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A robust iterative unfolding method for signal processing

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Abstract

It is a common problem in signal processing to remove a non-ideal detector resolution from a measured probability density function of some physical quantity. This process is called unfolding (a special case is the deconvolution), and it would involve the inversion of the integral operator describing the folding (i.e. the smearing of the detector). Currently, there is no unbiased method known in the literature for this issue (here, by unbiased we mean those approaches which do not assume an ansatz for the unknown probability density function). There is a well-known series expansion (Neumann series) in functional analysis for perturbative inversion of specific operators on Banach spaces. However, operators that appear in signal processing (e.g. folding and convolution of probability density functions), in general, do not satisfy the usual convergence condition of that series expansion. This paper provides some theorems on the convergence criteria of a similar series expansion for this more general case, which is not yet covered by the literature. The main result is that a series expansion provides a robust unbiased unfolding and deconvolution method. For the case of the deconvolution, such a series expansion can always be applied, and the method always recovers the maximum possible information about the initial probability density function, thus the method is optimal in this sense. A very significant advantage of the presented method is that one does not have to introduce *ad hoc* frequency regulations etc, as in the case of usual naive deconvolution methods. For the case of general unfolding problems, we present a computer-testable sufficient condition for the convergence of the series expansion in question. Some test examples and physics applications are also given. The most important physics example shall be (which originally motivated our survey on this topic) the case of $\pi^0 \to \gamma + \gamma$ particle decay: we show that one can recover the initial π^0 momentum density function form the measured single γ momentum density function by our series expansion.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In experimental physics, one commonly faces the following problem. The probability density function of a given physical quantity is to be measured (e.g. by histogramming) with an experimental apparatus, but a non-ideal detector smears the signal. The question arises: if one knows the behaviour of the detector quite well (i.e. one knows the response function of the detector), how can one reconstruct the original undistorted probability density function of the given physical quantity? Specially: there is an unknown probability density function $x \mapsto f(x)$ (this is the unknown probability density function of the undistorted physical quantity), and the measured density function is obtained by $y \mapsto g(y) = \int \rho(y|x) f(x)\,\mathrm{d}x$ (where the conditional density function $(y, x) \mapsto \rho(y|x)$ describes the smearing of the measurement apparatus, also called the *response function*); then under which conditions, and how, can one re-obtain (i.e. *unfold*) the original probability density function f by measuring g and by knowing ρ? We formalize this problem below. (In the text we shall abbreviate *probability density function* by *pdf*, *conditional probability density function* by *cpdf*, and the notion *Lebesgue almost everywhere* or *Lebesgue almost every*, known in measure theory, by a.e.)

Let X and Y be two finite-dimensional real vector spaces, each equipped with the Lebesgue measure (which is unique up to a global positive constant factor). Then $L^1(X)$ and $L^1(Y)$ denote the space of Lebesgue integrable function classes $X \to \mathbb{C}$ and $Y \to \mathbb{C}$, respectively.

Definition 1. *Let $\rho : Y \times X \to \mathbb{R}^+_0$, $(y, x) \mapsto \rho(y|x)$ is a cpdf over the product space $Y \times X$, (i.e. it is a non-negative valued Lebesgue measurable function on the product space which satisfies for all $x \in X : \int \rho(y|x)\,\mathrm{d}y = 1$). Then the linear operator*

$$A_\rho : L^1(X) \to L^1(Y), \qquad (x \mapsto f(x)) \mapsto \left(y \mapsto \int \rho(y|x) f(x)\,\mathrm{d}x\right),$$

is called the <u>*folding operator by ρ*</u>.

Remark 2. The remarks below are trivial.

(1) By Fubini's theorem, this linear operator is well defined.
(2) By the monotonicity of integration, such an operator is continuous:

$$\|A_\rho f\|_{L^1(Y)} = \int \left| \int \rho(y|x) f(x)\,\mathrm{d}x \right| \mathrm{d}y \leqslant \int\int \rho(y|x)|f(x)|\,\mathrm{d}x\,\mathrm{d}y = \|f\|_{L^1(X)}.$$

It is also trivial that we can saturate the above inequality by taking a.e. non-negative function f, thus $\|A_\rho\| = 1$ also follows.

Our main interest will be the questions: when is the operator A_ρ invertible, and how can the inverse operator be evaluated on given pdfs in a constructive way?

1.1. A special case: deconvolution problem

A special case of the unfolding problem is the so-called *deconvolution*, i.e. when $Y = X$ and the cpdf ρ is translation invariant in the sense that for all $a \in X$ and for all $y, x \in X : \rho(y|x + a) = \rho(y - a|x)$. In this case, the cpdf ρ can be expressed by a pdf η in the way $\rho(y|x) = \eta(y - x)$ for all $x, y \in X$.

Definition 3. *Let η be a pdf (i.e. it is a non-negative valued Lebesgue integrable function on X such that $\int \eta(x)\,\mathrm{d}x = 1$). Then the linear operator*

$$A_\eta : L^1(X) \to L^1(X), \qquad f \mapsto \eta \star f = \left(y \mapsto \int \eta(y-x) f(x)\,\mathrm{d}x \right).$$

is called the convolution operator by η.

We will state here a few properties of a convolution operator (see e.g. [1, 2]).

(1) A convolution operator is not onto, and its image is not closed.
(2) The range of a convolution operator is dense if and only if the Fourier transform of the convolver function is nowhere zero (*Wiener's approximation theorem*).
(3) A convolution operator is one-to-one if and only if the set of zeros of the Fourier transform of the convolver function has zero Lebesgue measure.

Remark 4. As a consequence, the inverse of a convolution operator—if it exists at all—is *not* continuous. Indeed, the convolution operator is everywhere defined and continuous, so it is closed, thus its inverse is closed as well; since the domain of the inverse is not closed, the inverse cannot be continuous by Banach's closed graph theorem.

We see that the characterization of a convolution operator is strongly related to the Fourier operators:

$$F_\pm : L^1(X) \to C^0_\infty(X^*), \qquad (x \mapsto f(x)) \mapsto \left(y \mapsto \int \mathrm{e}^{\pm \mathrm{i} \langle y|x \rangle} f(x)\,\mathrm{d}x \right).$$

We denote by $C^0_\infty(X^*)$ the space of continuous functions $X^* \to \mathbb{C}$ which have zero limit at the infinity. Here X^* is the dual space of X, and for any $y \in X^*$ and $x \in X$ the number $\langle y|x \rangle$ means the value of the covector y on the vector x.

The Fourier operators have the following basic properties [7]:

(1) $C^0_\infty(X^*)$ is a Banach space with the maximum norm, $F_\pm$ is continuous and $\|F_\pm\| = 1$.
(2) The Fourier operators are one-to-one. Thus, the inverse Fourier operators $F_\pm^{-1}$ exist.
(3) The range of $F_\pm$ is dense in $C^0_\infty(X^*)$, however it is not the whole space. Thus, again by Banach's closed graph theorem, we infer that the operator $F_\pm^{-1}$ is *not* continuous.
(4) If $f, g \in L^1(X)$, then $F_\pm(f \star g) = F_\pm(f) \cdot F_\pm(g)$ (*convolution theorem*).

The naive deconvolution procedure then goes in the following way:

(1) take the Fourier transform of the convolution, $F_\pm(\eta \star f)$,
(2) divide the above function by $F_\pm \eta$,
(3) calculate the inverse Fourier transform;

$$f = F_\pm^{-1} \left(\frac{F_\pm(\eta \star f)}{F_\pm \eta} \right).$$

The listed properties of the convolution operator, however, make it practically impossible to apply the deconvolution procedure in signal processing. The reason is that the measured density function (which is approximated by a normalized histogram in general) is not in the range of the convolution operator: it can be considered as the sum of a pdf in the range of the operator, plus a noise (e.g. Poissonian noise, originating from the statistical fluctuations of the entries in the histogram bins) outside the range of the operator in general. When applying the deconvolution procedure, the inverse operator can be calculated on the first term; however the deconvolution would give a nonsense result on the noise term, as it is not in the range of the convolution operator, thus leading to a nonsense result on the whole. Various noise suppression methods (high frequency cutoffs) are introduced as symptomatic treatment of this problem; however these solutions are based on rather intuitive approaches not on sound mathematics, and are highly non-unique (thus the derived solutions depend on the noise suppression approach). This is because the non-continuity of the inverse of the convolution operator: a small change caused by the high frequency regulation in the Fourier spectrum is not guaranteed to stay small after the deconvolution. This effect, in general, is referred to as follows: the deconvolution problem (or unfolding problem) is ill-posed, i.e. one cannot get a robust method to do the deconvolution (or unfolding). Furthermore, if the Fourier transform of the convolver pdf has zeros in the finite, then the naive deconvolution becomes even more ambiguous: one has to introduce regulation procedures even at certain finite frequencies (at the zeros of the Fourier transform of the convolver pdf).

