On the running and the UV limit of Wilsonian renormalization group flows

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Outline

I. On Wilsonian RG flow of correlators (arbitrary signature):

- On manifolds: nice topological vector space behavior
- On flat spacetime for bosonic fields: $\exists$ of UV limit
- Is that true on manifolds?

[*Class.Quant.Grav.* **39**(2022)185004]

II. On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields):

- $\exists$ of UV limit Feynman measure
- $\exists$ of UV limit interaction potential
- A new kind of renormalizability condition

[*manuscript in preparation*]

Part 0:

Notations, introduction

Recap on distribution theory

Will consider only scalar and bosonic fields for simplicity.

Will consider only flat (affine) spacetime manifold for simplicity.

- $\mathcal{E}$: space of all smooth fields over spacetime.
  They form a vector space with a topology: collection of “open” sets
  $\varphi_i \in \mathcal{E}$ $(i \in \mathbb{N}) \to 0$ iff all derivatives locally uniformly converge to zero.
- $\mathcal{S}$: space of rapidly decreasing smooth fields (Schwartz fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{S}$ $(i \in \mathbb{N}) \to 0$ iff all derivatives $\times$ all polynomials uniformly converge to zero.
- $\mathcal{D}$: space of compactly supported smooth fields (test fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{D}$ $(i \in \mathbb{N}) \to 0$ iff they stay within a compact set and $\to 0$ in $\mathcal{E}$ sense.

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

Distributions are continuous duals of $\mathcal{E}$, $\mathcal{S}$, $\mathcal{D}$.

- $\mathcal{E}'$: continuous $\mathcal{E} \to \mathbb{R}$ linear functionals.
  They are the compactly supported distributions.
- $\mathcal{S}'$: continuous $\mathcal{S} \to \mathbb{R}$ linear functionals.
  They are the tempered or Schwartz distributions.
- $\mathcal{D}'$: continuous $\mathcal{D} \to \mathbb{R}$ linear functionals.
  They are the space of all distributions.

They carry a corresponding natural topology (notion of “open” sets).

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \quad\subset\quad \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \quad\subset\quad \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

[Of course, functions are also distributions, e.g. $\mathcal{D} \subset \mathcal{E}'$ and $\mathcal{E} \subset \mathcal{D}'$ etc.]

Recap on measure / integration / probability theory

- Let X be a set (is elements called elementary events).

- Let Σ be a collection of subsets of X such that:
  - X is in Σ,
  - for all A in Σ, its complement is in Σ.
  - for all max countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N})$, the union $\underset{i\in\mathbb{N}}{\cup} A_i$ is in Σ.

  Then, Σ is called a sigma-algebra (collection of composite events or measurable sets). When X carries open sets (topology), the sigma-alg generated by them is used (Borel). (X, Σ) is called measurable space.

- Let $\mu : \Sigma \to \mathbb{R}_0^+ \cup \{\infty\}$ be a weight-assigning function to sets, such that:
  - $\mu(\emptyset) = 0$,
  - for all max countably inf. disjoint system $A_i \in \Sigma$ $(i \in \mathbb{N})$: $\mu\big(\underset{i\in\mathbb{N}}{\cup} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$,
  - $\exists$ some max countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N})$ with $\mu(A_i) < \infty$: $X = \underset{i\in\mathbb{N}}{\cup} A_i$.

  Then, μ is called measure.

(X, Σ, μ) is called measure space. [E.g. probability measure space iff $\mu(X) =$ finite.]

- A function $f : X \to \mathbb{C}$ is called measurable iff in good terms with mesure theory: for all $B \in \mathrm{Borel}(\mathbb{C})$, one has $\overset{-1}{f}(B) \in \Sigma$ of X.
  Theorem: f is measurable iff approximable pointwise by "histograms" with bins from Σ.

- The integral $\int\limits_{\phi\in X} f(\phi)\,\mathrm{d}\mu(\phi)$ is defined via the histogram "area" approximations.
  Theorem: this is well-defined.

- Let (X, Σ, μ) be a measure space and (Y, Δ) a measurable space.
  Let $C : X \to Y$ be a measurable mapping.
  Then, one can define the pushforward (or marginal) measure $C_*\mu$ on Y.
  [For all $B \in \Delta$ one defines $(C_*\mu)(B) := \mu\big(\overset{-1}{C}(B)\big)$.]

- Pushforward (marginal) measure means simply transformation of integration variable.
  If forgetful transformation, the "forgotten" d.o.f. are "integrated out".

- If μ is a probability measure e.g. on $X = \mathcal{E}, \mathcal{S}, \mathcal{D}, \mathcal{E}', \mathcal{S}', \mathcal{D}'$, then
  $Z(j) := \int\limits_{\phi\in X} \mathrm{e}^{\mathrm{i}(j\,|\,\phi)}\,\mathrm{d}\mu(\phi)$ is its Fourier transform (partition function in QFT).

Ideology of Euclidean Wilsonian renormalization

- Take an Euclidean action $S = T + V$, with kinetic + potential term splitting. Say, $T(\varphi) = \int \varphi\,(-\Delta + m^2)\varphi$, and $V(\varphi) = g\int \varphi^4$.

- Then T, i.e. $(-\Delta + m^2)$ has a propagator $K(\cdot,\cdot)$ which is positive definite:
  - $(-\Delta + m^2)_x\, K(x,y) = \delta_y(x)$,
  - for all $j \in \mathcal{S}$ rapidly decreasing sources: $(K|j\otimes j) \geq 0$.

- Due to above, the function $Z_T(j) := \mathrm{e}^{-(K|j\otimes j)}$ $(j \in \mathcal{S})$ has “quite nice” properties.

- Bochner-Minlos theorem: because of
  - “quite nice” properties of Z_T,
  - “quite nice” properties of the space $\mathcal{S}$,

  $\exists|$ measure γ_T on $\mathcal{S}'$, whose Fourier transform is Z_T.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{S}'} (\dots)\,\mathrm{d}\gamma_T(\phi) = \int\limits_{\phi\in\mathcal{S}'} (\dots)\,\mathrm{e}^{-T(\phi)}\,\text{“}\mathrm{d}\phi\text{”}$.

