beta}(z)$ are Jacobi polynomials. For vector-type perturbations the boundary condition is (14), and solutions exist with frequencies $\omega = l + 2 + 2n$, for $n \geq 0$ integers, and then

$$p = \sin^{l+1} x \cos x \frac{n!}{(l + \frac{3}{2})_n} P_n^{(l+\frac{1}{2}, \frac{1}{2})}(\cos(2x)). \tag{23}$$

In both cases n gives the number of radial nodes, i.e. the number of zero crossings.

For each (l, m, n), where $l \geq 2$, $|m| \leq l$, $n \geq 0$ integers, there is a scalar- and a vector-type linear geon mode with arbitrary amplitude. Since the frequencies of all these modes are integers, an arbitrary linear combination of them is still a time-periodic solution with $\omega = 1$. This shows that there is an infinite-parameter family of linear geons, which are all valid to first order. However, it appears that the nonlinear system only has one-parameter families of AdS geon solutions. This is true for all cases studied by the higher order nonlinear expansion formalism, where we have to start with finite number of parameters, and also supported by direct numerical search for time-periodic solutions of Einstein's equations, however we are not aware of a proof of it.

3.4. Higher Orders in the Expansion

Let us now consider the inhomogeneous scalar equation (13) at higher orders in the ε expansion. Then the homogeneous left hand side has solutions with frequency $\omega = l + 1 + 2n$ in the scalar-type case, and $\omega = l + 2 + 2n$ in the vector-type case. The inhomogeneous right hand side, in general, consists of sums of source terms, where $\bar{\Phi} = p_0 \sin(\omega_0 t)$ or $\bar{\Phi} = p_0 \cos(\omega_0 t)$, for various ω_0 integer frequencies, and functions p_0 depending on x. We can consider these source terms one by one, and add the corresponding solutions for Φ in the end. Picking a term with a specific ω_0, if $\omega \neq \omega_0$ for all $n \geq 0$ integers, where $\omega = l + 1 + 2n$ in the scalar-type case, and $\omega = l + 2 + 2n$ in the vector-type case, there are always time-periodic

