# A natural extension of the conformal Lorentz group in a field theory context

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# Introduction

- Well-known in theory QFT models: simple unification of spacetime and internal (gauge) symmetries is prohibited by Coleman-Mandula no-go theorem.
- Its assumptions about the gauge group: direct product of copies of $\mathrm{U}(1)$ and a real semisimple compact Lie group.
  - Theoretical conveniences: existence of an invariant symmetric non-degenerate positive definite bilinear form over the Lie algebra of internal symmetries.
  - Experimental justification: Standard Model has $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ internal (gauge) symmetries.
- Consequence for total symmetries at a fix point of spacetime or momentum space (point symmetry group): it is direct product of copies of $\mathrm{U}(1)$ and a real semisimple Lie group.

  [E.g. in Standard Model it is $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SL}(2, \mathbb{C})$. In conformally invariant versions of Standard Model it is $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{D}(1) \times \mathrm{SL}(2, \mathbb{C})$.]
- SUSY was constructed in order to circuimvent Coleman-Mandula theorem. Internal symmetries (or part of) are mixed with the Poincaré group via coupling to translations. $\Rightarrow$ It is semi-direct product of those internal symmetries and spacetime symmetries.

- Is is widely believed that SUSY is the only way to connect (part of) the internal symmetries to spacetime symmetries.
- Note: SUSY is not yet supported by experimental evidence, up to LHC energies.

## Is SUSY really the only physically plausible mathematical way?

- We intend to show a simple, physically plausible mathematical example for a point symmetry group, which is some nontrivial extension of the (conformal) Lorentz group (actually, of its covering group).
- It is a point symmetry group, i.e. spacetime point (or, equivalently, momentum space point) is fixed.
  $\Rightarrow$ coupling of (part of) internal symmetries to spacetime symmetries is not done through translation generators, but purely inside point symmetry group.
- It circuimvents Coleman-Mandula theorem via non-semisimpleness of the group.
- Traditionally, non-semisimple Lie groups are considered “unphysical”, but we shall show direct physical interpretation of the proposed point symmetry group.

Physical content of the proposed point symmetry group:

- The proposed point symmetry group can be viewed as the automorphisms of the creation operator algebra for a spin $1/2$ particle along with its antiparticle, at a point (spacetime or momentum space).
- The regular part is $\mathrm{U}(1) \times$ covering group of conformal Lorentz group.
- The non-semisimple part can be regarded as “dressing transformations” making “dressed” states from pure one-particle states. It is a so called nilpotent subgroup (meaning that its Lie algebra is nilpotent).

# Preliminaries: automorphisms of a Grassmann algebra

**Definition 1.** *(Grassmann algebra, canonical generators) A finite dimensional complex associative algebra $G$ with unit is called a **Grassmann algebra** if there exists a minimal generating system $(e_1, \ldots, e_n)$ of $G$ such that*

$$
\begin{aligned}
e_i e_j + e_j e_i &= 0 && (i, j = 1, \ldots, n) \text{ and} \\
e_{i_1} e_{i_2} \ldots e_{i_k} & && (1 \leq i_1 < i_2 < \cdots < i_k \leq n,\ 0 \leq k \leq n) \\
& && \text{are linearly independent.}
\end{aligned}
$$

*Such a minimal generating system is called a **canonical generator system**.*

*(Automorphisms of $G$: $G \to G$ invertible complex linear maps preserving algebraic product, "alg. symmetries".)*

**Example 2.** *An exterior algebra*

$$
\Lambda(V) \quad := \quad \bigoplus_{k=0}^{\dim(V)} \wedge^k V
$$

*of some finite dimensional complex vector space $V$ is a Grassmann algebra. In fact, all Grassmann algebras are isomorphic (not naturally) to an exterior algebra with corresponding number of generators.*

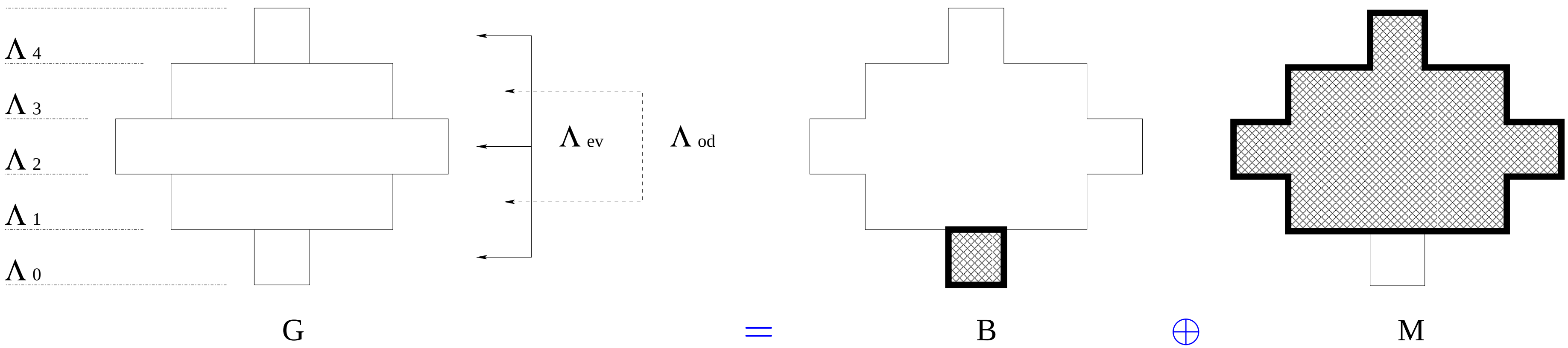
**Remark 3.** *Algebra automorphisms of $\Lambda(V)$ are not only the ones generated by $\mathrm{GL}(V)$.*

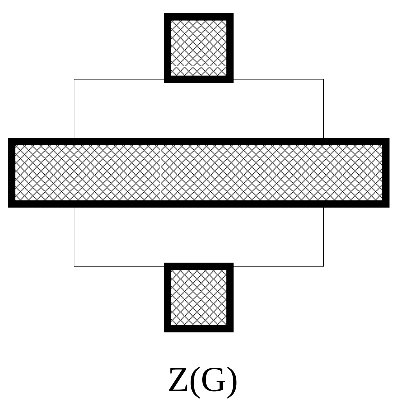
**Remark 4.** *Physically, an $n$-generator Grassmann algebra can be though of as the algebra of creation operator polynomials in a formal QFT at a point of a fermion particle with $n$ internal degrees of freedom.*

