AN ITERATION FORMULA FOR FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND.*

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1. Introduction. Neumann's Method of solving Fredholm integral equations of the *second* kind by iteration is of great practical and theoretical value. For Fredholm integral equations of the *first* kind, on the other hand, Hellinger and Toeplitz [1] remark that a method of solution by iteration is not available.

Physical problems often lead to an integral equation of the first kind to which a good first approximation may be derived by physical reasoning. An example of this is the problem of determining an axial source-sink or doublet distribution which would yield the axially-symmetric potential flow about a body of revolution in a uniform stream. This problem leads to an integral equation of the *first* kind, $\frac{1}{2} = \int_0^1 [(x-t)^2 + y(x)^2]^{-3/2} m(t)\,dt$, where the axis of the body coincides with the x-axis from $x=0$ to $x=1$, $y(x)$ is a known function, representing the ordinates of the intersection of the given surface with a meridian plane and $m(x)$ is an unknown function, representing the distribution of the doublet strength per unit length along the axis. A well-known, excellent, first approximation to the doublet distribution for elongated bodies of revolution is [2] $m_0(x) = [y(x)]^2/4$. In cases such as this it would be highly desirable to have a method of successive approximations for improving upon this approximation.

The theories of Schmidt and Picard furnish expressions for solutions to integral equations of the first kind. However, these expressions are of little practical value since they involve the characteristic numbers and functions of an arbitrary kernel, and the methods for obtaining these are both tedious and approximate.

It is proposed to present an iteration formula for obtaining successive approximations to the solution of Fredholm integral equations of the *first* kind, and to prove the convergence of the successive approximations under various conditions.

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2. The integral equation of the first kind; theory of Schmidt and Picard. We are concerned with solutions and approximations to solutions of the integral equation of the first kind

$$f(x) = \int_a^b k(x, y) g(y) dy, \tag{1}$$

where $f(x)$ and $k(x, y)$ are given continuous real functions in $a \leqq x, y \leqq b$, and $g(y)$ is an unknown function. As is well known, (1) may be transformed into the integral equation with a symmetric kernel,

$$F(x) = \int_a^b K(x, y) g(y) dy, \tag{2}$$

where

$$K(x, y) = \int_a^b k(t, x) k(t, y) dt \tag{3}$$

and

$$F(x) = \int_a^b k(y, x) f(y) dy. \tag{4}$$

A theory due to E. Schmidt [3] shows that there exists a set $\{\lambda_i\}$ of positive characteristic numbers, which may be supposed arranged in increasing order of magnitude, and corresponding adjoint sets $\phi_i(x)$ and $\psi_i(x)$ of real, continuous, orthonormalized characteristic functions, $(i = 1, 2, \cdots)$, such that

$$\phi_i(x) = \lambda_i \int_a^b k(x, y) \psi_i(y) dy, \qquad \psi_i(x) = \lambda_i \int_a^b k(y, x) \phi_i(y) dy. \tag{5}$$

It will be convenient, hereafter, to employ the customary operator notation for integral transforms, viz., $kg \equiv \int_a^b k(x, y)g(y)dy$, $Kg \equiv \int_a^b K(x, y)g(y)dy$; furthermore, since the range of variation and the integration limits will always be from a to b, specific reference to these limits will be omitted and we will frequently write integrals in an abbreviated form, viz., $\int_a^b f(x)\phi_i(x)dx \equiv \int f\phi_i$.

If the kernel $k(x, y)$ is degenerate, the number of characteristic functions is finite and they can be found by a well known procedure [4]. If $f(x)$ is expressible in the form $f(x) = \sum_{i=1}^n a_i\phi_i(x)$, the solution of (1) is

$$g(x) = \sum_{i=1}^n \lambda_i a_i \psi_i(x), \qquad a_i = \int f\phi_i. \tag{6}$$

If $f(x)$ is not of the above form, then (6) gives the best approximate solution of (1) in the least square sense, as can easily be shown. If the kernel $k(x, y)$

is non-degenerate, the sets $\{\lambda_i\}$, $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ are infinite. Since the degenerate case is readily disposed of, only the non-degenerate case will be considered hereafter.