- Tempting definition for Feynman measure of interacting theory:

$$\int\limits_{\phi\in\mathcal{S}'} (\dots)\,\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_T(\phi) \qquad \left[= \int\limits_{\phi\in\mathcal{S}'} (\dots)\,\underbrace{\mathrm{e}^{-(T(\phi)+V(\phi))}}_{=\mathrm{e}^{-S(\phi)}}\,\text{“}\mathrm{d}\phi\text{”}\right]$$

- Problem, the interacting Feynman measure $\mu := \mathrm{e}^{-V}\cdot\gamma_T$ is undefined:

$$\int_{\phi\in\mathcal{S}'} (\ldots)\ \underbrace{\mathrm{d}\mu(\phi)}_{\text{wannabe Feynman measure}} \quad := \quad \int_{\phi\in\mathcal{S}'} (\ldots)\ \underbrace{\mathrm{e}^{-V(\phi)}}_{\text{lives on function sense fields}}\ \underbrace{\mathrm{d}\gamma_T(\phi)}_{\text{lives on distribution sense fields}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi) = g\int\varphi^4$.
  How to bring e^{-V} and γ_T to common grounds?

- Physicist workaround: Wilsonian regularization.
  Take a continuous linear mapping $C:$ (distributional fields) → (function sense fields).
  Take the pushforward Gaussian measure $C_*\gamma_T$, which lives on $\mathrm{Ran}(C)$.
  Those are functions, so safe to integrate e^{-V} there:

$$\int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-V(\varphi)}\ \mathrm{d}\big(C_*\gamma_T\big)(\varphi) \qquad \Big[= \int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\ \mathrm{e}^{-(T_C(\varphi)+V(\varphi))}\ \text{``}\mathrm{d}\varphi\text{''}\Big]$$

a space of UV regularized fields

[Schwartz kernel theorem: C is convolution by a test function, if translationally invariant.
I.e., it is a momentum space damping, or coarse-graining of fields.]

- What do we do with the C-dependence? What is the physics / mathematics behind?
- Take a family V_C ($C \in \{$coarse-grainings$\}$) of interaction terms. $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot C_* \gamma_T$
  We say that it is a Wilsonian renormalization group (RG) flow iff:
  $\exists$ some continuous functional $z : \{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')_*\,\mu_{C''} \;=\; z(C)_*\,C'_*\,\mu_C$$
  [z is called the running wave function renormalization factor.]
- If $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \dots)$ are the moments of μ_C, then
  $\exists$ some continuous functional $z : \{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $$z(C'')^n\,\mathcal{G}_{C''}^{(n)} \;=\; z(C)^n \otimes^n C'\,\mathcal{G}_C^{(n)} \quad \text{for all } n = 0, 1, 2, \dots$$
  [Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

[We can always set $z(C) = 1$, by rescaling fields: $\tilde{\mu}_C := z(C)_*\,\mu_C$ or $\tilde{\mathcal{G}}_C^{(n)} := z(C)^n\,\mathcal{G}_C^{(n)}$.]

Part I:

On Wilsonian RG flow of correlators (arbitrary signature)

[*Class.Quant.Grav.***39**(2022)185004]

Clean definition:

A family of smooth correlators $\mathcal{G}_C$ $(C \in$ coarse-grainings$)$ is Wilsonian RG flow iff

$\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$ one has that

$\mathcal{G}_{C''}^{(n)} \;=\; \otimes^n C'\,\mathcal{G}_C^{(n)}$ holds $(n=0,1,2,...)$. ⟵ rigorous RGE in any signature

Space of Wilsonian RG flows is nonempty:

For any distributional correlator G, the family

$$\mathcal{G}_C^{(n)} \quad := \quad \otimes^n C\,G^{(n)} \qquad\qquad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lászó, Z.Tarcsay *Class.Quant.Grav.***41**(2024)125009]:

1. On manifolds it is “quite nice” topological vector space, similar to distributions.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.

$\downarrow$

UV limit.

Sketch of proofs.

1. On manifolds it is “quite nice” topological vector space, similar to distributions.
   [It is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.]

- Coarse-grainings have a natural ordering of being less UV than an other: $C'' \preceq C$ iff $C'' = C$ or $\exists\, C' :\ C'' = C'\,C$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
- Check known properties of $\mathcal{T}(\mathcal{E})$, some of them are preserved by projective limit.

2. On flat spacetime for bosonic fields, all Wilsonian RG flows are $\mathcal{G}^{(n)}_{C} = \otimes^{n} C\, G^{(n)}$.

- On flat spacetime, convolution ops by test functions $C_\eta := \eta \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_\eta}$ is n-order homogeneous polynomial in η.

  That is, $\exists|\ \mathcal{G}^{(n)}_{\eta_1,\dots,\eta_n}$ symmetric n-linear map in $\eta_1, ..., \eta_n$, such that $\mathcal{G}^{(n)}_{C_\eta} = \mathcal{G}^{(n)}_{\eta,\dots,\eta}$.
- Due to RGE, commutativity of convolution ops, and a Banach-Steinhaus thm variant, $\mathcal{G}^{(n)}_{\eta^t_1,\dots,\eta^t_n}\Big|_0$ extends to an n-variate distribution, it will do the job as $(G^{(n)}\,|\,\eta_1 \otimes ... \otimes \eta_n)$.

[A Banach-Steinhaus theorem variant (the key lemma – A.László, Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n \mathcal{D}$, then also on full $\mathcal{D}_n$.]

So, it turns out that Wilsonian RG flow of correlators $\leftrightarrow$ distributional correlators.
(under mild conditions)

Executive summary:

- In QFT, the fundamental objects of interest are distributional field correlators.
- Physical ones selected by a “field equation”, the master Dyson-Schwinger equation. Through their smoothed (Wilsonian regularized) instances [*CQG* **39**(2022)185004].

Academic question:

- What about Wilsonian RG flow of measures? (In Euclidean signature QFT.) Manuscript in preparation about that.

Part II:

On Wilsonian RG flows of Feynman measures (Euclidean signature, flat spacetime, bosonic fields)

[*manuscript in preparation*]

Wilsonian renormalization in Euclidean signature

Let us come back to Euclidean Feynman measures on flat spacetime, for bosonic fields.
[We work on $\mathcal{S}$ and $\mathcal{S}'$, because we can.]

Take a family V_C $(C \in \{\text{coarse-grainings}\})$ of interaction terms $\leftrightarrow \mu_C := \mathrm{e}^{-V_C} \cdot C_* \gamma_T$.
Let it be a Wilsonian RG flow:
∀ coarse-grainings C, C', C'' with $C'' = C'\,C$:

$$\mu_{C''} \quad = \quad C'_*\, \mu_C$$

Space of Wilsonian RG flow of Feynman measures is nonempty:
For any Feynman measure μ on $\mathcal{S}'$, the family

$$\mu_C \quad := \quad C_*\, \mu \qquad (*)$$

is a Wilsonian RG flow.