Some properties of a Grassmann algebra $G$ with $n$ canonical generators:

- An automorphism $\alpha \in \mathrm{Aut}(G)$ can uniquely be characterized by its action on a fixed system of canonical generators.
- Given a fixed system of canonical generators, their pure $k$-th order polynomials are called $k$-forms, denoted by $\Lambda_k$. The pure even (odd) polynomials are called even (odd) forms, denoted by $\Lambda_{\mathrm{ev}}$ ($\Lambda_{\mathrm{od}}$). One has $G = \bigoplus_{k=0}^{n} \Lambda_k$ and $G = \Lambda_{\mathrm{ev}} \oplus \Lambda_{\mathrm{od}}$, these are called the $\mathbb{Z}$ and $\mathbb{Z}_2$-grading of $G$.
- The unity $\mathbb{1}$ is $\mathrm{Aut}(G)$-invariant. Its span is denoted by $B$, and thus is $\mathrm{Aut}(G)$-invariant.
- The subspace $M := \bigoplus_{k=1}^{n} \Lambda_k$ is the maximal ideal of $G$, and thus is $\mathrm{Aut}(G)$-invariant.
- Therefore we have the $\mathrm{Aut}(G)$-invariant splitting $G = B \oplus M$ with corresponding invariant projection operators $I - m$ and $m$. Because of invariance of unity, $I - m = \mathbb{1}\, b$ with uniquely determined $\mathrm{Aut}(G)$-invariant complex-linear map $b:\ G \to \mathbb{C}$.
- Since $M$ is $\mathrm{Aut}(G)$-invariant, its powers $M^l = \bigoplus_{k=l}^{n} \Lambda_k$ are also.
- The center $Z(G)$ of $G$, consisting of elements commuting with $G$, is $\mathrm{Aut}(G)$-invariant.

E.g. with 4 generators:



**Theorem 5.** *(D. Z. Djokovic: Can. J. Math.* ***30*** *(1978) 1336)*
*Let $G$ be a Grassmann algebra, and $(e_1, \dots, e_n)$ a system of canonical generators.*

*(i) Let $\mathrm{Aut}_{\mathbb{Z}}(G)$ be the $\mathbb{Z}$-grading preserving automorphisms. These are of the form $e_i \mapsto \sum_{j=1}^{n} \alpha_{ij} e_j$ ($i \in \{1, \dots, n\}$) with $(\alpha_{ij})_{i,j\in\{1,\dots,n\}} \in \mathrm{GL}(\mathbb{C}^n)$.*

*(ii) Let $N_{\mathrm{ev}}$ be those $\mathbb{Z}_2$-grading preserving automorphisms which act as unity on the factor space $M^1/M^2$. These are of the form $e_i \mapsto e_i + b_i$ with $b_i \in M^3 \cap \Lambda_{\mathrm{od}}$ ($i \in \{1, \dots, n\}$).*

*(iii) Let $\mathrm{InAut}(G)$ be the subgroup of inner automorphisms, i.e. the ones of the form $\exp(a)(\cdot)\exp(a)^{-1}$ (with some $a \in G$). These are of the form $e_i \mapsto e_i + [a, e_i]$ ($i \in \{1, \dots, n\}$) with some $a \in G$.*

*With these, the semi-direct product splitting*

$$\mathrm{Aut}(G) \quad = \quad \mathrm{InAut}(G) \rtimes N_{\mathrm{ev}} \rtimes \mathrm{Aut}_{\mathbb{Z}}(G)$$

*holds.*

$\Rightarrow$ The only $\mathrm{Aut}(G)$-invariant splitting is $G = B \oplus M$ to $0$-particle and to at-least-$1$-particle states, because every $\mathrm{Aut}(G)$-invariant subspace not containing $B$ contains $M^n$.
This is because the nilpotent normal subgroup $N := \mathrm{InAut}(G) \rtimes N_{\mathrm{ev}}$ of “dressing transformations” mixes higher particle content to lower particle states, in particular $M^n$.

# Spin algebra and its group of automorphisms

**Definition 6.** *($^+$-algebra) A finite dimensional complex associative algebra $A$ with unit shall be called* $^+$***-algebra*** *if it is equipped with a conjugate-linear involution $(\cdot)^+$ satisfying $(xy)^+ = x^+y^+$ for all $x, y \in A$.*

Important: the $^+$-involution does not reverse product, i.e. it is not a $^*$-involution.

**Definition 7.** *(Spin algebra) A complex associative $^+$-algebra $A$ with unit is called a **spin algebra** if there exists a minimal generating system $(e_1, e_2, e_3, e_4)$ of $A$ such that*

$$\begin{aligned} e_ie_j + e_je_i &= 0 && (i,j \in \{1,2\} \text{ or } i,j \in \{3,4\}), \\ e_ie_j - e_je_i &= 0 && (i \in \{1,2\} \text{ and } j \in \{3,4\}), \\ e_{i_1}e_{i_2}\dots e_{i_k} & && (1 \leq i_1 < i_2 < \cdots < i_k \leq 4,\ 0 \leq k \leq 4) \\ & && \text{are linearly independent}, \\ e_3 = e_1^+,\ \ e_4 = e_2^+. & && \end{aligned}$$

*Such a minimal generating system is called a **canonical generator system**.*

*In the followings $n := 4$ is occasionally used.*

Automorphisms of a spin algebra $A$: the $A \to A$ invertible complex linear maps preserving algebraic product and $^+$-involution, i.e. the “algebra symmetries”.