Despite the above difficulties, we have developed a robust perturbative method which solves the problem. Our method of series expansion gives a robust and stable method for deconvolution. Using this method, the problem of zeros of the Fourier transform of the convolver pdf in the finite does not arise at all; furthermore one does not have to reconsider any high frequency regulations on a case-by-case intuitive basis. Plus, our series expansion is optimal in the sense that it recovers the maximum possible information about the initial pdf even in the case when the convolution in question is not even invertible.

2. Inverse operator by a series expansion

There exists a basic theorem providing a perturbative method to obtain the inverse of continuous linear operators on a Banach space which are not too far from the identity operator. That theorem in its original form, however, does not apply to the case of convolution (or folding) operators. The main result of this paper is a generalization of that theorem to the case of convolution operators.

Now we recall the series expansion (also called Neumann series) for the inverse of an operator.

Let A be a continuous linear operator on a Banach space such that $\|I-A\| < 1$, where I is the identity operator. Then the operator A is one-to-one and onto and its inverse is continuous, and the series $N \mapsto \sum_{n=0}^{N}(I-A)^n$ is absolutely convergent to A^{-1}.

The proof is pretty simple and can be found in any textbooks of functional analysis (e.g. [8, 9]). It will be instructive, however, to cite the proof, as later we will strengthen this theorem.

First, it is easily shown by induction that $\sum_{n=0}^{N}(I-A)^n A = A\sum_{n=0}^{N}(I-A)^n = I-(I-A)^{N+1}$. The condition $\|I-A\| < 1$ guarantees that the sequence $N \mapsto (I-A)^{N+1}$ converges to zero in the operator norm, and the absolute convergence of the series $N \mapsto \sum_{n=0}^{N}(I-A)^n$, thus $\left(\sum_{n=0}^{\infty}(I-A)^n\right)A = A\left(\sum_{n=0}^{\infty}(I-A)^n\right) = I$, i.e. $A^{-1} = \sum_{n=0}^{\infty}(I-A)^n$. As A^{-1} is expressed as a limit of a series of continuous operators which is convergent in the operator norm, we infer that A^{-1} is continuous.

Remark 5. The conditions of the above series expansion theorem fail for any folding operator A_ρ.

(1) We can observe that the series expansion is only meaningful for the case of a folding operator only when the spaces X and Y are the same.
(2) Let us assume that $Y = X$. Then, it is easily obtained that a folding operator A_ρ does not satisfy the required condition $\|I-A_\rho\| < 1$. It is trivial by the triangle inequality of norms that $\|I-A_\rho\| \leqslant 2$. We will show now that this inequality can be saturated for a wide class of cpdfs. Let us choose an arbitrary point $y \in X$, and consider the series

of pdfs $n \mapsto \frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}$, where $K_n(y)$ are compact sets having non-zero Lebesgue measure $\lambda(K_n(y))$, such that $K_{n+1}(y) \subset K_n(y)$ for all $n \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} K_n(y) = \{y\}$. Then,

$$\left\|(I - A_\rho)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}\right\| = \int_{z\notin K_n(y)} \int \rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z$$
$$+ \int_{z\in K_n(y)} \left|\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(z) - \int \rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\right| \mathrm{d}z.$$

By making use of the fact that the integral of any pdf is 1, one can write

$$\int_{z\notin K_n(y)} \int \rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z$$
$$= 1 - \iint \chi_{K_n(y)}(z)\rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z$$

for the first term. For the second term, one can use the monotonity of integration

$$\int_{z\in K_n(y)} \left|\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(z) - \int \rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\right| \mathrm{d}z$$
$$\geqslant \left|\int_{z\in K_n(y)} \left(\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(z) - \int \rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\right) \mathrm{d}z\right|$$
$$= \left|\int \frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(z)\,\mathrm{d}z - \iint \chi_{K_n(y)}(z)\rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z\right|$$
$$= \left|1 - \iint \chi_{K_n(y)}(z)\rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z\right|$$
$$= 1 - \iint \chi_{K_n(y)}(z)\rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z.$$

Here, at the second equality $\int \frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(z)\,\mathrm{d}z = 1$ was used, and the fact that the integral of any pdf over a Borel set is smaller or equal to 1 was used at the third equality. Thus, we infer the inequality

$$\left\|(I - A_\rho)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}\right\| \geqslant 2\cdot\left(1 - \iint \chi_{K_n(y)}(z)\rho(z|x)\frac{1}{\lambda(K_n(y))}\chi_{K_n(y)}(x)\,\mathrm{d}x\,\mathrm{d}z\right).$$

If the point $(y, y) \in X \times X$ is a Lebesgue point of ρ, then we will show that the integral term goes to zero when n goes to infinity, thus saturating our inequality in question. If a function $g : X \to \mathbb{C}$ is locally integrable, then a point $y \in X$ is called a *Lebesgue point of* g if $\lim_{n\to\infty}\frac{1}{\lambda(K_n(y))}\int_{K_n(y)} |g(x) - g(y)|\,\mathrm{d}x = 0$. If $y \in X$ is a Lebesgue point for g, then by the monotonity of integration it also follows that $\lim_{n\to\infty}\frac{1}{\lambda(K_n(y))}\int_{K_n(y)} g(x)\,\mathrm{d}x = g(y)$. Applying this result for ρ on the product space $X \times X$ (assuming that the point $(y, y) \in X \times X$ is a Lebesgue point of ρ), we have that the sequence $n \mapsto \frac{1}{\lambda(K_n(y))}\frac{1}{\lambda(K_n(y))}\int_{K_n(y)}\int_{K_n(y)} \rho(z|x)\,\mathrm{d}x\,\mathrm{d}z$ is convergent to $\rho(y|y)$. Multiplying this sequence by the sequence $n \mapsto \lambda(K_n(y))$ (which is convergent to zero), we infer that $\lim_{n\to\infty}\frac{1}{\lambda(K_n(y))}\int_{K_n(y)}\int_{K_n(y)} \rho(z|x)\,\mathrm{d}x\,\mathrm{d}z = 0$. If ρ is continuous, then every point in $X \times X$ is a Lebesgue point of ρ. Thus, we have shown that if the cpdf ρ is continuous, then $\|I - A_\rho\| = 2$ holds; therefore the original theorem of Neumann cannot be applied directly for a folding operator with continuous cpdf.

Apart from the above remark, the reason is obvious for the obstruction of inverting the convolution on the operator level: as the convolution operators are not onto in general, one only can try to invert the operator on a function in the range of the operator. We try to modify the theorem for the case of convolution operators requiring, instead of convergence in the operator series, the convergence of the series $N \mapsto \sum_{n=0}^{N}(I-A)^n(Af)$ in some sense (equivalently, the convergence of the sequence $N \mapsto (I-A)^{N+1}f$ in the same sense), for any $f \in L^1(X)$.

For getting a convenient result, let us recall that the elements of $L^1(X)$ can be viewed as regular tempered distributions. The Fourier transformations can be extended to the space of tempered distributions, where they are one-to-one and onto, continuous, and their inverse is also continuous [8, 9]. The proof of convergence will be performed on the Fourier transforms of the functions, then the result will be brought back by using the continuity of the inverse Fourier transformation on the space of tempered distributions.