These characteristic numbers and adjoint functions have several properties which will be required in the following:

a) λ_i^2 and $\psi_i(x)$ are characteristic numbers and functions of $K(x, y)$ [5], i. e.,

$$\psi_i = \lambda_i^2 K \psi_i. \tag{7}$$

b) A positive lower bound for the set $\{\lambda_i\}$ is given by the inequality [3]

$$1/\lambda_1^2 < \int\int k^2(x, y)\, dxdy. \tag{8}$$

c) Expansion Theorems. Every function $f(x)$ of the form (1), where $g(y)$ is any piecewise-continuous function, can be expanded in the absolutely and uniformly convergent series [5]

$$f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x); \qquad a_i = \int f\phi_i = (\int g\psi_i)/\lambda_i. \tag{9}$$

Every function $F(x)$ of the form (4), where $f(x)$ is any piecewise-continuous function, can be expanded in the absolutely and uniformly convergent series

$$F(x) = \sum_{i=1}^{\infty} c_i \psi_i(x); \qquad c_i = \int F\psi_i = (\int f\phi_i)/\lambda_i. \tag{10}$$

If f is the same function in (9) and (10), the relations between the "Fourier" coefficients may be written

$$c_i = \int F\psi_i = (\int f\phi_i)/\lambda_i = (\int g\psi_i)/\lambda_i^2. \tag{11}$$

In general a solution of (1) does not exist. A theorem due to E. Picard [6] states that, *if the orthogonal set ϕ_i is complete, a solution of the integral equation* (1) *exists if and only if the series*

$$\sum_{i=1}^{\infty} \lambda_i^2 a_i^2, \; a_i = \int f\phi_i \tag{12}$$

is convergent.

In the Schmidt-Picard theory, the solution of (1) is intimately related to the sequence

$$\bar{g}_n = \sum_{i=1}^{n} \lambda_i a_i \psi_i(x), \qquad n = 1, 2, \cdots \tag{13}$$

as is expressed in the following theorems:

THEOREM 1. *The sequence* $\{k\bar{g}_n\}$ *converges in the mean to* $f(x)$ *if and only if the set* $\{\phi_i\}$ *is complete relative to* $f(x)$. *The sequence converges uniformly to* $f(x)$, *if a piecewise-continuous solution of the integral equation* (1) *exists.*

THEOREM 2. *If a piecewise-continuous solution* $g(x)$ *of* (1) *exists, the sequence* $\{\bar{g}_n\}$ *converges in the mean to* $g(x)$ *if and only if the set* $\{\psi_i\}$ *is complete relative to* $g(x)$. *If* $g(x)$ *is of the form* $\int k(y,x)h(y)dy$, *where* $h(y)$ *is any piecewise-continuous function, then the sequence* $\bar{g}_n$ *converges uniformly to* $g(x)$.

The completeness conditions on the sequences $\{\phi_i\}$ and $\{\psi_i\}$ in Theorems 1 and 2 refer to the so-called completeness relations

$$\int f^2 = \sum_{i=1}^{\infty} a_i^2, \quad a_i = \int f\phi_i \quad \text{and} \quad \int g^2 = \sum_{i=1}^{\infty} b_i^2, \quad b_i = \int g\psi_i. \tag{14}$$

The phrase "complete relative to $f(x)$" in Theorem 1 signifies that (14) need be satisfied only by the particular function $f(x)$, a condition which is considerably weaker than the assumption that the set $\{\phi_i\}$ is complete relative to a class of functions. Similarly (14) is assumed to apply only to the particular function $g(x)$ in Theorem 2.

The first part of Theorem 1 is of especial interest since it indicates that with increasing n, the error due to the assumption of $\bar{g}_n(x)$ as an approximate solution of (1) diminishes in a least square sense, even if a solution of (1) does not exist. However the disagreeable possibility exists that, beyond some value of n, the error may accumulate and increase at some values of x. Nevertheless, even in this case, such a sequence may give useful successive approximations in a particular problem, if the errors are observed at each step, and the approximations stopped when the error exceeds an acceptable value at any point.