**Example 8.** *If $S^*$ is a 2 dimensional complex vector space ("cospinor space"), then $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ is a spin algebra. In fact, a spin algebra is always isomorphic (not naturally) to $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$.*

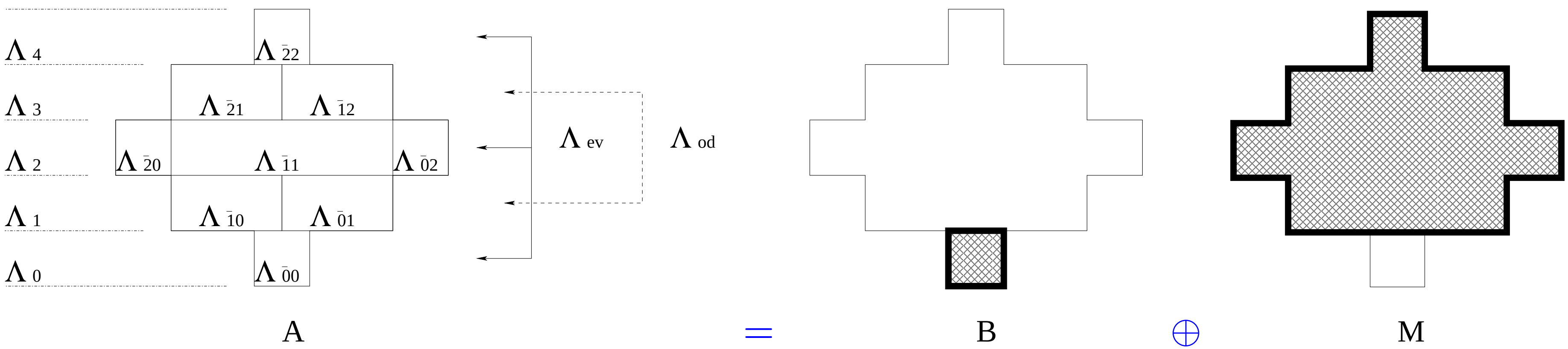
**Remark 9.** *Algebra automorphisms of $\Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ are not simply the ones generated by $\mathrm{GL}(S^*)$.*

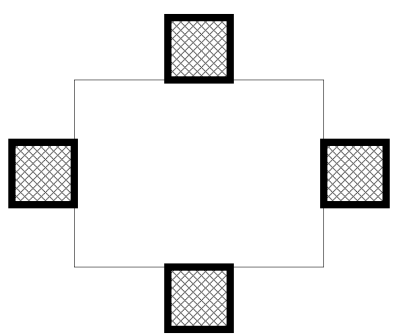
**Remark 10.** *Physically, a spin algebra can be though of as the algebra of creation operator polynomials in a formal QFT at a point of spacetime (or momentum space) of a spin 1/2 particle along with its antiparticle. Namely, $\{e_1, e_2\}$ can be though of as the creation operators of our spin 1/2 particle, $\{e_3, e_4\}$ can be thought of as the creation operators of its antiparticle, where the involution $(\cdot)^+$ exchanges them in a conjugate-linear way (charge conjugation). Note that in this construction, annihilation operators of the particle is not (yet) identified with the antiparticle creation operators (that involves an extra structure). We only have creation operators by now. Relation to canonical anticommutation relation (CAR) algebra will be discussed later.*

Some properties of a spin algebra $A$:

- An automorphism $\alpha \in \mathrm{Aut}(A)$ can uniquely be characterized by its action on a fixed system of canonical generators.
- Given a fixed system of canonical generators $(e_1, e_2, e_3, e_4)$, their pure polynomials of $p$-th order in $\{e_1, e_2\}$ and $q$-th order in $\{e_3, e_4\}$ is called $p, q$-forms, denoted by $\Lambda_{\bar{p}q}$. The $k$-th order polynomials made of these with $p + q = k$ are called $k$-forms, denoted by $\Lambda_k$. The pure even (odd) polynomials made of thes are called even (odd) forms, denoted by $\Lambda_{\mathrm{ev}}$ ($\Lambda_{\mathrm{od}}$). One has $A = \bigoplus_{p,q=0}^{2} \Lambda_{\bar{p}q}$, $A = \bigoplus_{k=0}^{4} \Lambda_k$ and $A = \Lambda_{\mathrm{ev}} \oplus \Lambda_{\mathrm{od}}$, these are called the $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}$ and $\mathbb{Z}_2$-grading of $A$.
- The unity $\mathbb{1}$ is $\mathrm{Aut}(A)$-invariant. Its span is denoted by $B$, and thus is $\mathrm{Aut}(A)$-invariant.
- The subspace $M := \bigoplus_{k=1}^{n} \Lambda_k$ is the maximal ideal of $A$, and thus is $\mathrm{Aut}(A)$-invariant.
- Therefore we have the $\mathrm{Aut}(A)$-invariant splitting $A = B \oplus M$ with corresponding invariant projection operators $I - m$ and $m$. Because of invariance of unity, $I - m = \mathbb{1}\, b$ with uniquely determined $\mathrm{Aut}(A)$-invariant complex-linear map $b :\ A \to \mathbb{C}$.
- Since $M$ is $\mathrm{Aut}(A)$-invariant, its powers $M^l = \bigoplus_{k=l}^{n} \Lambda_k$ is also.
- The center $Z(A)$ of $A$, consisting of elements commuting with $A$, is $\mathrm{Aut}(A)$-invariant. In fact, $Z(A) = \Lambda_{\bar{0}0} \oplus \Lambda_{\bar{2}0} \oplus \Lambda_{\bar{0}2} \oplus \Lambda_{\bar{2}2}$.

Fundamental structure of a spin algebra $A$:



Z(A)

**Theorem 11.** *(A. László: `arXiv:1507.08039`)*
*Let $A$ be a spin algebra, and $(e_1, e_2, e_3, e_4)$ a system of canonical generators.*

*(i) The $\alpha \in \mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A)$ automorphisms are defined by the matrix action*

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto \alpha \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} := \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ 0 & 0 & \bar{\alpha}_{11} & \bar{\alpha}_{12} \\ 0 & 0 & \bar{\alpha}_{21} & \bar{\alpha}_{22} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$

*over the generators with* $\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^2)$.

*(ii) We define the subgroup $\mathcal{J} := \{I, J\}$ such that*

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \mapsto J \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},$$

*i.e. $J$ is the particle-antiparticle label exchanging linear transformation.*

(iii) *We introduce the subgroup $\tilde{N}_{\rm ev}$ which acts on the generators as*

$$\begin{array}{rcl} e_1 & \mapsto & e_1 + b_1, \\ e_2 & \mapsto & e_2 + b_2, \\ e_3 & \mapsto & e_3 + b_1^+, \\ e_4 & \mapsto & e_4 + b_2^+ \end{array}$$

*with* $b_1, b_2 \in \Lambda_{\bar{1}2}$.

(iv) *Inner automorphisms* $\mathrm{In}_a \in \mathrm{InAut}(A)$ *are defined by* $\mathrm{In}_a(\cdot) := \exp(a)(\cdot)\exp(a)^{-1}$ ($a \in \mathrm{Re}(A)$):

$$e_i \quad \mapsto \quad \mathrm{In}_a(e_i) = e_i + [a, e_i] + \frac{1}{2}[a,[a,e_i]] \qquad (i = 1, \ldots, 4),$$

*because for any* $x \in A$ *one has* $\mathrm{In}_a x := \exp(a) x \exp(a)^{-1} = x + [a, x] + \frac{1}{2}[a,[a,x]]$.