Theorem[A.Lászó, Z.Tarcsay *manuscript in prep.*]:

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $(*)$. ← UV limit
2. There exists some measurable potential $V : \mathcal{S}' \to \mathbb{R} \cup \{\pm\infty\}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma_T$.
3. For all above coarse-grainings C, one has $V_C(C\,\phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.
4. If $V_C : C[\mathcal{S}'] \to \mathbb{R} \cup \{\pm\infty\}$ bounded from below, then V is γ_T-ess.bounded from below.

Sketch of proofs.

1. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form $\mu_C = C_* \, \mu$.

- We prove it for Fourier transforms (partition functions), and then use Bochner-Minlos.
  We use that $\mathcal{S} \star \mathcal{S} = \mathcal{S}$, moreover
  that for all $K \subset \mathcal{S}$ compact $\exists\, \chi \in \mathcal{S}$ and $L \subset \mathcal{S}$ compact such that $K = \chi \star L$.

2. There exists some measurable potential $V:\, \mathcal{S}' \to \mathbb{R} \cup \{\pm\infty\}$, such that $\mu = \mathrm{e}^{-V} \cdot \gamma_T$.

- We apply Radon-Nikodym theorem, the fact that $\mathcal{S}'$ is so-called Souslin space,
  and that for $\eta \in \mathcal{S}$ with $F(\eta) > 0$ the coarse-graining $C_\eta := \eta \star (\cdot)$ is injective.

3. For all coarse-grainings C, one has $V_C(C\,\phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.

- Fundamental formula of integration variable substitution vs pusforward, Souslin-ness of $\mathcal{S}'$,
  injectivity of coarse-graining $C_\eta := \eta \star (\cdot)$ with $\eta \in \mathcal{S},\ F(\eta) > 0$.

4. If $V_C:\, C[\mathcal{S}'] \to \mathbb{R} \cup \{\pm\infty\}$ bounded from below, then V is γ_T-ess.bounded from below.
- Trivial from 3.

Relation to usual RG theory:

Fix some $\eta \in \mathcal{S}$ such that $\int \eta = 1$ and $F(\eta) > 0$.
Introduce scaled η, that is $\eta_\Lambda(x) := \Lambda^N \eta(\Lambda\, x)$ (for all $x \in \mathbb{R}^N$ and scaling $1 \leq \Lambda < \infty$).
One has $\eta_\Lambda \xrightarrow{\mathcal{S}'} \delta$ as $\Lambda \longrightarrow \infty$.

By our theorem, for all Λ, one has $V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = V(\phi)$ for γ_T-a.e. $\phi \in \mathcal{S}'$.

$\Downarrow$

Informally: ODE for $V_{C_{\eta_\Lambda}}$, namely $\frac{\mathrm{d}}{\mathrm{d}\Lambda} V_{C_{\eta_\Lambda}}(C_{\eta_\Lambda}\, \phi) = 0$ for $1 \leq \Lambda < \infty$.

QFT people try to solve such flow equation, given initial data $V_{C_\Lambda}\big|_{\Lambda=1}$.

But why bother? By our theorem, all RG flows of such kind has some V at the UV end.
Look directly for V?

What really the game is about?

Original problem:

- We had $\mathcal{V}:\ \{\text{function sense fields}\}\to\mathbb{R}\cup\{\pm\infty\}$, say $\mathcal{V}(\varphi)=g\int\varphi^4$.
- We would need to integrate it against γ_{T}, but that lives on $\mathcal{S}'$ fields.
- γ_{T} known to be supported “sparsely”, i.e. not on function fields, but really on $\mathcal{S}'$.
- So, we really need to extend $\mathcal{V}$ at least γ_{T}-a.e. to make sense of $\mu:=\mathrm{e}^{-V}\cdot\gamma_{T}$.

Caution by physicists: this may be impossible.

- We are afraid that V on S' might not exist.
- Instead, let us push γ_{T} to smooth fields by C, do there $\mu_C:=\mathrm{e}^{-V_C}\cdot C_*\,\gamma_{T}$.
- Then, get rid of C-dependence of μ_C by concept of Wilsonian RG flow.
  Maybe even $\mu_C\to\mu$ could exist as $C\to\delta$ if we are lucky...

Our result: we are back to the start.

- The UV limit Feynman measure μ then indeed exists.
- But we just proved that then there must exist some extension V of $\mathcal{V}$ to $\mathcal{S}'$, γ_{T}-a.e.
- So, we’d better look for that ominous extension V.
- For bounded from below $\mathcal{V}$, bounded from below measurable V needed.
  If we find one, $\mu:=\mathrm{e}^{-V}\cdot\gamma_{T}$ is then finite measure automatically.
  Only pathology: overlap integral of e^{-V} and γ_{T} expected small, maybe zero.
  We only need to make sure that $\int_{\phi\in\mathcal{S}'}\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_{T}(\phi)>0$!

A natural extension[A.László, Z.Tarcsay *manuscript in prep.*]:

If $\mathcal{V}$ is bounded from below, there is an optimal extension, the “greedy” extension.

$V(\cdot) \;:=\; (\gamma_T)\inf_{\{\eta_n\to\delta\}}\;\liminf_{\eta_n\to\delta}\;\mathcal{V}(\eta_n\star\;\cdot\;)$

This is the lower bound of extensions, i.e. overlap of e^{-V} and γ_T largest.

But is V measurable at all? Not evident.

Theorem[A.László, Z.Tarcsay *manuscript in prep.*]:

1. The “greedy extension” is measurable.
2. The interacting Feynman measure $\mu := \mathrm{e}^{-V}\cdot\gamma_T$ by greedy extension is nonzero iff

$$\exists\;\eta_n\to\delta\;:\qquad \int\limits_{\phi\in\mathcal{S}'}\limsup_{n\to\infty}\;\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma_T(\phi)\quad>\quad 0.$$

Sufficient condition:

$$\exists\;\eta_n\to\delta\;:\qquad \lim_{n\to\infty}\;\int\limits_{\phi\in\mathcal{S}'}\mathrm{e}^{-\mathcal{V}(\eta_n\star\phi)}\;\mathrm{d}\gamma_T(\phi)\quad>\quad 0.$$

This is actually a calculable condition for concrete models!

Summary

- Wilsonian RG flow of correlators can be defined
  in any signature and on manifolds.
  Have nice function space properties like distributions.

- Under mild conditions,
  they originate from a distributional correlator (UV limit).