Theorem 6. *Let A_η be a convolution operator for some $\eta \in L^1(X)$. Let Z be the set of zeros of the function $F_\pm\eta$. If the inequality*

$$|1 - F_\pm\eta| < 1$$

is satisfied everywhere outside Z, then for all $f \in L^1(X)$ the series

$$N \mapsto \sum_{n=0}^{N}(I - A_\eta)^n(A_\eta f)$$

is convergent in the space of tempered distributions, and

$$\sum_{n=0}^{\infty}(I - A_\eta)^n(A_\eta f) = f - F_\pm^{-1}(\chi_Z F_\pm f).$$

Proof. Assume that $|1 - F_\pm\eta| < 1$ holds everywhere outside Z. Let V denote the subset of X^*, where $F_\pm\eta$ is nonzero. It is clear that V and Z are disjoint Lebesgue measurable sets and $X^* = V \cup Z$. Trivially, the sequence $N \mapsto |1 - F_\pm\eta|^{N+1}$ converges pointwise to 0 on V, furthermore $|1 - F_\pm\eta|^{N+1} = 1$ on Z for all N. For every $f \in L^1(X)$ and rapidly decreasing test function φ on X^*, we have

$$\begin{aligned}
&\left|\int (1 - F_\pm\eta(y))^{N+1} F_\pm f(y)\cdot\varphi(y)\,\mathrm{d}y - \int \chi_Z \cdot F_\pm f(y)\cdot\varphi(y)\,\mathrm{d}y\right| \\
&\qquad = \left|\int_V (1 - F_\pm\eta(y))^{N+1} F_\pm f(y)\cdot\varphi(y)\,\mathrm{d}y\right| \\
&\qquad \leqslant \int_V |1 - F_\pm\eta(y)|^{N+1}|F_\pm f(y)|\cdot|\varphi(y)|\,\mathrm{d}y.
\end{aligned}$$

The series of Lebesgue integrable functions $N \mapsto |1-F_\pm\eta|^{N+1}|F_\pm f|\cdot|\varphi|$ converges pointwise to zero on V, and $|1 - F_\pm\eta|^{N+1}|F_\pm f|\cdot|\varphi| \leqslant |1 - F_\pm\eta|^1|F_\pm f|\cdot|\varphi|$ for all N; thus by Lebesgue's theorem of dominated convergence the last term of the inequality tends to zero when N goes to infinity. Therefore, the function series $N \mapsto (1-F_\pm\eta)^{N+1}(F_\pm f)$ is convergent in the space of tempered distributions to the function $\chi_Z F_\pm f$. Applying the inverse Fourier transformation $F_\pm^{-1}$ and using the continuity of the inverse Fourier transformation in the space of tempered distributions, we get the desired result, as by the convolution theorem we have $F_\pm^{-1}((1 - F_\pm\eta)^{N+1}(F_\pm f)) = (I - A_\eta)^{N+1}f$, and because

$$f - \sum_{n=0}^{N}(I - A_\eta)^n(A_\eta f) = (I - A_\eta)^{N+1}f$$

for all N. □

Remark 7. Let us assume that the condition of the theorem holds. Then it is quite evident that

(1) If Z has zero Lebesgue measure (which holds if and only if A_η is one-to-one), then $F_\pm^{-1}(\chi_Z F_\pm f) = 0$. This means that the series in question always restores the arbitrarily chosen original function f if and only if A_η is one-to-one, i.e. if and only if $F_\pm \eta$ is a.e. nonzero.
(2) If Z has nonzero Lebesgue measure, our series also converges and restores the maximum possible information about the original function f, namely the tempered distribution $f - F_\pm^{-1}(\chi_Z F_\pm f)$. However, this tempered distribution may not be a function in general. If the function $\chi_Z F_\pm f$ is not a continuous function which tends to zero at the infinity, then $F_\pm^{-1}(\chi_Z F_\pm f)$ cannot be an integrable function. As we shall see in the following section, if the function $\chi_Z F_\pm f$ is not a continuous function which is bounded, then $F_\pm^{-1}(\chi_Z F_\pm f)$ cannot even be a measure with finite variation.
(3) Let now η and f be pdfs and suppose that $F_\pm^{-1}(\chi_Z F_\pm f) = 0$. Then our convergence result has the following meaning in probability theory: the series converges in the sense that the expectation values of all rapidly decreasing test functions on X are restored. Namely, for any rapidly decreasing test function ψ on X we have that

$$\lim_{n\to\infty} \int \left(\sum_{n=0}^{N} (I - A_\eta)^n (A_\eta f) \right)(x) \cdot \psi(x)\,\mathrm{d}x = \int f(x) \cdot \psi(x)\,\mathrm{d}x.$$

It can be easily observed that the condition of our previous theorem is not always satisfied for a pdf η. For example, if η is a Gaussian pdf centred to zero, then it is satisfied, but, e.g., if η is a uniform pdf on a rectangular domain centred to zero, then the condition is not satisfied. Therefore, one could think that the applicability of our deconvolution theorem is rather limited. This is not the case, however, as stated in the following theorem.

Theorem 8. *Let η be a pdf on X. Then for any $f \in L^1(X)$ the series*

$$N \mapsto \sum_{n=0}^{N} (I - A_{P\eta} A_\eta)^n A_{P\eta}(A_\eta f)$$

is convergent in the space of tempered distributions, and

$$\sum_{n=0}^{\infty} (I - A_{P\eta} A_\eta)^n A_{P\eta}(A_\eta f) = f - F_\pm^{-1}(\chi_Z F_\pm f),$$

where $Z := \{y \in X^* | F_\pm \eta(y) = 0\}$. Here P is the parity operator on $L^1(X)$, namely $Pf(x) := f(-x)$ for all $f \in L^1(X)$ and $x \in X$.

Proof. Let us observe that, if $F_\pm \eta$ is real valued and non-negative for a pdf η, then $|1 - F_\pm \eta| < 1$ is automatically satisfied outside Z. This is because

(1) by our assumption $0 < F_\pm \eta$ outside Z, thus we conclude that $1 - F_\pm \eta < 1$ outside Z and
(2) by the inequality $|F_\pm \eta| \leqslant \|\eta\| = 1$, we conclude that $0 \leqslant 1 - |F_\pm \eta| = 1 - F_\pm \eta$.

It is easy to see that $F_\pm P\eta = \overline{F_\pm \eta}$ (where the bar denotes complex conjugation) for a pdf η, because η is real valued. Thus, we have that $F_\pm(P\eta \star \eta) = |F_\pm \eta|^2$ is real valued and non-negative; consequently, by our previous observation, the inequality $|1 - F_\pm(P\eta \star \eta)| < 1$ holds outside Z, i.e. our previous theorem can be applied by replacing the convolution operator A_η with the double convolution operator $A_{P\eta} A_\eta$. □

When applying this theorem in practice, one should take into account that the measured pdf (which is obtained by histogramming in general) is not in the range of the convolution operator, but it can be viewed as the sum of a pdf in the range of the convolution operator (if our model is accurate enough) and a noise term. By the above theorem, the series expansion will be convergent on the pdf in the range of the convolution operator, but will be divergent (most probably) on the noise term, as it is not in the range of the convolution operator (in general). Thus, the problem is that when to stop the series expansion: one should let the series go far enough to restore the original (unknown) pdf, but should stop the series expansion early enough to prevent the divergence arising from the noise term. This truncation procedure can be viewed as a very elegant way to do the high frequency regulation. Note, however, that the regulation problem at the finite frequencies (at the zeros of the Fourier transform of the convolver pdf) does not arise at all, with this method.

The only remaining question is: at which index should one stop to keep the noise content lower than a given threshold?

When working in practice, our density functions are discrete in general (e.g. histograms), thus we may view them as a vector of random variables (e.g. in the case of histogramming, these random variables are the number of entries in the histogram bins). Let us denote it by v. If A is a linear operator (i.e. a matrix here), then we have that $\mathrm{E}(Av) = A\mathrm{E}(v)$ and $\mathrm{Covar}(Av) = A\,\mathrm{Covar}(v)A^{+}$, where we denote expectation value by $\mathrm{E}(\cdot)$, covariance matrix by $\mathrm{Covar}(\cdot)$ and the adjoint matrix by $(\cdot)^{+}$. Thus, in the Nth step of the series expansion, we have

$$\mathrm{Covar}\left(\sum_{n=0}^{N}(I-A_\eta)^n v\right) = \left(\sum_{n=0}^{N}(I-A_\eta)^n\right)\mathrm{Covar}(v)\left(\sum_{n=0}^{N}(I-A_\eta)^n\right)^{+}.$$

This means that if we have an initial estimate for the covariance matrix $\mathrm{Covar}(v)$, we can calculate the covariance matrix at each step, thus can calculate the propagated errors at each order.