*With these, the semi-direct product splitting*

$$\mathrm{Aut}(A) \quad = \quad \mathrm{InAut}(A) \rtimes \tilde{N}_{\rm ev} \rtimes \mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \rtimes \mathcal{J}$$

*holds.*

Note: $\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \equiv GL(2,\mathbb{C}) \equiv \mathrm{D}(1)\times\mathrm{U}(1)\times\mathrm{SL}(2,\mathbb{C})$.

Notation:

$$\mathrm{Aut}(A) \quad = \quad \underbrace{\mathrm{InAut}(A) \ \rtimes \ \tilde{N}_{\mathrm{ev}}}_{=:N} \ \rtimes \ \underbrace{\mathrm{Aut}_{\mathbb{Z}\times\mathbb{Z}}(A) \ \rtimes \ \mathcal{J}}_{=:\mathrm{Aut}_{\mathbb{Z}}(A)}$$

Observation: the only $\mathrm{Aut}(A)$-invariant splitting is $A = B \oplus M$ to $0$-particle and to at-least-$1$-particle states, as an $\mathrm{Aut}(A)$-invariant subspace not containing $B$ contains $M^4$.

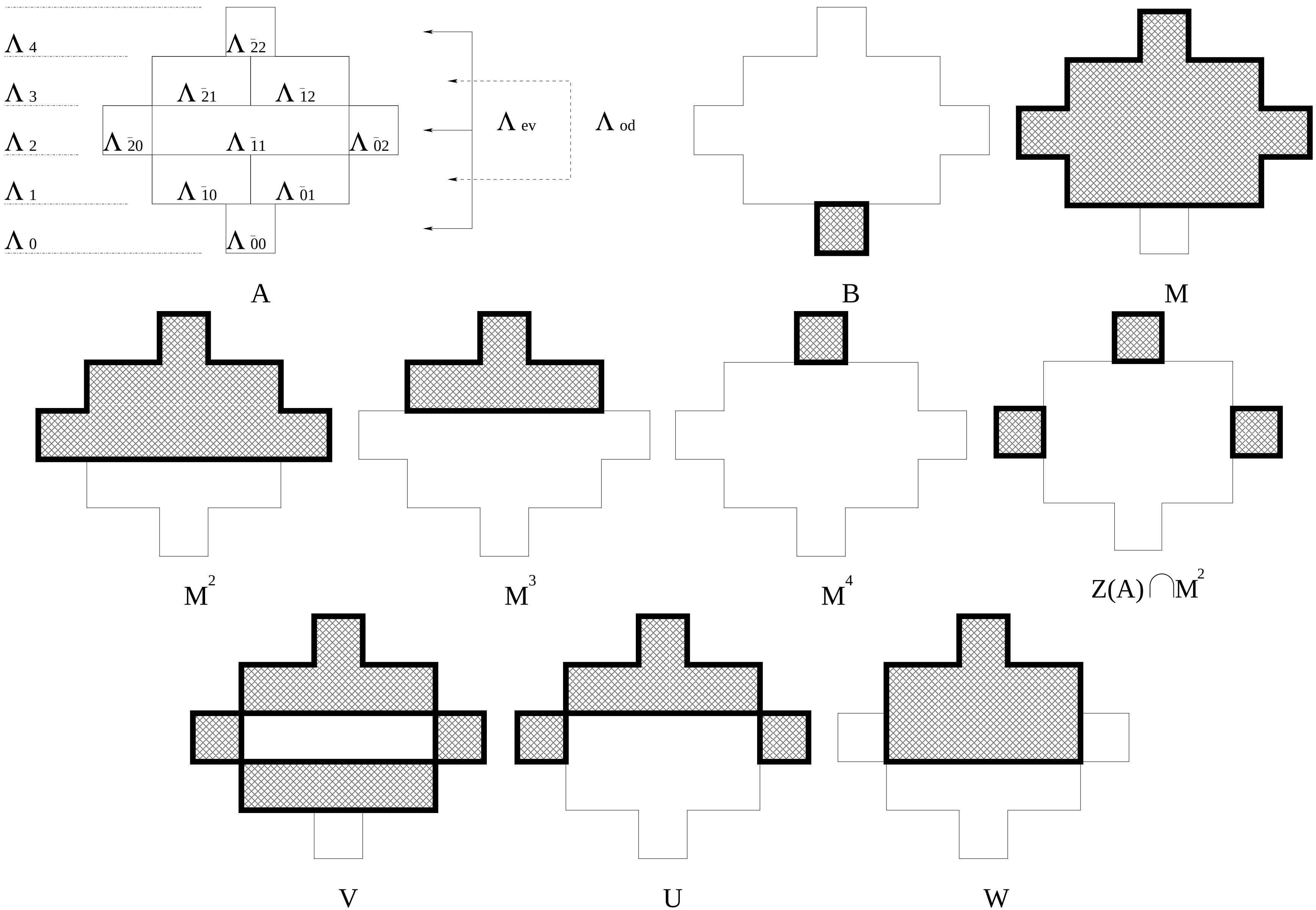
That is because the nilpotent normal subgroup $N$ of “dressing transformations” mixes higher particle content to lower particle states, in particular $M^4$.

If a fixed $\mathbb{Z}\times\mathbb{Z}$-grading is taken, $A \equiv \Lambda(\bar{S}^*)\otimes\Lambda(S^*)$ with $\dim_{\mathbb{C}}(S^*) = 2$ (“cospinor space”). Using spinor Penrose abstract indices, an element of $A$ may be represented as 9 fields:

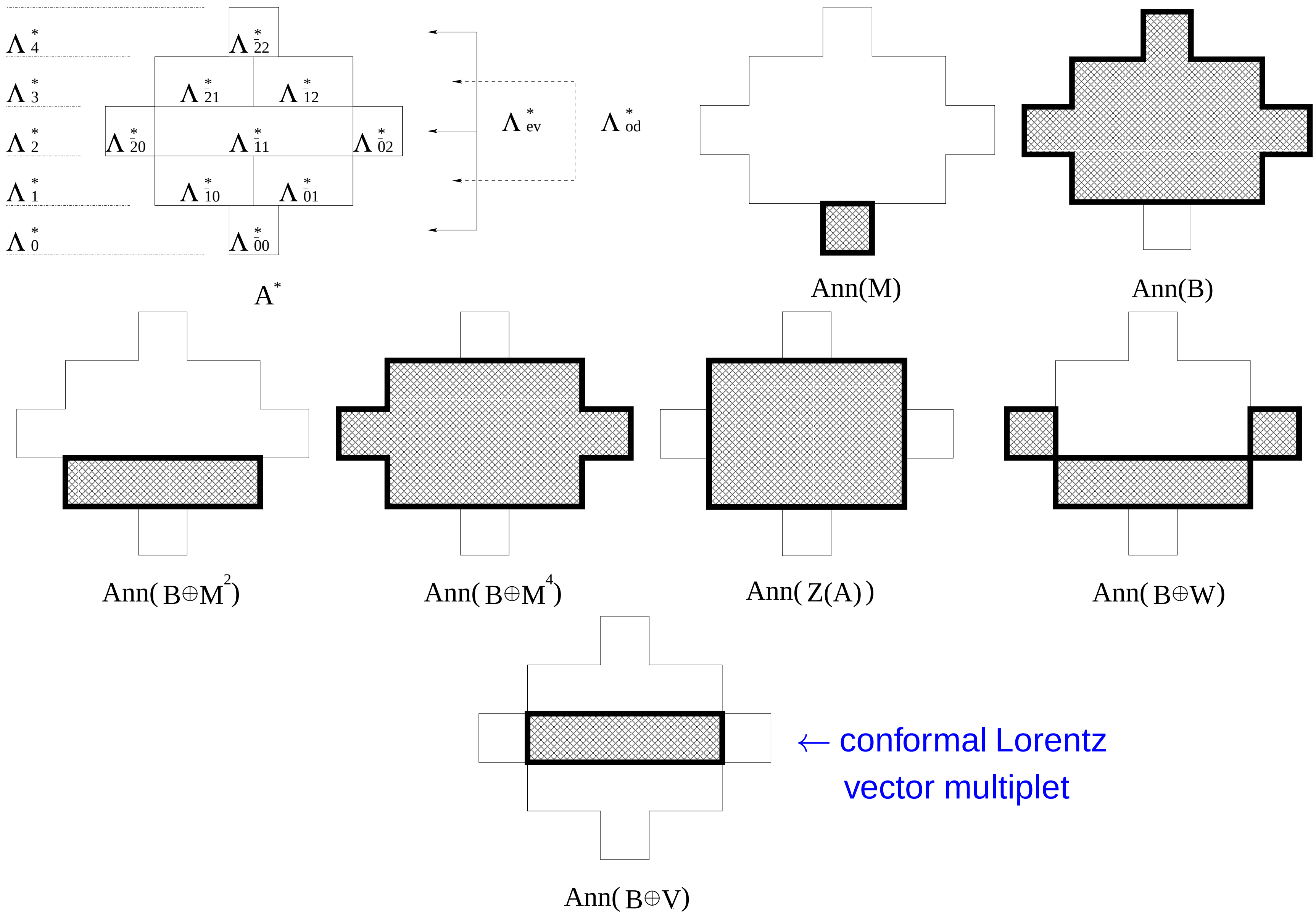
$$\begin{pmatrix}\varphi & \bar{\xi}_{(+)\,A'} & \xi_{(-)\,A} & \bar{\epsilon}_{(+)\,[A'B']} & v_{A'B} & \epsilon_{(-)\,[AB]} & \bar{\chi}_{(+)\,[C'D']A} & \chi_{(-)\,A'[CD]} & \omega_{[A'B'][CD]}\end{pmatrix}$$

using ordinary 2-spinor formalism. (Spinors and complex conjugate spinors do commute!)

# Indecomposable $\mathrm{Aut}(A)$-invariant subspaces of $A$:



## Indecomposable $\mathrm{Aut}(A)$-invariant subspaces of $A^*$ (dual vector space of $A$):



# Induced conformal Lorentz metric class by a spin algebra

$\mathrm{Re}(\mathrm{Ann}(B \oplus V)) \equiv \mathrm{Re}(\Lambda^*_{\bar{1}1})$ is a real $4$ dimensional $\mathrm{Aut}(A)$-invariant subspace of $A^*$.

Easily checked: $\mathrm{Aut}(A)$ acts as multiplication by positive real number on $M^4$.
$\Rightarrow \mathrm{Re}(M^4)$ can be split up to $\mathrm{Aut}(A)$-invariant cones of positive and negative max. forms: to $\mathrm{Re}_+(M^4)$ and $\mathrm{Re}_-(M^4)$. The former is positive multiples of $e_1\, e_2\, {e_1}^+ {e_2}^+$ by convention.

Given a fixed $\mathbb{Z}$-grading of $A$, it admits a Hopf algebra structure.

- Product: $\nabla :\ A \otimes A \to A$ (linear view of $A \times A \to A$ algebraic multiplication operation).
- Unit: $\eta :\ \mathbb{C} \to A$ (linear view of algebraic unity element $\mathbb{1} \in A$ as $\eta(\cdot) := (\cdot)\, \mathbb{1}$).
- Counit: $\varepsilon :\ A \to \mathbb{C}$ linear map, we set $\varepsilon := b$.
- Antipode: $S :\ A \to A$ linear map, we set $S(\cdot) := (\cdot)_{\mathrm{ev}} + (-1)(\cdot)_{\mathrm{od}}$.
- Swapping involution: $\mathcal{I} :\ A \otimes A \to A \otimes A$ linear map, we set $x \otimes y \mapsto \mathcal{I}(x \otimes y) := \sum_{p,q=0}^{2} \sum_{r,s=0}^{2} (-1)^{pr+qs} x_{\bar{p}q} \otimes y_{\bar{r}s}$ (for all $x, y \in A$).
- Coproduct: $\triangle :\ A \to A \otimes A$ linear map, we define it by $\triangle(\mathbb{1}) := \mathbb{1} \otimes \mathbb{1}$,
  $\triangle(x) := \mathbb{1} \otimes x + x \otimes \mathbb{1}$ (for all $x \in \Lambda_{\bar{1}0} \oplus \Lambda_{\bar{0}1}$),
  and that $\triangle$ is an $A \to A \otimes A$ algebra homomorphism.
  (Where product on $A \otimes A$ is the skew-natural product: $(\nabla \otimes \nabla) \circ (I \otimes (J \circ \mathcal{I}) \otimes I)$,
  $J :\ A \otimes A \to A \otimes A$ being the swapping map.)