  [$\sim$ existence theorem for multiplicative renormalization.]

- Likely to be generically true
  (on manifolds, in any signature).

- In Euclidean signature, similar for Feynman measures.
  + a new condition for renormalizability.

Backup slides

Followed guidelines

Do not use (unless emphasized):

- Structures specific to an affine spacetime manifold.
- Known fixed spacetime metric / causal structure.
- Known splitting of Lagrangian to free + interaction term.

Consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics. (No Schwartz functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.
- Can only use generic, differential geometrically natural objects.

Outline

Will attempt to set up eom for the key ingredient for the quantum probability space of QFT.

I. On Wilsonian regularized Feynman functional integral formulation:

- Can be substituted by regularized master Dyson-Schwinger equation for correlators.
- For conformally invariant or flat spacetime Lagrangians, showed an existence condition for regularized MDS solutions, provides convergent iterative solver method.

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II. On Wilsonian renormalization group flows of correlators:

- They form a topological vector space which is
  Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
- On flat spacetime for bosonic fields: in bijection with distributional correlators.

[**`arXiv:2303.03740`** *with Zsigmond Tarcsay*]

Part I:

On Wilsonian regularized Feynman functional integral formulation

The classical field theory scene

$\mathcal{M}$ a smooth orientable oriented manifold (wannabe spacetime, but no metric, yet).

$V(\mathcal{M})$ a vector bundle over it (its smooth sections are matter fields + metric if dynamical).

Field configurations:

$$\underbrace{(v,\nabla)}_{=:\ \psi} \ \in\ \underbrace{\Gamma\Big(V(\mathcal{M})\times_{\mathcal{M}}\mathrm{CovDeriv}\big(V(\mathcal{M})\big)\Big)}_{=:\ \boldsymbol{\mathcal{E}}}$$

Real topological affine space with the $\mathcal{E}$ smooth function topology.

Field variations:

$$\underbrace{(\delta v,\delta C)}_{=:\ \delta\psi} \ \in\ \underbrace{\Gamma\Big(V(\mathcal{M})\times_{\mathcal{M}}T^{*}(\mathcal{M})\otimes V(\mathcal{M})\otimes V^{*}(\mathcal{M})\Big)}_{=:\ \mathcal{E}}$$

Real topological vector space with the $\mathcal{E}$ smooth function topology.

Test field variations: $\delta\psi_{T}\in\mathcal{D}$, compactly supported ones from $\mathcal{E}$ with $\mathcal{D}$ test funct. top.

Informal Feynman functional integral in Lorentz signature

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ for bringing the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$, and take $J_1, \ldots, J_n \in \mathcal{E}'$. Then, $\psi \mapsto (J_1|\psi{-}\psi_0) \cdot \ldots \cdot (J_n|\psi{-}\psi_0)$ defines a $\boldsymbol{\mathcal{E}} \to \mathbb{R}$ polynomial observable.

Feynman type quantum vacuum expectation value of this is postulated as:

$$\int\limits_{\psi \in \boldsymbol{\mathcal{E}}} (J_1|\psi{-}\psi_0) \cdot \ldots \cdot (J_n|\psi{-}\psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, \mathrm{d}\lambda(\psi) \quad \Big/ \quad \int\limits_{\psi \in \boldsymbol{\mathcal{E}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, \mathrm{d}\lambda(\psi)$$

Partition function often invoked to book-keep these (formal Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S} \lambda$):

$$Z_{\psi_0} : \quad \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int\limits_{\psi \in \boldsymbol{\mathcal{E}}} \mathrm{e}^{\mathrm{i}\,(J|\psi-\psi_0)} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S(\psi)} \, \mathrm{d}\lambda(\psi),$$

and from this one can define

$$G^{(n)}_{\psi_0} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0}(J)} \, \partial^{(n)}_J Z_{\psi_0}(J) \right) \Big|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0} := \left(G^{(0)}_{\psi_0}, G^{(1)}_{\psi_0}, \ldots, G^{(n)}_{\psi_0}, \ldots\right) \in \bigoplus\limits_{n \in \mathbb{N}_0} \overset{n}{\otimes} \mathcal{E}$.

Above quantum expectation value expressable via distribution pairing: $\left(J_1 \otimes \ldots \otimes J_n \,\middle|\, G^{(n)}_{\psi_0}\right)$.

Well known problems:

- No “Lebesgue” measure λ in infinite dimensions.
- Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ is meaningful. (Can be repaired to some extent in Euclidean signature.)
- Neither the Fourier transform of this undefined measure is meaningful.

Rules in informal QFT:

- as if λ existed as *translation invariant* (Lebesgue) measure,
- as if $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ existed as *finite measure*, with *finite moments* and *analytic Fourier transform*.

Textbook “theorem”: because of above rules, one has
$Z:\,\mathcal{E}'\to\mathbb{C}$ is Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ “$\Longleftrightarrow$” it satisfies master-Dyson-Schwinger eq

$$\Big(\,\mathbf{E}\big((-\mathrm{i})\partial_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathcal{E}')$$

where $E(\psi):=DS(\psi)$ is the Euler-Lagrange functional at $\psi\in\boldsymbol{\mathcal{E}}$.

Does this informal PDE have a meaning? [Yes, on the correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.]

Rigorous definition of Euler-Lagrange functional

- Let a Lagrange form be given, which is

$$\mathrm{L}:\ V(\mathcal{M})\ \oplus\ T^*(\mathcal{M})\otimes V(\mathcal{M})\ \oplus\ T^*(\mathcal{M})\wedge T^*(\mathcal{M})\otimes V(\mathcal{M})\otimes V^*(\mathcal{M})\ \longrightarrow\ \overset{\dim(\mathcal{M})}{\wedge}\ T^*(\mathcal{M})$$

pointwise bundle homomorphism.

- Lagrangian expression:

$$\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}}\mathrm{CovDeriv}\big(V(\mathcal{M})\big)\big)\ \longrightarrow\ \Gamma\big(\overset{\dim(\mathcal{M})}{\wedge}\ T^*(\mathcal{M})\big),\quad (v,\nabla)\ \longmapsto\ \mathrm{L}(v,\nabla v,F(\nabla))$$

where $F(\nabla)$ is the curvature tensor.