When using the method of histograming, as the entries in the histogram bins are known to obey independent Poisson distributions, the initial undistorted estimates $\mathrm{E}(v_i) \approx N_i$ ($i \in \{1, \ldots, M\}$) and $\mathrm{Covar}(v) \approx \mathrm{diag}(N_1, \ldots, N_M)$ will be valid, where we consider our histogram to be a mapping $H : \{1, \ldots, M\} \to \mathbb{N}_0, i \mapsto N_i$. The squared standard deviations are the diagonal elements of the covariance matrix, thus we can have an estimate on the L^1 norm of the noise term at each Nth order by taking $\frac{1}{\sum_{j=1}^{M} N_j}\sum_{i=1}^{M}\sqrt{\mathrm{Covar}_{ii}\left(\sum_{n=0}^{N}(I-A_\eta)^n v\right)}$. By stopping the series expansion when this noise content exceeds a certain predefined threshold, we get the desired truncation of the series expansion.

Remark 9. We show an other (iterative) form of our series expansion which may be more intuitive for physicists. Namely, take the initial conditions

$$f_0 := A_{P\eta}H, \qquad \hat{C}_0 := A_{P\eta}\,\mathrm{diag}(H), \qquad C_0 := \left(A_{P\eta}\hat{C}_0^{+}\right)^{+}.$$

Then, perform the iteration steps

$$f_{N+1} := f_N + f_0 - A_{P\eta}A_\eta f_N,$$
$$\hat{C}_{N+1} := \hat{C}_N + \hat{C}_0 - A_{P\eta}A_\eta\hat{C}_N, \qquad C_{N+1} := \left(\hat{C}_N^{+} + \hat{C}_0^{+} - A_{P\eta}A_\eta\hat{C}_N^{+}\right)^{+}.$$

Here H means the initial (measured) histogram, f_N is the deconvolved histogram at the Nth step and $A_{P\eta}A_\eta$ is the discrete version of the double convolution operator. The quantity $\hat{C}_N$ is a supplementary quantity, and C_N is the covariance matrix at each step. The noise content can be written as $\frac{1}{\sum_{j=1}^{M} N_j}\sum_{i=1}^{M}\sqrt{(C_N)_{ii}}$, which should be kept under a certain predefined threshold.

Remark 10. As pointed out in the previous remark, one can exactly follow the error propagation during the iteration. However, to store and to process the whole covariance matrix can cost a lot of memory and CPU time. Therefore, one may rely on a slightly more pessimistic but less costly approximation of the error propagation, namely on the Gaussian error propagation. This means that at each step one assumes the covariance matrix to be approximately diagonal, i.e. this method is based on the neglection of correlation of entries (which, indeed, holds initially) that slightly will overestimate the error content. Gaussian error propagation means that when calculating the action of the operators in questions, we apply the following two rules:

(1) if v is a random variable (histogram entry) and a is a number, then $\sigma(a \cdot v) := |a| \cdot \sigma(v)$ (this is exact, of course) and
(2) if v_1 and v_2 are random variables (histogram entries), then $\sigma^2(v_1+v_2) := \sigma^2(v_1) + \sigma^2(v_2)$ (which is exact only if v_1 and v_2 are uncorrelated). Here σ means standard deviation.

Remark 11. Even if the convergence condition for the deconvolution by series expansion is satisfied for A_η, it is better to use the double deconvolution procedure by $A_{P\eta}A_\eta$, for the following reason. In practice the measured pdf corresponds to a pdf in the range of A_η plus a noise term. When convolving the measured pdf by $P\eta$ before the iteration, the noise level is reduced by orders of magnitudes (the convolution by $P\eta$ smooths out the statistical fluctuations). As a thumb rule, one iteration step is lost with the convolution by $P\eta$, but several iteration steps are gained, as we start the iteration from a much lower noise level.

3. The general case of unfolding

For the case of general unfolding problems, a series expansion will become even more interesting, as there are no known alternative methods like the naive deconvolution in the case of deconvolution problems.

Unfortunately, for the general case of unfolding, we cannot state such a strong result as for the case of deconvolution. This is because our theorem on the deconvolution strongly relies on the relation of convolutions and Fourier transformation. However, we can state a sufficient condition for the convergence of a series expansion for the general case of unfolding. To state this theorem, we have to perform studies not only on pdfs, but also on probability measures. The spaces X and Y are going to denote finite-dimensional vector spaces again.

A complex measure P on X is a complex valued σ-additive set function defined on the Borel σ-algebra of X. The variation of the complex measure P is the non-negative measure $|P|$ defined as follows: if E is a Borel set, then $|P|(E)$ is the supremum of $\sum_{k=1}^{n} |P(E_k)|$ for all splitting $(E_1, \ldots, E_n)$ of E, i.e. for all such $(E_1, \ldots, E_n)$ finite system of disjoint Borel sets whose union totals up to E [6, 8]. The measures with finite variation (i.e. the complex measures P for which $|P|(X) < \infty$) form a Banach space with the norm being the value of the variation on X, i.e. $\|P\| := |P|(X)$. Let us denote this space by $M(X)$.

Recall that a probability measure P on an X is a non-negative measure on the Borel σ-algebra of X, with $P(X) = 1$. Thus, a probability measure is evidently in $M(X)$.

Definition 12. *We shall call a mapping $Q : X \to M(Y), x \mapsto Q(\cdot|x)$ a folding measure if for every $x \in X$ the measure $Q(\cdot|x)$ is a probability measure on Y, and for every Borel set E in Y the function $x \mapsto Q(E|x)$ is measurable.*

Note, that Q may be viewed as a conditional probability measure on the product space $Y \times X$. Evidently, if ρ is a cpdf, then $Q_\rho(E|x) := \int_E \rho(y|x)\,\mathrm{d}y$ defines a folding measure.

Definition 13. *Let Q be a folding measure Q. Then the linear map*

$$A_Q : M(X) \to M(Y), \qquad P \mapsto \left(\int Q(\cdot|x)\,\mathrm{d}P(x) \right),$$

will be called the <u>*folding operator by Q*</u>.

Remark 14. The following remarks are trivial.

(1) Such an operator is well defined, as for all points $x \in X$ and Borel sets E the inequality $Q(E|x) \leqslant 1$ holds, thus the function $x \mapsto Q(E|x)$ is integrable by any measure with finite variation.
(2) By the monotonicity of integration, such an operator is continuous and $\|A_Q\| = 1$, just as in the L^1 case.
(3) The folding operator defined above can be viewed as a generalization of the folding operator $A_\rho : L^1(X) \to L^1(Y)$ defined by a cpdf ρ. This is because $L^1(X)$ can naturally be embedded into $M(X)$ by assigning to each $f \in L^1(X)$ the measure $E \mapsto P_f(E) := \int_E f(x)\,\mathrm{d}x$. Of course, if the folding measure Q_ρ is defined by a cpdf ρ, then the restriction of A_{Q_ρ} to $L^1(X)$ is just A_ρ as defined before.

First, we generalize our deconvolution results to the space of measures with finite variation.

Remark 15. The convolution of two measures $F, G \in M(X)$ can be defined by

$$F \star G : E \mapsto \int F(E - x)\,\mathrm{d}G(x),$$

where E runs over all the Borel sets. (Of course, $P_f \star P_g = P_{f\star g}$ for any $f, g \in L^1(X)$.)

The Fourier transformations can also be defined on $M(X)$, and have the same properties as in the L^1 case, except that the Riemann–Lebesgue lemma does not hold (i.e. the Fourier transform of a measure is a bounded continuous function but does not tend to zero at the infinity). Therefore, our previous results on the series expansion for the deconvolution (theorem 8) can directly be generalized to the probability measures, as the elements of $M(X)$ can also be viewed as tempered distributions.