The second part of Theorem 1 asserts that, for sufficiently large n, $\bar{g}_n$ satisfies the integral equation (1) as closely as desired. It is noteworthy that no assumptions are made with regard to the convergence of the sequence $\{\bar{g}_n\}$. Indeed, Theorem 2 shows that an additional condition is necessary to assure even convergence in the mean.

The expression (13) for $\bar{g}_n$, however, is of little practical value since it is expressed in terms of the characteristic numbers and functions of the kernel $k(x,y)$. Principally for these reasons the Fredholm integral equation of the first kind has been considered to be of little value [7]. On the other hand, another readily calculable sequence of functions $\{g_n(x)\}$ will be defined, which, it will be shown, has properties relative to a solution of the integral equation (1) identical to those of $\bar{g}_n(x)$.

3. The iteration formula. Let us now extend the operator notation, denoting $K^r g \equiv \int \cdots \int K(x, y_r) K(y_r, y_{r-1}) \cdots K(y_2 y_1) g(y_1) \, dy_r dy_{r-1} \cdots dy_1$. This notation is appropriate since the relation $K^r(K^s g) \equiv K^{r+s} g$ is satisfied, as is easily verified.

Let $g_0(x)$ be an assumed, approximate, piecewise-continuous solution of the integral equation (1). Then a set of continuous functions $g_1(x), g_2(x), \cdots$ is defined by the iteration formula

$$g_n = g_{n-1} + F - K g_{n-1} \tag{15}$$

where K and F are the functions defined in equations (3) and (4). The convergence of this sequence of functions and the applicability of its members as successive approximations to a solution of the integral equation (1) is the subject of the subsequent discussion.

The recurrence formula (15) can be readily solved for g_n in terms of g_0. First put

$$\gamma_n = g_n - g_{n-1}. \tag{16}$$

Then

$$g_n = g_0 + \sum_{i=1}^{n} \gamma_i \tag{17}$$

and also (15) may be written as

$$\gamma_n = F - K g_{n-1}. \tag{18}$$

Thus the γ_n's are not only the differences between successive g_n's but also serve as measures of the errors corresponding to the g_n's as approximate solutions of the iterated integral equation (2). Now, from (18), we have $\gamma_n - \gamma_{n-1} = -K\gamma_{n-1}$ or, in operation notation, $\gamma_n = (1 - K)\gamma_{n-1}$. Hence, since the operator K satisfies the associative laws of multiplication, we obtain

$$\gamma_n = (1 - K)^{n-1} \gamma_1, \tag{19}$$

where $(1 - K)^{n-1}$ is to be formally expanded by the binomial theorem before operating on γ_1. Substituting for the γ_1 in equation (17) from equation (19), and performing the indicated summation, we obtain

$$g_n = g_0 + \{[1 - (i - K)^n]/K\}(F - K g_0), \tag{20}$$

where, in the fractional operator, $(1 - K)^n$ is to be expanded by the binomial theorem and a factor K in the numerator cancelled with the denominator before operating on $(F - K g_0)$.

If the sequence $\{g_n(x)\}$ converges uniformly, it is clear from (15), that $\lim g_n$ is a solution of the iterated integral equation (2). However, since an integral equation of the first kind has a solution only under special circumstances, $\{g_n(x)\}$ may not converge uniformly, and indeed may not converge at all. Nevertheless the g_n's may serve as useful approximations to a solution of (1) and (2) as will be evident on the basis of the convergence theorems in the next section.

4. Convergence theorems. It will be assumed hereafter that

$$\int_a^b \int_a^b k^2(x, y)\, dx dy \leqq 2. \tag{21}$$

This is no restriction since $k(x, y)$ can always be modified, so as to satisfy (21), by multiplying (1) by a suitable factor and, in the right member of the equation, incorporating the factor into the kernel.

The convergence theorems will first be stated and discussed briefly before their proofs are presented.