Hopf algebraness means “self-dualness”:
given a $\mathbb{Z}$-grading, the dual vector space $A^*$ may also be equipped to be a spin algebra.

Coproduct $\triangle$ splits up elements in all possible ways as if were algebraic products.

$$
\begin{aligned}
\triangle(\mathbb{1}) &= \mathbb{1}\otimes\mathbb{1},\\
\triangle(e_1) &= \mathbb{1}\otimes e_1 + e_1\otimes\mathbb{1},\\
\triangle(e_2) &= \mathbb{1}\otimes e_2 + e_2\otimes\mathbb{1},\\
\triangle(e_3) &= \mathbb{1}\otimes e_3 + e_3\otimes\mathbb{1},\\
\triangle(e_4) &= \mathbb{1}\otimes e_4 + e_4\otimes\mathbb{1},\\
\triangle(e_1e_2) &= \mathbb{1}\otimes e_1e_2 + e_1\otimes e_2 - e_2\otimes e_1 + e_1e_2\otimes\mathbb{1},\\
\triangle(e_1e_3) &= \mathbb{1}\otimes e_1e_3 + e_1\otimes e_3 + e_3\otimes e_1 + e_1e_3\otimes\mathbb{1},\\
\triangle(e_1e_4) &= \mathbb{1}\otimes e_1e_4 + e_1\otimes e_4 + e_4\otimes e_1 + e_1e_4\otimes\mathbb{1},\\
\triangle(e_2e_3) &= \mathbb{1}\otimes e_2e_3 + e_2\otimes e_3 + e_3\otimes e_2 + e_2e_3\otimes\mathbb{1},\\
\triangle(e_2e_4) &= \mathbb{1}\otimes e_2e_4 + e_2\otimes e_4 + e_4\otimes e_2 + e_2e_4\otimes\mathbb{1},\\
\triangle(e_3e_4) &= \mathbb{1}\otimes e_3e_4 + e_3\otimes e_4 - e_4\otimes e_3 + e_3e_4\otimes\mathbb{1},\\
\triangle(e_1e_2e_3) &= \mathbb{1}\otimes e_1e_2e_3 + e_1\otimes e_2e_3 - e_2\otimes e_1e_3 + e_1e_2\otimes e_3\\
&\quad + e_1e_3\otimes e_2 + e_3\otimes e_1e_2 - e_2e_3\otimes e_1 + e_1e_2e_3\otimes\mathbb{1},\\
\triangle(e_1e_2e_4) &= \mathbb{1}\otimes e_1e_2e_4 + e_1\otimes e_2e_4 - e_2\otimes e_1e_4 + e_1e_2\otimes e_4\\
&\quad + e_1e_4\otimes e_2 + e_4\otimes e_1e_2 - e_2e_4\otimes e_1 + e_1e_2e_4\otimes\mathbb{1},\\
\triangle(e_1e_3e_4) &= \mathbb{1}\otimes e_1e_3e_4 + e_1\otimes e_3e_4 + e_3\otimes e_1e_4 + e_1e_3\otimes e_4\\
&\quad - e_1e_4\otimes e_3 - e_4\otimes e_1e_3 + e_3e_4\otimes e_1 + e_1e_3e_4\otimes\mathbb{1},\\
\triangle(e_2e_3e_4) &= \mathbb{1}\otimes e_2e_3e_4 + e_2\otimes e_3e_4 + e_3\otimes e_2e_4 + e_2e_3\otimes e_4\\
&\quad - e_2e_4\otimes e_3 - e_4\otimes e_2e_3 + e_3e_4\otimes e_2 + e_2e_3e_4\otimes\mathbb{1},\\
\triangle(e_1e_2e_3e_4) &= \mathbb{1}\otimes e_1e_2e_3e_4 + e_3\otimes e_1e_2e_4 - e_4\otimes e_1e_2e_3 + e_3e_4\otimes e_1e_2\\
&\quad + e_1\otimes e_2e_3e_4 + e_1e_3\otimes e_2e_4 - e_1e_4\otimes e_2e_3 + e_1e_3e_4\otimes e_2\\
&\quad - e_2\otimes e_1e_3e_4 - e_2e_3\otimes e_1e_4 + e_2e_4\otimes e_1e_3 - e_2e_3e_4\otimes e_1\\
&\quad + e_1e_2\otimes e_3e_4 + e_1e_2e_3\otimes e_4 - e_1e_2e_4\otimes e_3 + e_1e_2e_3e_4\otimes\mathbb{1}
\end{aligned}
$$

Given a fixed $\mathbb{Z}$-grading, it can be checked e.g. by direct calculations that indeed a spin algebra $(A,(\cdot)^{+},\eta,\bigtriangledown)$ may be equipped as a Hopf algebra $(A,(\cdot)^{+},\eta,\bigtriangledown,\varepsilon,\triangle,S,\mathcal{I})$ (and the construction is invariant to $\mathrm{Aut}_{\mathbb{Z}}(A)$).

Note: the full automorphism group $\mathrm{Aut}(A)$ of the spin algebra $(A,(\cdot)^{+},\eta,\bigtriangledown)$ part does not preserve the coalgebra operations $(\varepsilon,\triangle,S,\mathcal{I})$. Indeed, a dressing transformation from $N$ deforms the part $(\varepsilon,\triangle,S,\mathcal{I})$ to an other compatible coalgebra structure $(\varepsilon,\triangle',S',\mathcal{I}')$.

**Theorem 12.** *(A. László:* `arXiv:1507.08039`*)*
*Let $\omega\in\mathrm{Re}_{+}(M^{4})\setminus\{0\}$. Then, the bilinear form*

$$G(\omega):\ \mathrm{Re}(\mathrm{Ann}(B\oplus V))\,\times\,\mathrm{Re}(\mathrm{Ann}(B\oplus V))\,\to\,\mathbb{R},\ (a,b)\,\mapsto\,G(\omega)(a,b):=(a\otimes b|\triangle(\omega))$$

*is a Lorentz signature metric, where $(\cdot|\cdot)$ denotes duality pairing. The action of $\mathrm{Aut}(A)$ preserves $G(\omega)$ up to a positive real scaling factor, i.e. $\mathrm{Aut}(A)$ acts on $\mathrm{Re}(\mathrm{Ann}(B\oplus V))$ as the conformal Lorentz group. The construction does not depend on the choice of the coproduct $\triangle$, or equivalently, on the choice of $\mathbb{Z}$-grading. (We would get the same $G(\omega)$ using a coproduct $\triangle'$, deformed by a dressing transformation from $N$.)*

(This fact can already be suspected from the relation to 2-spinor calculus.)