As we remarked above for the deconvolution case, we have a powerful result also in the more general framework of measures with finite variation. However, we are still lacking an answer for the general cases of unfolding.

Remark 16. The conditions of the original Neumann series expansion theorem fail also in the case of measures.

(1) We can observe that our series expansion is only meaningful for the case of a folding operator only when the spaces X and Y are the same. (Just as in the L^1 case.)
(2) Let us assume that $Y = X$. Then, it is easily obtained that a folding operator A_Q does not satisfy the required condition $\|I - A_Q\| < 1$, in general. It is trivial by the triangle inequality of norms that $\|I - A_Q\| \leqslant 2$. We will show now that this inequality can be saturated for a wide class of folding measures. Let $K_n(y)$ ($n \in \mathbb{N}$) be a sequence of compact sets with nonzero Lebesgue measure, such that $K_{n+1}(y) \subset K_n(y)$ for each $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} K_n(y) = \{y\}$. Let us denote the complement of a set $K_n(y)$ by $K_n^{\complement}(y)$. Clearly, by considering the splitting $\big(K_n(y), K_n^{\complement}(y)\big)$ of the Borel set X, one has

$$\begin{aligned}|(I - A_Q)\delta_y|(X) &\geqslant |\delta_y(K_n(y)) - Q(K_n(y)|y)| + \left|\delta_y\big(K_n^{\complement}(y)\big) - Q\big(K_n^{\complement}(y)\big|y\big)\right| \\ &= |1 - Q(K_n(y)|y)| + Q\big(K_n^{\complement}(y)|y\big).\end{aligned}$$

At the equality, $\delta_y(K_n(y)) = 1$ and $\delta_y\big(K_n^{\complement}(y)\big) = 0$ was used. Let us take the limit $n \to \infty$ on the right-hand side. By the monotone continuity of measures, we have that $\lim_{n\to\infty} Q(K_n(y)|y) = Q(\{y\}|y)$ and $\lim_{n\to\infty} Q\big(K_n^{\complement}(y)\big|y\big) = Q(X\backslash\{y\}|y)$; furthermore by the subtractivity of measures we have $Q(X \setminus \{y\}|y) = Q(X|y) - Q(\{y\}|y)$. As $Q(\cdot|y)$ is a probability measure, we also have $Q(X|y) = 1$. Thus,

$$|(I - A_Q)\delta_y|(X) \geqslant |1 - Q(\{y\}|y)| + (1 - Q(\{y\}|y)).$$

As the measure $Q(\cdot|y)$ cannot take up larger values then 1 on any Borel set, we conclude that

$$\|I - A_Q\| \geqslant 2 \cdot (1 - Q(\{y\}|y)).$$

Thus, if there exists such a point $y \in X$, where $Q(\{y\}|y) = 0$, then $\|I - A_Q\| = 2$. When the folding measure Q_ρ is defined by a cpdf ρ, then $Q_\rho(\{y\}|y) = 0$ always holds (this is because a measure of the form P_f—for any function $f \in L^1(X)$—cannot have sharp points, i.e. such points where $P_f(\{y\}) \neq 0$). Thus, $\big\|I - A_{Q_\rho}\big\| = 2$ holds for any cpdf ρ, therefore the Neumann series cannot converge for A_{Q_ρ} in the $M(X)$ operator norm. (But of course, even $Q(\{y\}|y) \leqslant \frac{1}{2}$ is enough to violate $\|I - A_Q\| < 1$.)

Just like in the convolution case, our strategy will be to require much weaker notions of convergence. By intuition, one would think that if for all $x \in X$ the Dirac-measures δ_x are restored by the method (in some sense of convergence), then this would be enough for the restoration of any other arbitrary measures with finite variation. We provide a similar result with slightly stronger conditions. The theorem below is a trivial consequence of Lebesgue's theorem of dominated convergence.

Theorem 17. *Let A_Q be a folding operator for some folding measure Q. Let us fix a Borel set E in X. If for all $x \in X$ the sequence*

$$N \mapsto ((I - A_Q)^{N+1}\delta_x)(E)$$

converges to zero, furthermore

$$\sup_{N\in\mathbb{N}} \sup_{x\in X} |((I - A_Q)^{N+1}\delta_x)(E)| < \infty$$

holds, then for any $P \in M(X)$ the series

$$N \mapsto \left(\sum_{n=0}^{N}(I - A_Q)^n A_Q P\right)(E)$$

is convergent and

$$\left(\sum_{n=0}^{\infty}(I - A_Q)^n A_Q P\right)(E) = P(E).$$

Proof. First, we note that for any index N the measurable function $x \mapsto |((I - A_Q)^{N+1}\delta_x)(E)|$ can be bounded by 2^{N+1}, thus these functions are integrable by any measure with finite variation.

We know that for all $x \in X$ the relation $\lim_{N\to\infty}((I - A_Q)^{N+1}\delta_x)(E) = 0$ holds, furthermore $\sup_{N\in\mathbb{N}} \sup_{x\in X}|((I - A_Q)^{N+1}\delta_x)(E)| < \infty$. The integral $\int((I - A_Q)^{N+1}\delta_x)(E)\,\mathrm{d}P(x)$ exists for all N and the integrands converge pointwise to zero as N tends to infinity. As the integrands are dominated by a constant independent of N which is clearly $|P|$-integrable, by Lebesgue's theorem of dominated convergence, the limit and the integration can be interchanged: $\lim_{N\to\infty}\int((I-A_Q)^{N+1}\delta_x)(E)\,\mathrm{d}P(x)=\int\lim_{N\to\infty}((I-A_Q)^{N+1}\delta_x)(E)\,\mathrm{d}P(x)=0$. On the left-hand side, $(I-A_Q)$ can be interchanged with the integration, because I is the identity operator and because A_Q itself is an integral: we can interchange the integrals by Fubini's theorem, namely $\int\big(A_Q^N\delta_x\big)(E)\,\mathrm{d}P(x)=\int\int\cdots\int Q(E|y_N)\,\mathrm{d}Q(y_N|y_{N-1})\cdots\mathrm{d}Q(y_1|x)\,\mathrm{d}P(x)=\big(A_Q^N P\big)(E)$, for arbitrary power N. Thus, $\lim_{N\to\infty}((I-A_Q)^{N+1}P)(E)=0$.

Using the equality $\big(P-\sum_{n=0}^{N}(I-A_Q)^n A_Q P\big)(E)=((I-A_Q)^{N+1}P)(E)$, we get the desired result. □

Remark 18. Assume that the condition of our theorem holds.

(1) The condition $\sup_{N\in\mathbb{N}}\sup_{x\in X}|((I-A_Q)^{N+1}\delta_x)(E)|<\infty$ (i.e. the condition of boundedness) is crucial for the proof in order to be able to interchange the limit and the integration. In other words: the restoration of the Dirac-measures δ_x for all $x\in X$ is not enough.

(2) If P is a probability measure, then the meaning of our convergence result is that the probability of the event (Borel set) E is restored:

$$\lim_{N\to\infty}\left(\sum_{n=0}^{N}(I-A_Q)^n A_Q P\right)(E)=P(E).$$

The present theorem is weaker than that for deconvolution, nevertheless it provides a computer-testable condition of convergence for any unfolding problem (which may not be expressed as convolution). In the following section, we shall provide some physical examples which show the method in operation. Of course, the iteration procedure goes just the same as discussed at the end of the previous section.

Remark 19. If we are testing the convergence criterion by computer, some measure theory trivialities are useful. Namely, if the condition holds for disjoint sets, then it also holds for the union of them. Thus, in practice (e.g. when handling histograms), it is enough to confirm the condition when the Borel sets E are the histogram bins, because then the condition will automatically hold for any set built up from the histogram bins. Of course, we cannot go below the granulation of our histogram binning, but if our granulation is fine enough, the numerical test of convergence condition can give an accurate answer.

The disadvantage of our presented convergence criterion is that it is rather expensive even for a simple one-dimensional case (however, for a given folding measure Q, this condition has to be shown only once). It may be better to only show the convergence for the *given* unfolding problem, i.e. on a case-by-case basis, and not for the general case of every $P\in M(X)$. (The disadvantage of such a convergence condition is that surely it will be violated after a certain iteration step, because of the divergence arising from the noise term.) Such a condition of convergence may be obtained by Cauchy's root criterion.