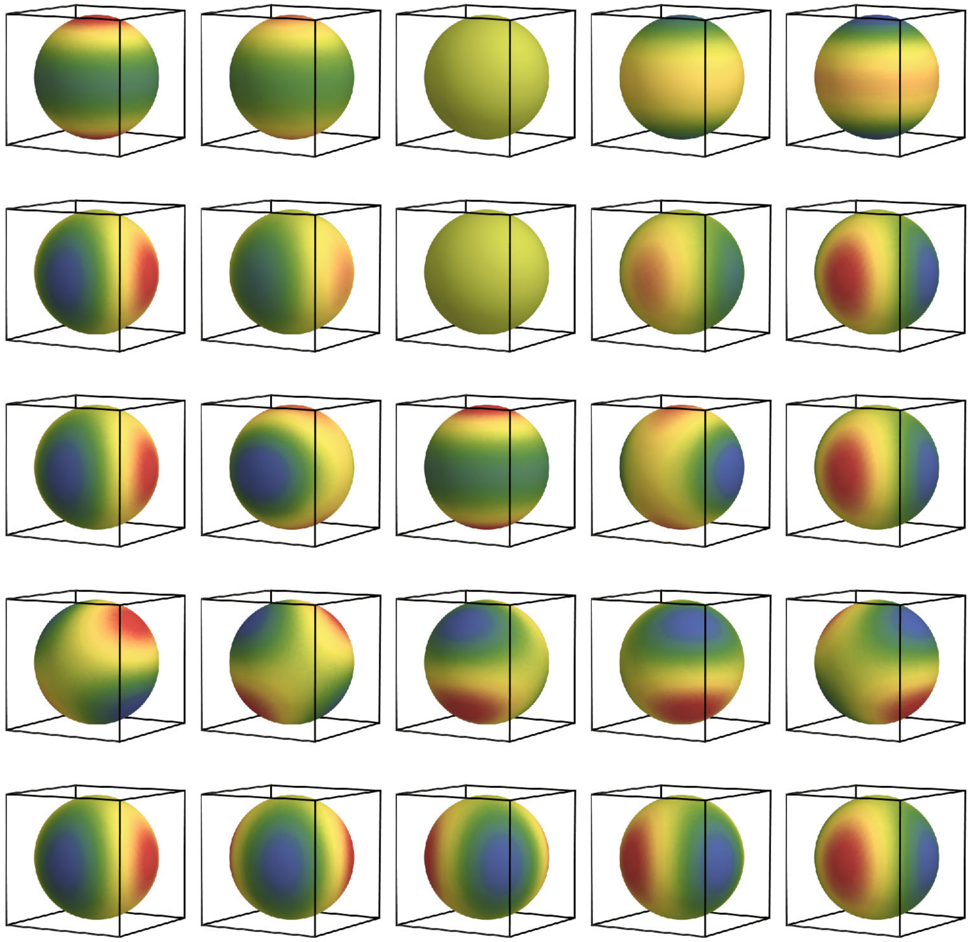

Fig. 2. Snapshots of the angular dependence of five $\omega = 3$ geon families. The colors indicate the value of the scalar field ϕ. Red is positive, blue is negative, light yellowish-green corresponds to zero. The different families are in the five rows, and the columns correspond to the moments of time $t = 0$, $\frac{\pi}{12}$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$. The first row is an axially symmetric solution corresponding to the $(2, 0, 0, 3)_S$ mode. The second row is also a nonrotating solution belonging to $(2, 2, 0, 3)_S$. The third row shows a third nonrotating solution, emerging from the combination of time-shifted $(2, 2, 0, 3)_S$ and $(2, 0, 0, 3)_S$ modes, and oscillating between them. The fourth row is a rotating solution emerging from the combination of the $(2, 1, 0, 3)_S$ and $(2, -1, 0, 3)_S$ modes. The last row is also a rotating solution corresponding to the $(2, 2, 0, 3)_S$ and $(2, -2, 0, 3)_S$ modes.

Φ solutions of (13) which are asymptotically AdS and have a regular center. However, if $\omega = \omega_0$ for some n, then $\bar{\Phi}$ is a resonant source term. Generally, taking a resonant source term, regular asymptotically AdS solutions for Φ in (13) are blow-up solutions of the type $t\cos(\omega t)$. Appropriate time-periodic solution for Φ corresponding to a resonant source term can only exist if a consistency condition holds. For each resonant source term, having the form $\bar{\Phi} = p_0 \sin(\omega_0 t)$ or $\bar{\Phi} = p_0 \cos(\omega_0 t)$, the consistency condition is

$$\int_0^{\frac{\pi}{2}} \frac{p_{l,n} p_0}{\sin^2 x} \mathrm{d}x = 0, \qquad (24)$$

where $p_{l,n}$ is the regular solution of the left hand side homogeneous equation.

The consistency conditions determine the change of physical frequency $\bar{\omega}$ as a function of the amplitude ε, and they also determine the ratio of the modes that were included initially at the linear order. If the consistency conditions cannot be satisfied, then there are terms with linearly increasing amplitude $t\cos(\omega t)$. Since these are generally higher harmonics, this means shift of energy towards higher frequency modes, which can be the starting point to turbulent instability, that may lead eventually to black hole formation.

When building up an AdS geon by the expansion procedure, the natural simplest choice is to start with only one mode at linear order. There is a scalar and a vector mode for each $l \geq 2$, $|m| \leq l$, and $n \geq 0$ integers. Let us represent the scalar mode by the four numbers $(l, m, n, \omega_s)_S$, where $\omega_s = l + 1 +$

$2n$. Similarly, let us denote the vector-mode by $(l, m, n, \omega_v)_V$, where $\omega_v = l + 2 + 2n$. It was realized early on, that for some single linear modes there is no corresponding nonlinear AdS geon solution [22], since the consistency conditions cannot be solved at ε^3 order. Examples are: $(2, 0, 1, 5)_S$, $(4, 0, 0, 5)_S$, $(3, 2, 0, 4)_S$, $(2, 2, 0, 4)_V$. The resolution for the problem is that we have to start with more than one mode, and take the linear combination of same-frequency modes at linear order in ε [24]. For example, including the $(2, 0, 1, 5)_S$ mode with amplitude α, and the $(4, 0, 0, 5)_S$ mode with amplitude β, the consistency conditions at third order will fix the ratio $\alpha/\beta \approx 0.12909$ or -152.52. Consequently, there will be two non-trivial one-parameter families arising from these two linear modes with frequency $\omega = 5$.

A further surprising result is that there are non-rotating non-axially symmetric geons [35]. An example is the geon emerging from the linear mode $(l, m, n, \omega_s)_S = (2, 2, 0, 3)_S$. Although the angular dependence of the linearized solution is $\cos(2\varphi)$, there is a corresponding nonlinear solution which has zero angular momentum. When we take identical-amplitude linear combination of the $(2, 2, 0, 3)_S$ and $(2, -2, 0, 3)_S$ modes with a shift in time phase, we get a rotating linearized solution, which generates a rotating nonlinear geon with a helical Killing vector.

The above results show that same-frequency linear modes should be treated together. The lowest possible frequency is $\omega = 3$, which belongs to the $l = 2$, $n = 0$ scalar modes. We have constructed all AdS geon solutions that in the small-amplitude limit reduce to $\omega = 3$ modes only. There are five such linear modes, belonging to $m = -2, -1, 0, 1, 2$. Since each of them can have $\cos(3t)$ or $\sin(3t)$ time dependence, there are 10 independent amplitude constants. Solutions are considered equivalent if they can be transformed into each other by time shift and spatial rotation. The result of our detailed analysis up to ε^5 order shows that there are 5 nonequivalent one-parameter families that emerge from $\omega = 3$ frequency modes only [35]. The change of the angular behavior of the scalar field during a half oscillation period is shown on Fig. 2 for these five families.

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