- Action functional:

$$S:\quad \underbrace{\Gamma\big(V(\mathcal{M})\times_{\mathcal{M}}\mathrm{CovDeriv}(V(\mathcal{M}))\big)}_{=:\ \boldsymbol{\mathcal{E}}}\ \longrightarrow\ \mathrm{Meas}(\mathcal{M},\mathbb{R}),\quad \underbrace{(v,\nabla)}_{=:\ \psi}\ \longmapsto\ \Big(\mathcal{K}\mapsto S_{\mathcal{K}}(v,\nabla)\Big)$$

where $S_{\mathcal{K}}(v,\nabla):=\int\limits_{\mathcal{K}}\mathrm{L}(v,\nabla v,F(\nabla))$ for all $\mathcal{K}\subset\mathcal{M}$ compact.

Action functional $S : \boldsymbol{\mathcal{E}} \to \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : \quad \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi) \,\big|\, \delta\psi\big)\Big)$$

is the usual Euler-Lagrange integral on $\mathcal{K}$ $+$ usual boundary integral on $\partial\mathcal{K}$.
Jointly continuous in its variables, linear in second variable.

Euler-Lagrange functional:
We restrict DS from $\boldsymbol{\mathcal{E}} \times \mathcal{E}$ to $\boldsymbol{\mathcal{E}} \times \mathcal{D}$, to make the EL integral over full $\mathcal{M}$ finite.

$$E : \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad \big(\psi,\, \delta\psi_{T}\big) \longmapsto \big(E(\psi) \,\big|\, \delta\psi_{T}\big) := \big(DS_{\mathcal{M}}(\psi) \,\big|\, \delta\psi_{T}\big)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full $\mathcal{M}$, real valued.
Jointly sequentially continuous, linear in second variable. (Also, $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ continuous.)

Classical field equation is

$$\psi \in \boldsymbol{\mathcal{E}}\ ? \qquad \forall\, \delta\psi_{T} \in \mathcal{D} : \ \big(E(\psi) \,\big|\, \delta\psi_{T}\big) = 0.$$

Observables are the $O : \boldsymbol{\mathcal{E}} \to \mathbb{R}$ continuous maps.

Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on Z to n-field correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.
  These sit in the tensor algebra $\mathcal{T}(\mathcal{E}):=\bigoplus\limits_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}$ of field variations.

  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
  Naturally topologized: with Tychonoff topology, similar to $\mathcal{E}$, i.e. nuclear Fréchet.

- Algebraic tensor algebra $\mathcal{T}_a(\mathcal{E}'):=\bigoplus\limits_{n\in\mathbb{N}_0}\hat{\otimes}_\pi^n\,\mathcal{E}'$ of sources.

  Naturally topologized: loc.conv. direct sum topology, similar to $\mathcal{E}'$, i.e. dual nuclear Fréchet.

- Schwartz kernel thm gives some simplification: $\hat{\otimes}_\pi^n\,\mathcal{E}\equiv\mathcal{E}_n$ and $\hat{\otimes}_\pi^n\,\mathcal{E}'\equiv\mathcal{E}'_n$ (n-variate).

- One has $\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ and $\big(\mathcal{T}(\mathcal{E})\big)''\equiv\mathcal{T}(\mathcal{E})$ etc, “nice” properties.
  Moreover, tensor algebra of field variations is topological unital bialgebra.

  Unity $\mathbb{1}:=(1,0,0,0,\ldots)$.
  Left-multiplication L_x by a fix element x meaningful and continuous linear.
  Left-insertion $\boldsymbol{\ell}_p$ (tracing out) by $p\in\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ also meaningful, continuous linear.
  Usual graded-commutation: $\big(\,\boldsymbol{\ell}_p\,L_{\delta\psi}\,\pm\,L_{\delta\psi}\,\boldsymbol{\ell}_p\,\big)\,G\;=\;(p|\delta\psi)\,G\quad(\forall p\in\mathcal{E}',\;\delta\psi\in\mathcal{E},\;G\,)$.

Take a classical observable $O:\ \boldsymbol{\mathcal{E}}\to\mathbb{R},\ \psi\mapsto O(\psi)$, let $O_{\psi_0}:=O\circ(\mathrm{I}_{\mathcal{E}}+\psi_0)$.

That is, $O_{\psi_0}(\psi-\psi_0)\overset{!}{=}O(\psi)\quad(\forall\psi\in\boldsymbol{\mathcal{E}})$, with some fixed reference field $\psi_0\in\boldsymbol{\mathcal{E}}$.

We say that O is multipolynomial iff for some $\psi_0\in\boldsymbol{\mathcal{E}}$ there exists $\mathbf{O}_{\psi_0}\in\mathcal{T}_a(\mathcal{E}')$, such that

$$\forall\,\psi\in\boldsymbol{\mathcal{E}}:\quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=\,O(\psi)}\;=\;\Big(\mathbf{O}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi{-}\psi_0),\,\overset{2}{\otimes}(\psi{-}\psi_0),\,\ldots\big)\Big).$$

Similarly $E:\ \boldsymbol{\mathcal{E}}\to\mathcal{D}'$, $\psi\mapsto E(\psi)$, let $E_{\psi_0}:=E\circ(\mathrm{I}_{\mathcal{E}}+\psi_0)$ the same re-expressed on $\mathcal{E}$.

That is, $E_{\psi_0}(\psi-\psi_0)\overset{!}{=}E(\psi)\quad(\forall\psi\in\boldsymbol{\mathcal{E}})$, with some fixed reference field $\psi_0\in\boldsymbol{\mathcal{E}}$.

We say that E is multipolynomial iff $\exists\,\mathbf{E}_{\psi_0}\in\mathcal{T}_a(\mathcal{E}')\hat{\otimes}_\pi\mathcal{D}'$, such that

$$\forall\,\psi\in\boldsymbol{\mathcal{E}},\ \delta\psi_T\in\mathcal{D}:\ \underbrace{\Big(E_{\psi_0}(\psi-\psi_0)\,\Big|\,\delta\psi_T\Big)}_{=\big(E(\psi)\,\big|\,\delta\psi_T\big)}\ =\ \Big(\mathbf{E}_{\psi_0}\,\Big|\,\big(1,\,\overset{1}{\otimes}(\psi-\psi_0),\,\overset{2}{\otimes}(\psi-\psi_0),\,\ldots\big)\otimes\delta\psi_T\Big).$$

For fixed $\delta\psi_T\in\mathcal{D}$ one has $(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)\in\mathcal{T}_a(\mathcal{E}')$, i.e. one can left-insert with it:

$\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acts on $\mathcal{T}(\mathcal{E})$.