Theorem 20. *Let A_Q be a folding operator for some folding measure Q. Let us fix a measure $P\in M(X)$ and a Borel set E in X. If the inequality*

$$\limsup_{N}\sqrt[N]{|((I-A_Q)^N A_Q P)(E)|}<1$$

holds, then the series

$$N\mapsto\left(\sum_{n=0}^{N}(I-A_Q)^n A_Q P\right)(E)$$

is absolute convergent.

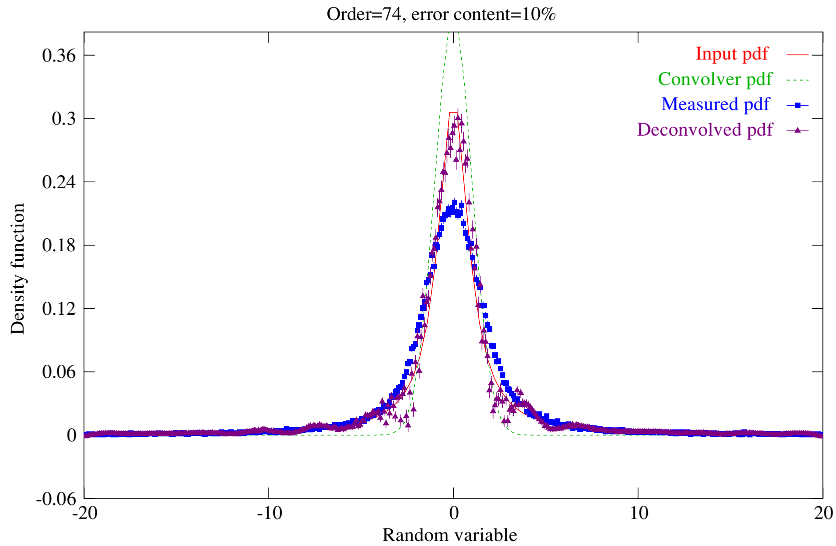

Figure 1. A Gauss⋆ Cauchy deconvolution by series expansion.

With the above condition one may control the convergence of the series iteration for a given measured pdf: the condition $\limsup_N \sup_E \sqrt[N]{|((I - A_Q)^N A_Q P)(E)|} < 1$ may be required as a condition of convergence, where the Borel sets E are the histogram bins. Given the order N, we shall call the number $\sup_E \sqrt[N]{|((I - A_Q)^N A_Q P)(E)|}$ the Cauchy index.

Remark 21. The iteration scheme is the same as discussed at the end of the previous section (*remark 9*). In the iteration scheme, the convolution operator A_{P_η} should be replaced by some folding operator A_G (used to artificially smear the measured histogram in order to reduce the noise content, as pointed out in remark 11—typically this may be chosen to be a convolution operator by a Gauss pdf centred to zero, or can be chosen to be the identity operator, if smoothing is not needed), and the convolution operator A_η should be replaced by the folding operator A_Q (describing the physical smearing process).

4. Examples and applications in physics

Our first test example will be a deconvolution problem of an initial Cauchy pdf of the form $x \mapsto \frac{1}{\pi} \cdot \frac{1}{\Gamma^2+x^2}$ and with a Gauss convolver pdf of the form $x \mapsto \frac{1}{\sqrt{2\pi \cdot \sigma^2}} \cdot \exp\left(-\frac{x^2}{2\cdot\sigma^2}\right)$ over the real numbers. We will choose $\Gamma = 1$ and $\sigma = 1$ in our example. By theorem 8 we can assure the convergence of the problem. The result is shown in figure 1.

Our second test example will be a deconvolution problem of an initial Cauchy pdf as in the previous example with a triangle convolver pdf of the form $x \mapsto \frac{1}{W^2} \cdot \chi_{[-W,W]}(x) \cdot |W - |x||$ over the real numbers. We will choose $W = 2$ in our example. By theorem 8 we can also assure the convergence of the problem. The result is shown in figure 2.

A signal smearing, caused by a measurement apparatus, is described by folding in general. In this case the cpdf in the folding integral is the response function of the device. Our series unfolding can be applied to remove the non-ideal detector smearing at the spectrum level.

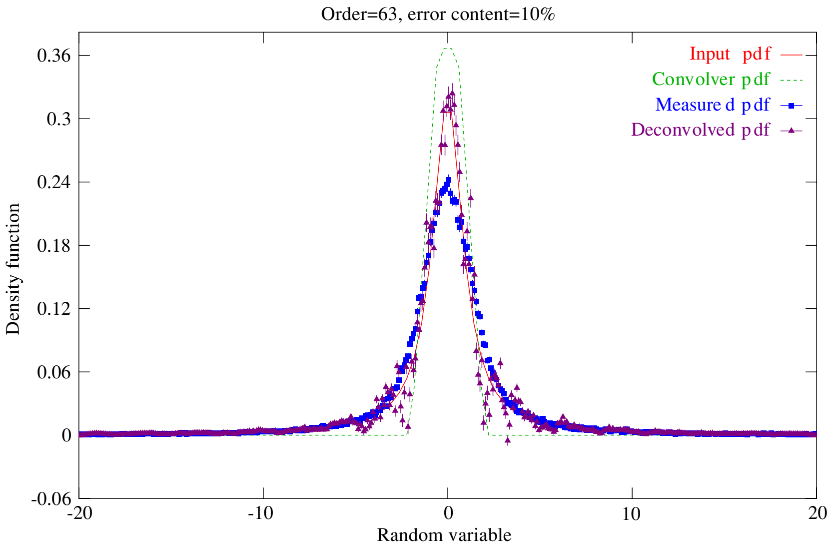

Figure 2. A triangle⋆ Cauchy deconvolution by series expansion.

This is a common issue in analysis of recorded data in experimental physics, which may be solved by our method.

Our physical example will be the π^0 decay. π^0-s are produced in high-energy particle collisions (e.g. in hadron or heavy-ion collisions). The particle π^0 decays through the channel $\pi^0 \to \gamma + \gamma$ decay (98.798% branching ratio). It has such a short lifetime (8.4×10^{-17} s) that even in the highest energy colliders it only travels at most micrometres before decay, thus from the detector's point of view, the resulting γ photons come from the collision point. The π^0 particles are detected via the resulting γ photon pairs. This is possible because the dominant part of the γ yield comes from π^0 decays in hadron or heavy-ion collisions. The γ candidate signals are paired to each other in every possible combination, and the mass of each pair is calculated from the hypothesis that they originate from a common π^0 decay. The combinatorial background is estimated by so-called event mixing techniques (by taking γ candidates from different events, thus these signals are completely independent). The π^0 yield as a function of momentum thus can be obtained, which plays an important role in high-energy particle physics.

However, in certain cases (e.g. in heavy-ion collisions) the reconstruction efficiency of π^0-s can be very low at certain momentum space regions; thus this straightforward reconstruction method is not always applicable for measuring the momentum distribution of the produced π^0-s.

A possible idea is to measure the single γ momentum distribution, and reconstruct the parent π^0 momentum distribution from it, somehow. The arising of the child γ photon momentum pdf from a parent π^0 momentum pdf is described by a folding, as will be discussed below. The task is: to unfold the original π^0 momentum pdf from the γ momentum pdf. This issue was also addressed in [4], however the answer given by the paper was not fully satisfactory. Firstly, the method described in the paper was very specific to the particular case of $\pi^0 \to \gamma + \gamma$ decay (and did not deal with the general problem of unfolding). Secondly, two kinematical kind of approximations were used which are mathematically ill-defined and have an unclear physical meaning. It seems, indeed, that our method gives a more realistic answer, as it will be shown.