THEOREM 3. *The sequence $\{Kg_n\}$ converges uniformly to $F(x)$.*

Theorem 3 is very strong. Without any restrictive assumptions about completeness, the existence of a solution, or the convergence of the sequence $\{g_n\}$, it asserts that, for sufficiently large n, g_n satisfies the iterated integral equation (2) as closely as desired. Basically, however, our interest is in the integral equation (1), rather than with (2). Concerning the suitability of the g_n's as approximate solutions of (1) we have the weaker theorems.

THEOREM 4. *The sequence $\{kg_n\}$ converges in the mean to $f(x)$ if and only if the set $\{\phi_i\}$ is complete relative to $f(x)$. The sequence converges uniformly to $f(x)$ if a piecewise-continuous solution of the integral equation (1) exists.*

It will now be supposed that the 0-th approximation $g_0(x)$ is chosen of the form

$$g_0(x) = \int k(y, x) h(y)\, dy, \tag{22}$$

where $h(y)$ is any piecewise-continuous function. The special case $h(y) \equiv 0$ is also allowed. Concerning the convergence of the sequence $\{g_n\}$ we then have

THEOREM 5. *If a piecewise-continuous solution $g(x)$ of (1) exists, the sequence $\{g_n\}$ converges in the mean to $g(x)$ if and only if the set $\{\psi_i\}$ is complete relative to $g(x)$. If $g(x)$ is of the form $\int k(y, x) h(y)\, dy$, where $h(y)$ is any piecewise-continuous function, then the sequence $\{g_n\}$ converges uniformly to $g(x)$.*

It should be noted that Theorems 4 and 5 are identical, word for word, with Theorems 1 and 2 except for the substitution of g_n for $\bar{g}_n$. Hence the remarks concerning the suitability of the $\bar{g}_n$'s as approximations to a solution of the integral equation (1) are applicable to the g_n's as well.

In order to prove the foregoing theorems it is first convenient to establish several lemmas. Put

$$(23) \qquad F_n(x) \equiv K g_n, \qquad f_n(x) \equiv k g_n.$$

The "Fourier" coefficients of F_n, f_n and g_n then satisfy the relations

$$(24) \qquad c_{in} = \int F_n \psi_i = (\int f_n \phi_i)/\lambda_i = (\int g_n \psi_i)/\lambda_i^2.$$

We then have

LEMMA 1. *$F_n(x)$ and $f_n(x)$ can be expanded in the absolutely and uniformly convergent series*

$$(25) \quad F_n(x) = \sum_{i=1}^{n} c_{in}\psi_i(x), \qquad f_n(x) = \sum_{i=1}^{n} \lambda_i c_{in}\phi_i(x), \qquad n = 0, 1, 2, \cdots.$$

If $g_0(x)$ is chosen of the form (22), then also $g_n(x)$ may be expanded in the absolutely and uniformly convergent series

$$(26) \qquad g_n(x) = \sum_{i=1}^{n} \lambda_i^2 c_{in}\psi_i(x), \qquad n = 0, 1, 2, \cdots.$$

Proof. It is clear, from their definitions in (23), that the expansion theorems apply to $F_n(x)$ and $f_n(x)$ and consequently the series (25) converge as stated in the lemma. In the case of the g_n's, it can be shown successively, from the iteration formula (15), that $g_1(x), g_2(x), \cdots$ are of the same form as $g_0(x)$. Thus we have

$$(27) \qquad g_1 = g_0 + F - K g_0.$$

But $g_0 = \int k(y,x)h(y)dy$; from (4), $F = \int k(y,x)f(y)dy$; and from (3), (23), $Kg_0 = \int k(y,x)f_0(y)dy$. Hence (27) becomes $g_1 = \int k(y,x)[h(y) + f(y) - f_0(y)]dy$. Hence the expansion theorem is applicable to $g_n(x)$ and the series (26) also converge, as stated.