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:

$$\underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} = 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\iota_{(\mathbf{E}_{\psi_0} \,|\, \delta\psi_T)} - \mathrm{i}\,\hbar\, L_{\delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\delta\psi_T}} G_{\psi_0} = 0.$$

This substitutes Feynman functional integral formulation, signature independently.
Also, no fixed background causal structure etc needed.

[Feynman type quantum vacuum expectation value of O is then $\big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0}\big)$.]

Example: ϕ^4 model.

Euler-Lagrange functional is

$$E: \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_T) \longmapsto \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y \psi(y)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \psi^3(y)\, \mathrm{v}(y).$$

$$\Downarrow$$

MDS operator at $\psi_0 = 0$ reads

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\, G\big)^{(n)}(x_1, \ldots, x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y G^{(n+1)}(y, x_1, \ldots, x_n)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, G^{(n+3)}(y, y, y, x_1, \ldots, x_n)\, \mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar\; \underbrace{n\, \tfrac{1}{n!} \textstyle\sum_{\pi\in\Pi_n} \delta\psi_T(x_{\pi(1)})\, G^{(n-1)}(x_{\pi(2)}, \ldots, x_{\pi(n)})}_{=\, (L_{\delta\psi_T}\, G)^{(n)}(x_1, \ldots, x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.

Theorem: no solutions with high differentiability (e.g. as smooth functions).

Theorem: for free Minkowski KG case, distributional solution only,
namely $G_{\psi_0} = \exp(K_{\psi_0})$, where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0} && \longleftarrow \text{(symmetric propagator)} \\ K^{(n)}_{\psi_0} &= 0 && (n \geq 2) \end{aligned}$$

So we expect distributional solutions only, at best.

How can one extend to distributions interaction term like $G^{(n+3)}(y, y, y, x_1, ..., x_n)$?

With sufficiency condition of H́ormander? (Theorem: not workable.)

Via approximation with functions, i.e. sequential closure? (Theorem: not workable.)

Workaround in QFT: Wilsonian regularization using coarse-graining (UV damping).

Wilsonian regularized master Dyson-Schwinger equation

- When $\mathcal{E}$ (resp $\mathcal{D}$) are smooth sections of some vector bundle,
  denote by $\mathcal{E}^{\times}$ (resp $\mathcal{D}^{\times}$) the smooth sections of its densitized dual vector bundle.
  Then, distributional sections are $\mathcal{D}^{\times\prime}$ (resp $\mathcal{E}^{\times\prime}$).

- A continuous linear map $C:\,\mathcal{E}^{\times\prime}\to\mathcal{E}$ is called smoothing operator.
  Schwartz kernel theorem: $C\ \longleftrightarrow$ its Schwartz kernel κ which is section over $\mathcal{M}\times\mathcal{M}$.

- C_κ is properly supported iff $\forall\,\mathcal{K}\subset\mathcal{M}$ compact: $\kappa|_{\mathcal{M}\times\mathcal{K}}$ and $\kappa|_{\mathcal{K}\times\mathcal{M}}$ has compact supp.
  It extends to $\mathcal{E}^{\times\prime}$, $\mathcal{E}$, $\mathcal{D}$, $\mathcal{D}^{\times\prime}$ and preserves compact support (the transpose similarly).

- A properly supported smoothing operator is coarse-graining iff
  injective as $\mathcal{E}^{\times\prime}\to\mathcal{E}$ and its transpose similarly.
  E.g. ordinary convolution by a nonzero test function over affine (Minkowski) spacetime.

Coarse-graining ops are natural generalization of convolution by test functions to manifolds.

Originally: Feynman integral “$\Longleftrightarrow$” MDS equation.

Wilsonian regularized Feynman integral:
integrate only on the image space $C_\kappa[\mathcal{D}^{\times\prime}] \subset \mathcal{E}$ of some coarse-graining operator C_κ.

Wilsonian regularized Feynman integral “$\Longleftrightarrow$” Wilsonian regularized MDS equation:

we search for $(\psi_0, \gamma(\kappa), \mathcal{G}_{\psi_0,\kappa})$ such that:

$$\underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=:\, b\,\mathcal{G}_{\psi_0,\kappa}} \quad = \quad 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\ell_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \mathcal{G}_{\psi_0,\kappa} \quad = \quad 0.$$

Brings back problem from distributions to smooth functions, but depends on regulator κ.

Smooth function solution to free KG regularized MDS eq: $\mathcal{G}_{\psi_0,\kappa} = \exp(\mathcal{K}_{\psi_0,\kappa})$ where

$$\begin{aligned}
\mathcal{K}^{(0)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(1)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0,\kappa} && \longleftarrow \text{(smoothed symmetric propagator)} \\
\mathcal{K}^{(n)}_{\psi_0,\kappa} &= 0 && (n \geq 2)
\end{aligned}$$

No problem to evaluate interaction term like $\mathcal{G}^{(n+3)}(y, y, y, x_1, ..., x_n)$ on functions.

[We proved a convergent iterative solution method at fix κ, see the paper or ask.]

But what we do with κ dependence? (Rigorous Wilsonian renormalization?)

Part II:

On Wilsonian RG flows of correlators

Informal Wilsonian RG flows of Feynman measures

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ to bring the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$.

Fix a coarse-graining C_κ defining a UV regularization strength.

Assume that one has an action $S_{\psi_0, C_\kappa} : \underbrace{C_\kappa[\mathcal{D}^{\times}{}']}_{\subset \mathcal{E}} \to \mathbb{R}$ for a coarse-graining C_κ.

Informally, one assumes a Lebesgue measure λ_{C_κ} on each subspace $C_\kappa[\mathcal{D}^{\times}{}']$ of $\mathcal{E}$.
(In Euclidean signature this inexactness can be remedied by Gaussian measure.)

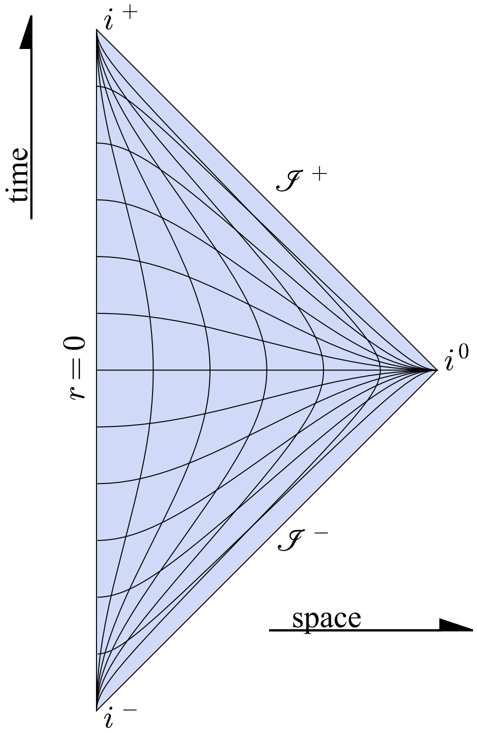
This defines the Wilsonian regularized Feynman measure $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\psi_0, C_\kappa}} \lambda_{C_\kappa}$.