Let us denote the momentum space by $(\mathbb{M}, g)$, where $\mathbb{M}$ is a four-dimensional real vector space and $g : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is a Lorentz form (with signature $1, -1, -1, -1$). Let us choose a time orientation on it. Let $V^+(0)$ denote the positive null cone (positive light cone), and let $V^+(m)$ be the positive mass shell with mass value m (m will now play the role of π^0 mass). The π^0 momentum pdf is defined over $V^+(m)$, and the γ photon momentum pdf is defined over $V^+(0)$. However, they also can be viewed as probability measures over $\mathbb{M}$, with their support in $V^+(m)$ and $V^+(0)$, respectively. Given a π^0 momentum, the γ momenta directions (decay axes) are uniformly distributed in the π^0 rest frame (this is the physical information put in). Namely, let us take the set

$$F := \left\{ (p, k) \in \mathbb{M} \times \mathbb{M} \;\middle|\; p \in V^+(m), k \in V^+(0), g\left(\frac{1}{\sqrt{g(p,p)}} p, k \right) = \frac{m}{2} \right\},$$

and let us define for every $p \in \mathbb{M}$ the set $F_p := \{k \in \mathbb{M} \mid (p, k) \in F\}$. Clearly, F_p is the set of possible γ photon momenta arising from a π^0 with momentum p (in other words: F_p is defined by the vectors in $V^+(0)$ which have energy $\frac{m}{2}$ in the rest frame of the π^0 with momentum p). We shall define our folding measure by the following: $Q(\cdot|p)$ is the measure over $\mathbb{M}$ for each p which describes the uniform distribution on F_p (as F_p is compact, it has finite measure, thus this is meaningful). If P is a probability measure over $\mathbb{M}$ describing the π^0 momentum distribution, then the γ photon momentum distribution is defined by the probability measure $A_Q P$. Thus, one may try to obtain the parent π^0 momentum distribution by unfolding the measured γ momentum distribution. This will be done explicitly below for a toy example.

Let us parameterize the momentum space with respect to an Einstein synchronized frame (e_0, e_1, e_2, e_3) that corresponds to the centre-of-mass system of the collision. We choose the collision axis (the beam axis) to be the third spatial coordinate axis which we also call the longitudinal direction. As the experimental setups of collisions are axially symmetric with respect to this axis, the single-particle momentum distributions are axially symmetric with respect to the longitudinal direction. Therefore, it is convenient to parameterize a π^0 momentum $p \in V^+(m)$ in the form $\left(g(e_3, p), \sqrt{g(e_1, p)^2 + g(e_2, p)^2}, \arctan\left(\frac{g(e_2,p)}{g(e_1,p)}\right)\right)$, and a γ momentum $k \in V^+(0)$ in the form $\left(g(e_3, k), \sqrt{g(e_1, k)^2 + g(e_2, k)^2}, \arctan\left(\frac{g(e_2,k)}{g(e_1,k)}\right)\right)$. The three coordinates are called longitudinal momentum, transverse momentum and azimuth, respectively. The axial symmetry means that the pdfs describing π^0 and γ momentum distributions only depend on the longitudinal and transverse momenta.

It is even more convenient to introduce a more sophisticated parameterization: if p_L is the longitudinal momentum and p_T is the transverse momentum, then $y := \operatorname{asinh}\left(\frac{p_L}{\sqrt{m^2+p_T^2}}\right)$ (longitudinal rapidity) and $E_T := \sqrt{m^2 + p_T^2}$ (transverse energy) can be introduced. The so-called longitudinal pseudorapidity $\eta := \operatorname{asinh}\left(\frac{p_L}{p_T}\right)$ is also useful for longitudinal parameterization. We shall present the pdfs in the (η, p_T) parameterization.

For demonstration, we take a realistic toy example of π^0 momentum pdf. The π^0 momentum pdf is characterized by as follows: the momentum pdf of the π^0 with respect to the Lorentz invariant measure of the mass shell $V^+(m)$ corresponds to a product of a Gaussian one in y and an exponential one in E_T (a typical experimental spectrum can be qualitatively described in this a way). The standard deviation of the y distribution was taken to be 0.5, and the inverse slope parameter of the E_T distribution was taken to be 0.5 GeV.

The initial π^0 momentum pdf is presented in figure 3 together with the arising γ momentum pdf. We used a sample of 10 000 000 Monte Carlo π^0 particles to generate the measured γ spectrum.

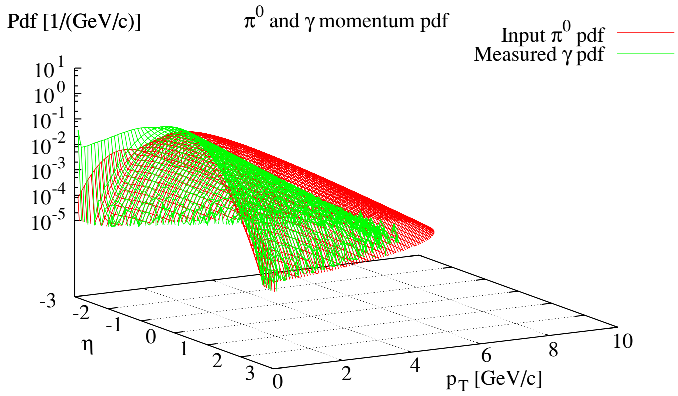

Figure 3. Input π^0 momentum pdf and measured γ momentum pdf.

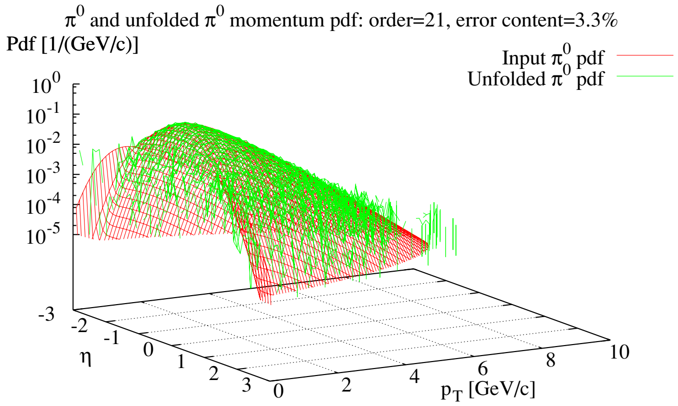

Figure 4. Input π^0 momentum pdf and unfolded π^0 momentum pdf.

The unfolded π^0 momentum pdf is presented in figure 4 together with the initial π^0 momentum pdf. Due to the high statistics, we did not apply smearing for noise reduction (as discussed in remark 21).

To demonstrate the capability of the method, we also included a smearing according to the CMS-ECAL detector's known energy and angular resolution function, when generating the measured gamma responses: the method also removes this detector effect from the momentum pdf. This fact is rather important in practice, because a non-ideal detector resolution changes the inverse slope parameter of the transverse momentum spectrum remarkably, which is used in heavy-ion physics to determine the temperature of the collided system.

Some sections of the previous pdfs are also presented at $\eta =$ constant slices in figures 5 and 6.

For completeness, we also show the answer given by Cahn's prescription (as described in [4]), in figures 7 and 8. Of course, here we did not include additional detector effects as in our unfolding case, as Cahn's method was not designed to undo detector effects. As one can see, the reconstructed π^0 momentum pdf given by Cahn's prescription is rather far from the

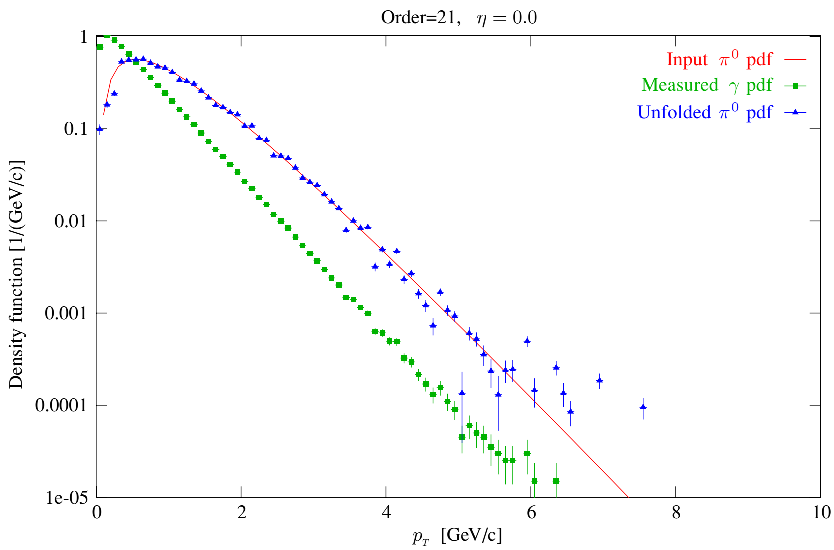

Figure 5. Input π^0 momentum pdf, measured γ momentum pdf, and reconstructed π^0 momentum pdf: taken at the $\eta = 0.0$ slice.