LEMMA 2.

$$(28) \qquad c_{in} - c_i = \mu_i^n (c_{i0} - c_i),$$

where $c_i = \int F\psi_i$, and the sequence μ_i is such that

$$(29) \qquad |\mu_i| < 1, \quad \mu_{i+1} \geqq \mu_i \text{ and } \lim_{i\to\infty} \mu_i = 1, \qquad i = 1, 2, \cdots.$$

Proof. We obtain, from (15) and (7), $\int g_n\psi_i = (1 - 1/\lambda_i^2)\int g_{n-1}\psi_i$

$+\int F\psi_i$. Put $\mu_i = 1 - 1/\lambda_i^2$. Then, by successive reduction, we obtain $\int g_n\psi_i = \mu_i^n \int g_0\psi_i + \lambda_i^2(1-\mu_i^n)\int F\psi_i$, which by (11) and (24), is seen to be equivalent to (28). Furthermore, from (8) and (21), we obtain $0 < 1/\lambda_i^2 < \int\int k^2(x,y)\,dxdy \leqq 2$, or $-1 < \mu_i < 1$. Thus, since the sequence $\{\lambda_i\}$ increases monotonically to infinity, it is seen that (29) is also satisfied. This completes the proof of Lemma 2.

LEMMA 3.

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_i^2(c_{in}-c_i)^2 = 0. \tag{30}$$

If a solution $g(x)$ of (1) or (2) exists, then also

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_i^4(c_{in}-c_i)^2 = 0. \tag{31}$$

Proof. We first note that the series $\sum_{i=1}^{\infty}(c_{i0}-c_i)^2$ converges since we have, from Bessel's inequality $\sum_{i=1}^{\infty}(c_{i0}-c_i)^2 \leqq \int(F_0-F)^2$; hence, $\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \sum_{i=1}^{\infty}\mu_i^{2n}(c_{i0}-c_i)^2$ is uniformly convergent in n, by (28) and the comparison test. Consequently, $\lim_{n\to\infty}\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \sum_{i=1}^{\infty}\lim_{n\to\infty}\mu_i^{2n}(c_{i0}-c_i)^2 = 0$. Similarly, applying Bessel's inequality to $f_0 - f$, and then to $g_0 - g$, when $g(x)$ is assumed to exist, we obtain (30) and (31), as desired.

LEMMA 4. *If the series $\Gamma_0(x) = \sum_{i=1}^{\infty} w_i(x)$, where the $w_i(x)$ are continuous functions, is absolutely and uniformly convergent, and if $\Gamma_n(x) = \sum_{i=1}^{\infty}\mu_i^n w_i(x)$, $n = 0, 1, 2, \cdots$, where μ_i satisfies condition (29), then the sequence $\Gamma_n(x)$ converges uniformly to 0.*

Proof. From the hypotheses on μ_i we have, for some sufficiently large r, $\mu_r \geqq |\mu_i|$, $r > i$. Also, considering the series for $\Gamma_0(x)$, given an $\epsilon > 0$, r can be chosen so large, and independent of x, that $\sum_{i=r+1}^{\infty}|w_i| < \epsilon/2$. Let r be chosen so that both conditions are satisfied. Further, we have $\sum_{i=1}^{r}|w_i| \leqq \sum_{i=1}^{\infty}|w_i| < M$, where M is an upper bound independent of x. Choose N sufficiently large so that $\mu_r^n < \epsilon/(2M)$ for $n > N$. Then $|\Gamma_n| \leqq \sum_{i=1}^{r}|\mu_i^n w_i| + \sum_{i=r+1}^{\infty}|\mu_i^n w_i|\ \mu_r^n M + \epsilon/2 < \epsilon$, when $n > N(\epsilon)$, as we wished to prove.

LEMMA 5. *If $G_n(x)$ can be expanded in a uniformly convergent series*

$$G_n(x) = \sum_{i=1}^{\infty} e_{in}\theta_i(x), \qquad n = 0, 1, 2, \cdots \tag{32}$$

in terms of the real, continuous, orthonormalized functions $\theta_i(x)$, $i = 1, 2, \cdots$ and if $G(x)$ is piecewise-continuous, with $e_i = \int G\theta_i$, then necessary and sufficient conditions for the sequence $G_n(x)$ to converge in the mean to $G(x)$ are that $\int G^2 dx = \sum_{i=1}^{\infty} e_i^2$ and $\lim_{n\to\infty} \sum_{i=1}^{\infty} (e_{in} - e_i)^2 = 0$.