A family of actions S_{ψ_0, C_κ} $(C_\kappa \in$ coarse-grainings$)$ is Wilsonian RG flow iff:
$\forall$ coarse-grainings C_κ, C_μ, C_ν with $C_\nu = C_\mu C_\kappa$ one has that
$\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\psi_0, C_\nu}} \lambda_{C_\nu}$ is the pushforward of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\psi_0, C_\kappa}} \lambda_{C_\kappa}$ by C_μ. ⟵ RGE

Rigorous definition will be this, but expressed on the formal moments (n-field correlators).

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E:\boldsymbol{\mathcal{E}}\to\mathcal{D}'$ conformally invariant: re-expressable on Penrose conformal compactification.



That is always a compact manifold, with cone condition boundary.

$E:\boldsymbol{\mathcal{E}}\to\mathcal{D}'$ reformulable over this base manifold.

In such situation, $\mathcal{E} = \mathcal{D}$ and have nice properties:
countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \supset F_1 \supset \ldots \supset F_m \supset \ldots \supset \mathcal{E}$$

(Intersection of shrinking Hilbert spaces F_m with Hilbert-Schmidt embedding.)

Theorem [Dubin,Hennings:*P.RIMS***25**(1989)971]:
without penalty, one can equip $\mathcal{T}(\mathcal{E})$ with a better topology, inheriting CHNF topology.

$$H_0 \supset H_1 \supset \ldots \supset H_m \supset \ldots \supset \mathcal{T}_h(\mathcal{E})$$

Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa}: \quad H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}\mathcal{G}$$

Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa}: \quad H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i\in\mathbb{N}_0} \mathbf{M}^\dagger_{\psi_0,\kappa,\delta\psi_T\,i}\mathbf{M}_{\psi_0,\kappa,\delta\psi_T\,i}\mathcal{G}$$

By construction: $\mathcal{G}$ is κ-regularized MDS solution $\Longleftrightarrow$ $b\,\mathcal{G} = 1$ and $\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G} = 0$.

Theorem [A.L.]:

(i) the iteration

$$\mathcal{G}_0 \;:=\; \mathbb{1} \;\text{ and }\; \mathcal{G}_{l+1} \;:=\; \mathcal{G}_l - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G}_l \qquad (l = 0, 1, 2, \dots)$$

is always convergent if $T >$ trace norm of $\hat{\mathbf{M}}^2_{\psi_0,\kappa}$.

(ii) the κ-regularized MDS solution space is nonempty iff

$$\lim_{l\to\infty} b\,\mathcal{G}_l \;\neq\; 0.$$

(iii) and in this case

$$\lim_{l\to\infty} \mathcal{G}_l$$

is an MDS solution, up to normalization factor.

Use for lattice-like numerical method in Lorentz signature?
(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space X are elementary events.
  - Collection of Borel sets Σ of X are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space $\mathcal{H}$ are elementary events, X.
  - Collection of all closed subspaces Σ of $\mathcal{H}$ are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S:\ F\to G$ is Fréchet-Hadamard differentiable at $\psi\in F$ iff:
there exists $DS(\psi):\ \mathbb{F}\to\mathbb{G}$ continuous linear, such that for all sequence $n\mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n\mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G})\lim_{n\to\infty}\left(\frac{S(\psi+t_n\,h_n)-S(\psi)}{t_n}-DS(\psi)\,h_n\right)\quad=\quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S:\ \boldsymbol{\mathcal{E}} \longrightarrow \mathrm{Meas}(\mathcal{M},\mathbb{R})$ is

$$DS:\ \boldsymbol{\mathcal{E}}\times\mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M},\mathbb{R}),\ (\psi,\delta\psi)\longmapsto \Big(\mathcal{K}\mapsto \big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v,\nabla)}_{=:\psi}\in\boldsymbol{\mathcal{E}}$ given,

$$\underbrace{(\delta v,\delta C)}_{=:\delta\psi}\ \mapsto\ \big(DS_{\mathcal{K}}(v,\nabla)\,\big|\,(\delta v,\delta C)\big)=$$

$$\int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))\,\delta v+D_2^a\mathrm{L}(v,\nabla v,P(\nabla))\,(\nabla_a\delta v+\delta C_a v)+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))\,\tilde{\nabla}_{[a}\delta C_{b]}\Big)$$

$$=\int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta v-\big(\tilde{\nabla}_a D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta v\Big)+$$

$$\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta C_a v-2\,\big(\tilde{\nabla}_a D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta C_b\Big)$$

$$+\ m\int\limits_{\partial\mathcal{K}}\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta v+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta C_b\Big)$$

$$(m:=\dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$W(\mathcal{M})$ vector bundle, $W^{\times}(\mathcal{M}) := W^{*}(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$ its densitized dual.
$W^{\times\times}(\mathcal{M}) \equiv W(\mathcal{M})$.

Correspondingly: $\mathcal{E}^{\times}$ and $\mathcal{D}^{\times}$ are densitized duals of $\mathcal{E}$ and $\mathcal{D}$.

$\mathcal{E} \times \mathcal{D}^{\times} \to \mathbb{R}$, $(\delta\psi, p_{T}) \mapsto \int\limits_{\mathcal{M}} \delta\psi \, p_{T}$ and $\mathcal{D} \times \mathcal{E}^{\times} \to \mathbb{R}$, $(\delta\psi_{T}, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_{T} \, p$ jointly sequentially continuous.

Therefore, continuous dense linear injections $\mathcal{E} \to \mathcal{E}^{\times\prime}$ and $\mathcal{D} \to \mathcal{D}^{\times\prime}$.
(hance the name, distributional sections)

Let $A : \mathcal{E} \to \mathcal{E}$ continuous linear.
It has formal transpose iff there exists $A^{t} : \mathcal{D}^{\times} \to \mathcal{D}^{\times}$ continuous linear, such that $\forall \delta\psi \in \mathcal{E}$ and $p_{T} \in \mathcal{D}^{\times}$: $\int\limits_{\mathcal{M}} (A\,\delta\psi)\, p_{T} = \int\limits_{\mathcal{M}} \delta\psi\, (A^{t}\, p_{T})$.
Topological transpose of formal transpose $(A^{t})' : (\mathcal{D}^{\times})' \to (\mathcal{D}^{\times})'$ is the distributional extension of A. Not always exists.