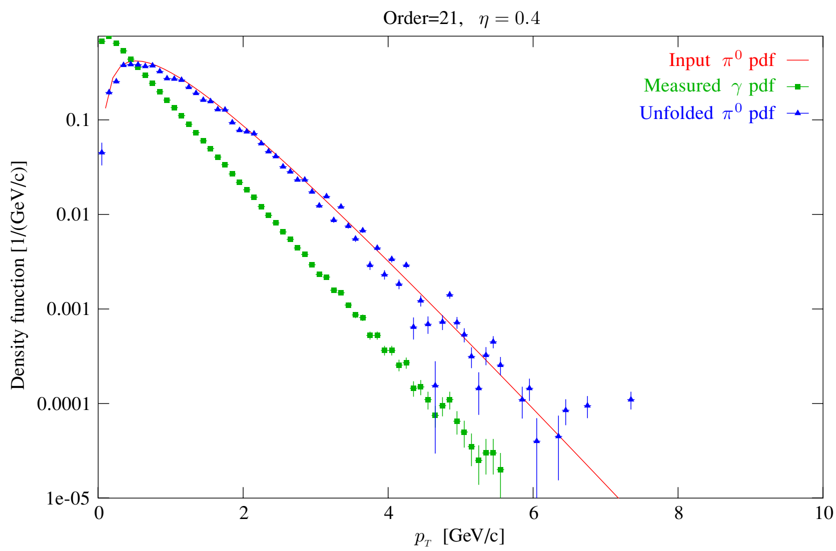

Figure 6. Input π^0 momentum pdf, measured γ momentum pdf, and reconstructed π^0 momentum pdf: taken at the $\eta = 0.4$ slice.

initial one, especially when compared to the answer given by our series expansion method, introduced in this paper.

Our remaining issue is to show the convergence of our series expansion for this $\pi^0 \to \gamma + \gamma$ decay unfolding problem. In figure 9 we plotted the Cauchy index as a function of the iteration

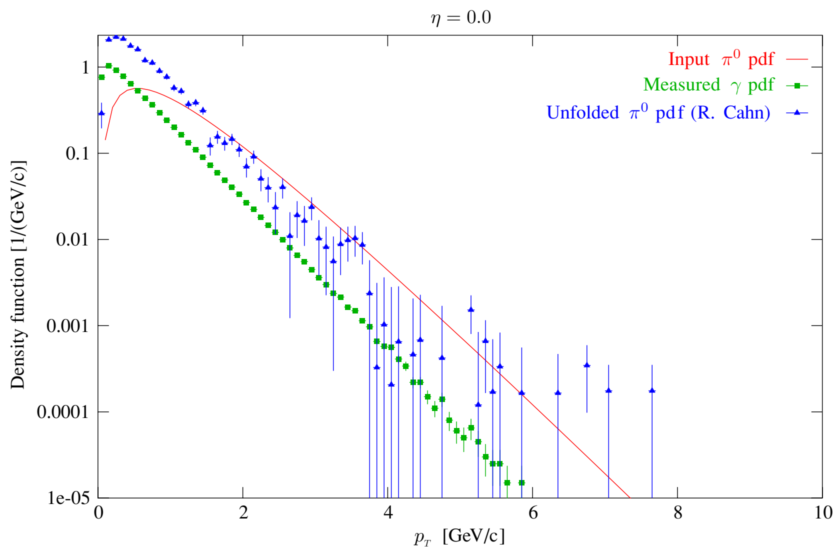

Figure 7. Input π^0 momentum pdf, measured γ momentum pdf, and reconstructed π^0 momentum pdf with R. Cahn's method: taken at the $\eta = 0.0$ slice.

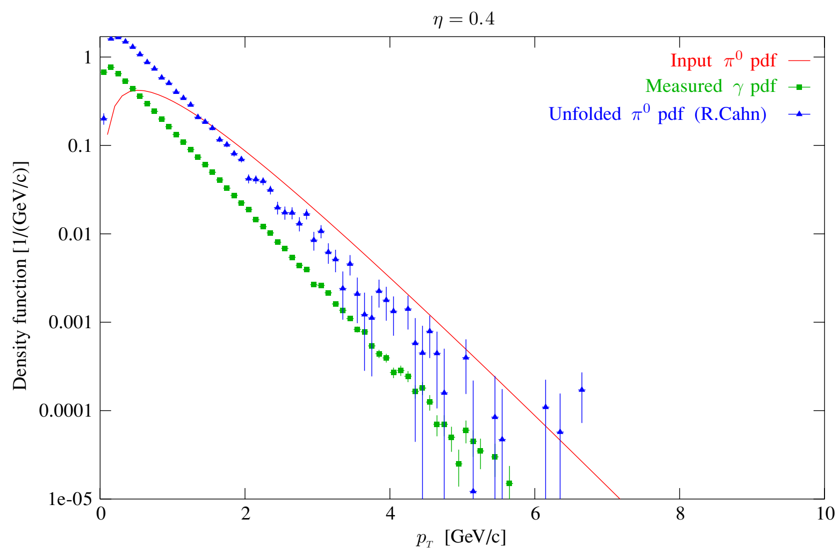

Figure 8. Input π^0 momentum pdf, measured γ momentum pdf, and reconstructed π^0 momentum pdf with R. Cahn's method: taken at the $\eta = 0.4$ slice.

order. It is clearly seen that the Cauchy indices are saturating to ≈ 0.8, thus the convergence is a consequence of theorem 20.

Remark 22. It is very important to note that when implementing the folding operator, one does not have to know the analytic form of the integral. In the $\pi^0 \to \gamma + \gamma$ case it is possible to calculate the integral formula analytically from kinematics, however, the integral becomes

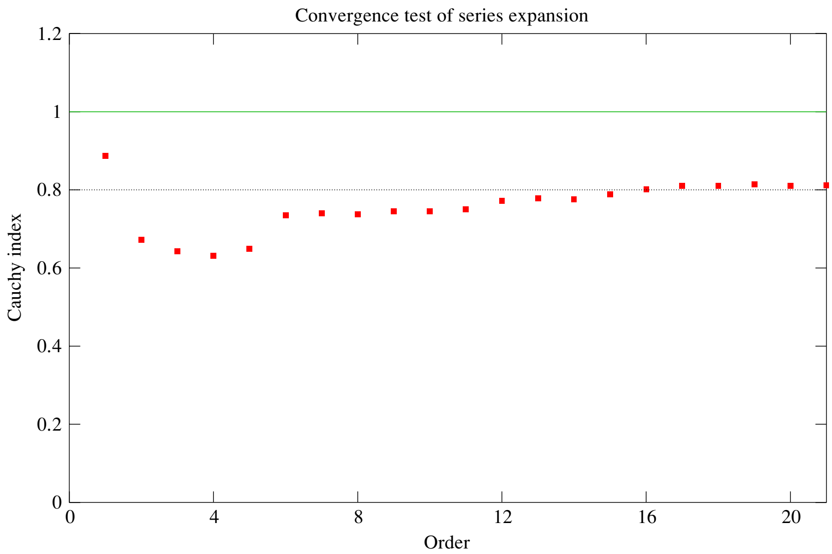

Figure 9. Cauchy convergence test of the series expansion for the $\pi^0 \to \gamma + \gamma$ problem.

very ugly in the (η, p_T) parameterization. Therefore we calculated the action of the folding operator by Monte Carlo simulation, which makes the method easy to implement.

5. Concluding remarks

A robust iterative deconvolution and unfolding method was developed for applications in signal processing. The method has three main advantages:

(1) it solves any deconvolution problem optimally,
(2) it also solves a wide class of more general unfolding problems (for which no general unbiased method was known previously) and
(3) the method is quite easy to implement even for sophisticated folding problems, if Monte Carlo integration method is applied.

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