This means that $\mathrm{Re}(\mathrm{Ann}(B \oplus V))$ admits an $\mathrm{Aut}(A)$-invariant notion of spacelike, timelike and null vectors.

Also there is a natural time orientation on $\mathrm{Re}(\mathrm{Ann}(B \oplus V))$, preserved by $\mathrm{Aut}(A)$: a timelike or null element $p \in \mathrm{Re}(\mathrm{Ann}(B \oplus V))$ shall be called future directed if for all $x \in M$ one has $p\,(x^+ x) > 0$.

[Analogy in 2-spinor calculus: future directed timelike or null elements of $\mathrm{Re}(\bar{S} \otimes S)$ are of the form $\bar{\xi}^{A'}\xi^A + \bar{\chi}^{A'}\chi^A$ (with $\xi^A, \chi^A \in S$).]

# Relation to Clifford algebras

**Definition 13.** *(Pauli embedding, Pauli injection, Pauli map)*
*Let us take a coproduct* $\triangle$ *on* $A$*. Then, the four real dimensional* $\mathrm{Aut}(A)$*-invariant subspace* $\mathrm{Re}(\mathrm{Ann}(B \oplus V))$ *of* $\mathrm{Re}(A^*)$ *may be embedded into* $\mathrm{Re}(\mathrm{Lin}(A))$ *using:* $s \mapsto (s \otimes I) \circ \triangle$ *for all* $s \in \mathrm{Re}(\mathrm{Ann}(B \oplus V))$*. This is called* ***Pauli embedding****.*
*Given a real four dimensional vector space* $T$ *("tangent space" or "momentum space"), a linear injection* $T \to \mathrm{Re}(\mathrm{Ann}(B \oplus V))$ *is called a* ***Pauli injection****.*
*The composition of a Pauli embedding with a Pauli injection is called a* ***Pauli map****, which thus is a* $T \to \mathrm{Re}(\mathrm{Lin}(A))$ *linear injection.*

If $\sigma : T \to \mathrm{Re}(\mathrm{Lin}(A))$ is a Pauli map, then $b\,\sigma : T \to \mathrm{Re}(\mathrm{Ann}(B \oplus V))$ is its underlying Pauli injection. It defines the action of $\mathrm{Aut}(A)$ over $T$ as an intertwining operator.

When given a fixed $\mathbb{Z} \times \mathbb{Z}$-grading and when $A$ is represented with 2-spinor calculus, a Pauli map $\sigma$ is the usual $\sigma_a^{A'A}$ intertwining operator ("soldering form") between $T$ and $\mathrm{Re}(\bar{S} \otimes S)$. (Index like ${}^a$ is $T$ index, while ${}_a$ is $T^*$ index.)

Given a Pauli map $\sigma$ and an $\omega$ positive maximal form, the tensor $g(\sigma, \omega)_{ab} := b\,\sigma_a\,\sigma_b\,\omega$ is a Lorentz metric on $T$. This is nothing but the pullback of $G(\omega)$ to $T$ by the Pauli injection $b\,\sigma_a$.

**Definition 14.** *(Dirac adjoint, Dirac gamma map)*

*Let us fix a Pauli map $\sigma : T \to \mathrm{Lin}(A)$ and a positive maximal form $\omega$. The conjugate-linear map*

$$\bar{(\cdot)} : A \to A^*,\; x \mapsto \bar{x} := \frac{1}{2} g(\sigma,\omega)^{ab}\, b\sigma_a \Big( x^+ \sigma_b(\cdot) + \sigma_b(x^+)(\cdot) \Big)$$

*is called the* ***Dirac adjoint****. The linear map*

$$\gamma(\sigma,\omega) : T \to \mathrm{Lin}(A),\; u \mapsto u^a \gamma(\sigma,\omega)_a(\cdot) := u^a \sqrt{2} \Big( \sigma_a(\cdot) + \sigma_a(\omega)(\cdot) \Big)$$

*is called the* ***Dirac gamma map****.*

**Theorem 15.** *(A. László:* `arXiv:1507.08039`*)*

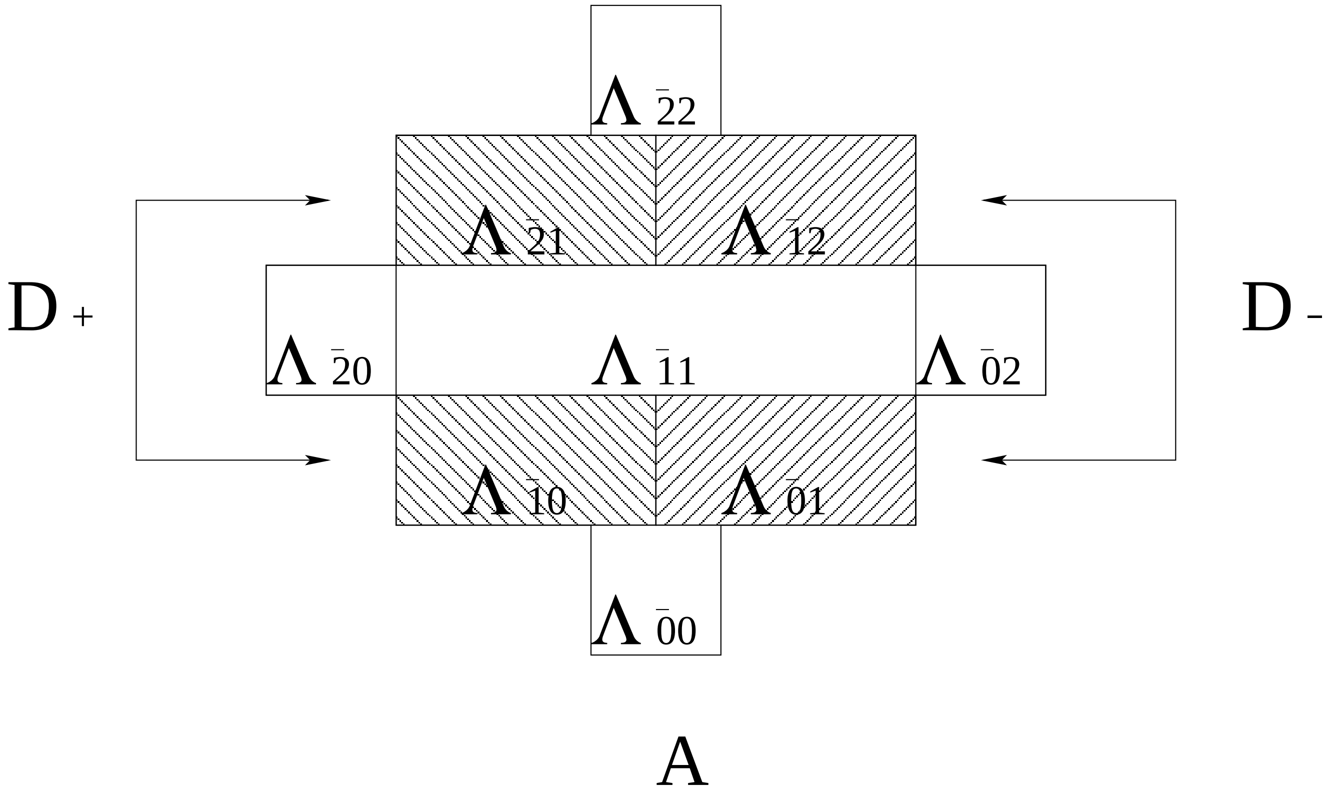
*Let us take the $\mathbb{Z}$-grading subordinate to the Pauli map $\sigma$. With this, introduce the subspaces $D_+ := \Lambda_{\bar{1}0} \oplus \Lambda_{\bar{2}1}$ and $D_- := \Lambda_{\bar{0}1} \oplus \Lambda_{\bar{1}2}$ of $A$. Then, the following properties hold.*