Fundamental solution on manifolds

Let $E:\ \boldsymbol{\mathcal{E}}\times\mathcal{D}\to\mathbb{R}$ be Euler-Lagrange functional, and $J\in\mathcal{D}'$.

$\mathtt{K}_{(J)}\in\boldsymbol{\mathcal{E}}$ is solution with source J, iff $\forall\delta\psi_{T}\in\mathcal{D}:\ \left(E(\mathtt{K}_{(J)})\,\middle|\,\delta\psi_{T}\right)=(J|\delta\psi_{T})$.

Specially: one can restrict to $J\in\mathcal{D}^{\times}\subset\mathcal{E}^{\times}\subset\mathcal{D}'$.

A continuous map $\mathtt{K}:\ \mathcal{D}^{\times}\to\boldsymbol{\mathcal{E}}$ is fundamental solution, iff for all $J\in\mathcal{D}^{\times}$ the field $\mathtt{K}(J)\in\boldsymbol{\mathcal{E}}$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0}:\ \mathcal{D}^{\times}\to\mathcal{E}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0}:\ \mathcal{E}\to\mathcal{D}'$):
$\mathtt{K}_{\psi_0}\in\mathcal{L}in(\mathcal{D}^{\times},\mathcal{E})\subset(\mathcal{D}^{\times})'\otimes(\mathcal{D}^{\times})'$ is distribution.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\, \mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$\begin{aligned} K^{(0)}_{\psi_0,\kappa} &= 0, \\ K^{(1)}_{\psi_0,\kappa} &= 0, \\ K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\, (C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \big\{y \in \mathbb{G} \,\big|\, \exists\, (x_n)_{n\in\mathbb{N}}$ in $\mathrm{Dom}(M)$ such that $\lim\limits_{n\to\infty} x_n = 0$ and $\lim\limits_{n\to\infty} M x_n = y\big\}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow$ $\mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow$ $\mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathcal{D}\otimes\mathcal{T}(\mathcal{E}) \to \mathcal{T}(\mathcal{E}), \quad G \mapsto \mathbf{M}\,G$$

linear, everywhere defined continuous. So,

$$\mathbf{M}: \quad \mathcal{T}(\mathcal{D}^{\times\prime}) \rightarrowtail \mathcal{D}'\otimes\mathcal{T}(\mathcal{D}^{\times\prime}), \quad G \mapsto \mathbf{M}\,G$$

linear, densely defined.

Similarly: $\mathbf{M}_\kappa$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ?$ $\qquad G^{(0)} = 1$ and $\exists\, \mathcal{G}_\kappa \to G$ approximator sequence, such that :

$$\lim_{\kappa \to \delta} \mathbf{M}\, \mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

Not good equation:

$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ?$ $\qquad G^{(0)} = 1$ and $\exists\, \mathcal{G}_\kappa \to G$ approximator sequence, such that :

$$\lim_{\kappa \to \delta} \mathbf{M}_\kappa\, \mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa :\ \mathbf{M}_\kappa\, \mathcal{G}_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.

Running coupling:
If in $\mathbf{M}_\kappa$ EL terms are combined with κ-dependent weights $\gamma(\kappa)$.
(Not just with real factors.)
E.g.:

$$(\gamma, G) \in \mathcal{T}(\mathcal{D}^{\times\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa :\ \mathbf{M}_{\gamma(\kappa),\kappa}\, \mathcal{G}_\kappa = 0.$$

Feynman integral “⟺” MDS equation.

Wilsonian regularized Feynman integral:
integrate not on $\mathcal{E}$, only on the image space $C_\kappa[\mathcal{E}]$ of a smoothing operator $C_\kappa : \mathcal{E} \to \mathcal{E}$.

[Smoothing operator: $\sim$ convolution, can be generalized to manifolds. Does UV damping.]
Automatically knows RGE relations.

Wilsonian regularized Feynman integral “⟺” regularized MDS equation + RGE:

$$\big(\psi_0, \kappa \mapsto \gamma(\kappa), \kappa \mapsto \mathcal{G}_{\psi_0,\kappa}\big) \;=\,?\; \text{ such that : } \qquad \underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=:\, b\,\mathcal{G}_{\psi_0,\kappa}} \;=\; 1,$$

$$\forall\kappa : \forall\,\delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\ell_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \mathcal{G}_{\psi_0,\kappa} \;=\; 0,$$

$$\text{RGE} \;\longrightarrow \qquad \forall\mu,\kappa : \quad \mathcal{G}^{(n)}_{\psi_0,(C_\mu\kappa)} \;=\; (\otimes^n C_\mu)\,\mathcal{G}^{(n)}_{\psi_0,\kappa}.$$

Running coupling is meaningful. Conjecture: RG flow of $\mathcal{G}_{\psi_0,\kappa} \leftrightarrow$ distributional G_{ψ_0}.
(Conjecture proved for flat spacetime for bosonic fields.)

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\begin{array}{ccccccc}
\hat{\otimes}_\pi^n \mathcal{E} & \equiv & \mathcal{E}_n & \equiv & (\hat{\otimes}_\pi^n \mathcal{E}')' & \equiv & \mathcal{L}in(\mathcal{E}', \hat{\otimes}_\pi^{n-1}\mathcal{E}) \\
 & & & & & & \\
(\hat{\otimes}_\pi^n \mathcal{E})' & \equiv & \mathcal{E}'_n & \equiv & \hat{\otimes}_\pi^n \mathcal{E}' & \equiv & \mathcal{L}in(\mathcal{E}, \hat{\otimes}_\pi^{n-1}\mathcal{E}') \\
 & & & & & & \\
\hat{\otimes}_\pi^n \mathcal{D} & \leftarrow & \mathcal{D}_n & \equiv & (\hat{\otimes}_\pi^n \mathcal{D}')' & & \\
 & \text{cont.bij.} & & & & & \\
(\hat{\otimes}_\pi^n \mathcal{D})' & \rightarrow & \mathcal{D}'_n & \equiv & \hat{\otimes}_\pi^n \mathcal{D}' & \equiv & \mathcal{L}in(\mathcal{D}, \hat{\otimes}_\pi^{n-1}\mathcal{D}')
\end{array}$$

$\mathcal{E} \times \mathcal{E} \to F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \to F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).