Proof. Since the series (32) is uniformly convergent, we have $\int GG_n = \sum_{i=1}^{\infty} e_{in} \int G\theta_i = \sum_{i=1}^{\infty} e_{in}e_i$, and similarly $\int G_n^2 = \sum_{i=1}^{\infty} e_{in}^2$. Hence

$$\int (G_n - G)^2 = \int G^2 + \sum_{i=1}^{\infty} (e_{in} - e_i)^2 - \sum_{i=1}^{\infty} e_i^2. \tag{33}$$

Now suppose the conditions of the lemma to be satisfied. Then $\int (G_n - G)^2 = \sum_{i=1}^{\infty} (e_{in} - e_i)^2$ and consequently by hypothesis, $\lim \int (G_n - G)^2 = 0$. This proves the first part of the lemma.

Now suppose that $\lim \int (G_n - G)^2 = 0$. From (33), $\int G^2 dx \leqq \sum_{i=1}^{\infty} e_i^2 + \int (G_n - G)^2$ for all n. Hence $\int G^2 \leqq \sum_{i=1}^{\infty} e_i^2$. But, by Bessel's inequality, $\int G^2 \geqq \sum_{i=1}^{\infty} e_i^2$. Hence $\int G^2 = \sum_{i=1}^{\infty} e_i^2$. Then, from (33), $\sum_{i=1}^{\infty} (e_{in} - e_i)^2 = \int (G_n - G)^2$, whence $\lim \sum_{i=1}^{\infty} (e_{in} - e_i)^2 = 0$. This completes the proof.

We can now proceed to the proof of the convergence theorems.

Proof of Theorem 3. By the expansion theorem and (11) and (24), the series $F_n - F = \sum_{i=1}^{\infty} (c_{in} - c_i)\psi_i$, $n = 0, 1, 2, \cdots$ are absolutely and uniformly convergent in x. Hence, by Lemma 2, the series $\sum_{i=1}^{\infty} \mu_i^n (c_{i0} - c_i)\psi_i$ are also absolutely and uniformly convergent in x. Hence the conditions of Lemma 4 are satisfied and the sequence $\{F_n - F\}$ converges uniformly to zero; or by (23), $\{Kg_n\}$ converges uniformly to F, as we wished to prove.

Proof of Theorem 4. By Lemmas 1 and 3 all the conditions of Lemma 5 are satisfied by the functions $f_n(x)$ and $f(x)$. Hence by (23) the first part of the theorem, concerning the convergence in the mean of $\{kg_n\}$ to $f(x)$, is proved.

In the second part of the theorem, since $g(x)$ exists by hypothesis, the expansion theorem may be applied to $f(x)$ as well as to $f_n(x)$. Hence by (11) and Lemmas 1 and 2, the series $f_n - f = \sum_{i=1}^{\infty} \mu_i^n \lambda_i (c_{i0} - c_i) \phi_i(x)$, $n = 0, 1, 2, \cdots$ are absolutely and uniformly convergent in x, and the conditions of Lemma 4 are satisfied. Hence the sequence $\{f_n - f\}$ converges uniformly to zero, or, by (23), $\{kg_n\}$ converges uniformly to $f(x)$. This completes the proof.

Proof of Theorem 5. Since $g_0(x)$ is of the form (22), Lemmas 1 and 3 indicate that the conditions of Lemma 5 are satisfied by the functions $g_n(x)$ and $g(x)$. Hence the first part of the theorem, concerning convergence in the mean of $\{g_n\}$ to $g(x)$, is proved.

In the second part of the theorem, the expansion theorem is applicable to $g(x)$, by hypothesis. Hence, by (11) and Lemmas 1 and 2, the series $g_n - g = \sum_{i=1}^{\infty} \mu_i^n \lambda_i^2 (c_{i0} - c_i) \psi_i(x)$, $n = 0, 1, 2, \cdots$ are absolutely and uniformly convergent in x, and the conditions of Lemma 4 are satisfied. Hence the sequence $\{g_n\}$ converges uniformly to $g(x)$, as we wished to prove.

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