*(i)* $\dim_{\mathbb{C}}(D_+) = \dim_{\mathbb{C}}(D_-) = 4$, $D_+ \cap D_- = \{0\}$, $(D_+)^+ = D_-$ *and* $(D_-)^+ = D_+$.

*(ii) The Dirac adjoint map $\bar{(\cdot)}$ becomes non-degenerate over $D_+$ and $D_-$.*

*(iii) The Dirac gamma map $\gamma(\sigma,\omega) : T \to \mathrm{Lin}(A)$ satisfies the Clifford relation over $D_+$ and $D_-$:*

$$\gamma(\sigma,\omega)_a \gamma(\sigma,\omega)_b + \gamma(\sigma,\omega)_b \gamma(\sigma,\omega)_a \quad = \quad 2\, I\, g(\sigma,\omega)_{ab}.$$




The embedded Dirac bispinor spaces are not preserved by $\mathrm{Aut}(A)$, but are deformed to other compatible embedded Dirac bispinor spaces by the dressing transformations $N$.

# Relation to $C^*$-algebras

Let $u \in T$ be a future directed timelike or null vector (modeling a momentum vector).

Then, the sesquilinear form

$$[\cdot|\cdot]_u \;:\; A \;\times\; A \;\to\; \mathbb{C}, \quad (x,y) \;\mapsto\; [x|y]_u := b\left(x^+ y\right) + u^a\, b\, \sigma_a\, \Big((x - \mathbb{1}\, b\, x)^+\, (y - \mathbb{1}\, b\, y)\Big)$$

is a positive semidefinite $\mathrm{Aut}(A)$-covariant inner product on $A$. (Depends on momentum $u^a$.)

Given a coproduct $\triangle$ on $A$, this induces a nondegerate scalar product $\langle\cdot|\cdot\rangle_{u,\triangle}$ on $A$, and with that $A$ becomes a finite dimensional Hilbert space ("Fock space").

The adjoining operation $(\cdot)^\dagger$ with respect to this scalar product identifies the creation operator of antiparticles with annihilation operator of particles.

The $^\dagger$-adjoining relates the spin algebra to $C^*$ and CAR algebras.

[The scalar product is not $\mathrm{Aut}(A)$-invariant, but $\mathrm{Aut}(A)$-covariant: an $\mathrm{Aut}(A)$ transformation induces a unitary equivalence between $\langle\cdot|\cdot\rangle_{u,\triangle}$ and $\langle\cdot|\cdot\rangle_{u',\triangle'}$.]

[In 2-spinor notation, the scalar product is induced by $u^a \sigma_a^{A'A}$ at given momentum vector $u$.]

# Relation to “superfield” formalism

If a fixed $\mathbb{Z} \times \mathbb{Z}$-grading is taken, $A \equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ with $\dim_{\mathbb{C}}(S^*) = 2$ (“cospinor space”). Using spinor Penrose abstract indices, an element of $A$ may be represented as:

$$\left(\varphi \quad \bar{\xi}_{(+)\ A'} \quad \xi_{(-)\ A} \quad \bar{\epsilon}_{(+)\ [A'B']} \quad v_{A'B} \quad \epsilon_{(-)\ [AB]} \quad \bar{\chi}_{(+)\ [C'D']A} \quad \chi_{(-)\ A'[CD]} \quad \omega_{[A'B'][CD]}\right)$$

using ordinary 2-spinor formalism. (Spinors and complex conjugate spinors commute!)

This might remind one about the algebra of superfields in SUSY at a fixed point of spacetime (or momentum space).

Note, however, that in order to satisfy SUSY relations, spinors and complex conjugate spinors in superfield algebra must anticommute (“Grassmann valued spinors”). Conseqently, superfield algebra (at a point) is isomorphic rather to $\Lambda(\bar{S}^* \oplus S^*)$, not to $\Lambda(\bar{S}^*) \otimes \Lambda(\bar{S})$.

And $\Lambda(\bar{S}^* \oplus S^*) \not\equiv \Lambda(\bar{S}^*) \otimes \Lambda(S^*)$ algebrawise.

$\Rightarrow$ Spin algebra $\not\equiv$ superfield algebra, but have the same dimensions.

# Summary

- A finite dimensional complex unital associative algebra is presented and its group of automorphisms were detailed.
- The pertinent associative algebra can be physically interpreted as the creation operator algebra of a spin 1/2 particle along with its antiparticle, at a fixed point of spacetime or momentum space.
- Its automorphism group was seen to be a semi-direct product of $\mathrm{U}(1) \times$ the covering group of the conformal Lorentz group and a nilpotent subgroup of “dressing transformations”.
- In this formalism, spacetime metric is a composite field.
- It may be used for a non-SUSY mixing of internal (gauge) and spacetime symmetries.

  [E.g. with tricks like in C. Furey: *Phys. Lett.* **B742** (2015) 195.]