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ON AN ITERATIVE METHOD FOR THE UNFOLDING OF SPECTRA

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ABSTRACT

In ref. [5] an iterative method has been formally proposed for the numerical solution of a special class of integral equations of the first kind, where one of the essential assumptions is the positivity of both the kernel and the right-hand side. Solving such an equation is also known as unfolding or deconvolution. In this paper, we report the main results of our study ref. [8] on a motivation for this iterative method and its convergence, taking into account a result ref. [7] on global convergence for a special discrete version of the iteration procedure. In presenting the results, priority has been given to the applicability of the method and not to its mathematical analysis, which may be found in ref. [8]. A numerical example from high-energy physics is presented.

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1. INTRODUCTION

A common procedure in high-energy physics and other fields of science is the measurement of various kinds of spectra, or, more generally, of some density function f which may depend on energy, effective mass, current, or some other parameter of one or more dimensions. The observed spectrum g is usually a distorted version of the true spectrum f. The distortions can come from various sources, such as the limited resolution of the measuring device (detector) as expressed by a resolution function r, the variable efficiency e with which events are detected, or statistical fluctuations in the measurements. It may also be the case that values of r and e can only be obtained approximately by using Monte Carlo simulation, which is another source of distortion. Confining ourselves to the one-dimensional case, the functions f, g, r, and e are related by an equation of the type

$$\int_0^1 k(x,y)f(y)\,dy = g(x), \qquad x \in [0,1], \tag{1.1}$$

where $k(x,y) := r(x,y)e(y)$. Equation (1.1) is an integral equation of the first kind in which all the functions are non-negative and f is unknown. The function k is called the kernel of the integral equation. Solving eq. (1.1) is also known as unfolding or deconvolution. It is well known that the unfolding problem is usually an ill-posed problem, which means that small changes in g may cause large changes in f.

In practice, g is never known exactly but is represented only by a finite number of approximate measured values. More precisely, let x_i, $i = 0(1)m$, be such that

$$0 \leqq x_0 < x_1 < \ldots < x_{m-1} < x_m \leqq 1\,. \tag{1.2}$$

We then consider the semidiscrete form of eq. (1.1):

$$\int_0^1 \tilde{k}_i(y)\tilde{f}(y)\,dy = \tilde{g}_i\,, \qquad i = 1(1)m\,, \tag{1.3}$$

where $\tilde{g}_i$ and $\tilde{k}_i(y)$ are approximations to g and $k(\cdot,y)$, respectively, on (x_{i-1},x_i). Examples of such a discretization are

$$\tilde{k}_i(y) := k(x_i,y)\,, \qquad \tilde{g}_i := g(x_i)\,,$$

or

$$\tilde{k}_i(y) := \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} k(x,y)\,dx\,, \qquad \tilde{g}_i := \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} g(x)\,dx\,, \qquad i = 1(1)m\,.$$

Equations (1.3) have no unique solution in the class of non-negative and continuous functions. If we know that f belongs to a given class of functions which depend on a finite number of parameters, it is advantageous to insert such a function $\tilde{f}$ into eqs. (1.3) in order to obtain a system of equations with a finite number of unknowns. This system of equations will in general be non-linear, and will not necessarily have a unique solution in the class of functions under consideration. When the system is overdetermined the method of least-squares is often used.

If we know only that f is non-negative and continuous we can use model functions as described, for example, in ref. [1]. Another possibility is to use quadrature formulae in order to replace eq. (1.3) by a system of linear equations of the form

$$\sum_{j=1}^{m} c_{ij}\tilde{\phi}_j = \tilde{g}_i\,, \qquad i = 1(1)m\,, \tag{1.4}$$

with coefficient matrix $(c_{ij}) := [c_j\tilde{k}_i(z_j)]$, where $z_j \in [0,1]$ are the knots and $c_j > 0$ the weights of the quadrature formula, i,j = 1(1)m. The $\tilde{\phi}_j$ are to be interpreted as approximations to $f(z_j)$.

There exists extensive literature on the numerical treatment of the unfolding problem, of which we mention only refs. [2–4], in which other references may be found. In particular, Kondor [5] has proposed an iterative method for solving eq. (1.1). He gives a purely formal description of the method and presents a few examples to illustrate it. A special discrete version of this iterative method had already been used by Shepp and Vardi [6], but this is not mentioned in Kondor's paper. Shepp and Vardi used the iterative scheme in order to solve a constrained maximization problem which they obtained from the maximum likelihood estimate of a finite number of unknown parameters of Poisson-distributed random variables. The unknown parameters were discretized values of an unknown emission density satisfying a two-dimensional integral equation of the form of eq. (1.1). The underlying physical problem arises in emission tomography, where counts observed by a detector can be used to approximate the right-hand side of eq. (1.1). Kondor did not study the convergence of the iterative method. Kaufman, Shepp and Vardi [6, 7] did investigate convergence for their discrete case.

In this paper we summarize the main results of our study (Mülthei and Schorr [8]) on the iterative method in both the continuous and the discrete case, taking into account the convergence theorem given in ref. [7]. Proofs of the theorems given below may be found in ref. [8].

2. THE ITERATIVE METHOD AND A MOTIVATION

In order to avoid unnecessary complications, we shall assume that $k \in C([0,1]^2)$, $g \in C([0,1])$, and that k and g are positive. These requirements can be weakened for practical cases. For instance, k may be allowed to have isolated integrable singularities and discontinuities. In what follows, the requirements to a solution f of eq. (1.1) are always explicitly stated.

The iterative method is defined by

$$f_{n+1} = G(f_n)\,, \qquad n \in \mathbb{N}_0\,, \tag{2.1}$$

where

$$G(f)(y) := f(y)T(f)(y)\,,$$

$$T(f)(y) := \frac{1}{a(y)} \int_0^1 \frac{g(x)k(x,y)}{F(x)}\,dx\,, \qquad F(x) := \int_0^1 k(x,y)f(y)\,dy\,,$$

$$a(y) := \int_0^1 k(x,y)\,dx\,, \qquad y \in [0,1]\,,$$

$$f, f_0 \in \mathcal{K} := \{h \in C[0,1]\setminus\{0\} : h(x) \geq 0 \quad \text{for all} \quad x \in [0,1]\}\,.$$

Obviously the operator G maps $\mathcal{K}$ into itself. Trivially, a solution $f \in \mathcal{K}$ of eq. (1.1) is also a function belonging to

$$\mathfrak{I} := \{h \in \mathcal{K} : T(h)(y) \leq 1 , \qquad y \in [0,1] , \\ \text{where equality holds for y with } h(y) > 0\} .$$

The significance of the set $\mathfrak{I}$ will become clear in the next section. For $f \in \mathfrak{I}$, the following fixed point equation holds:

$$f = G(f) . \tag{2.2}$$

In the sequel we shall derive a constrained maximization problem whose solution, under certain conditions, is equal to the solution of the continuous unfolding problem. We shall then show that the solution of this maximization problem can be obtained by the iterative method, if it converges. In order to derive the maximization problem, we use ideas which Kaufman, Shepp and Vardi [6, 7] applied to a discrete model of emission tomography. In their model, photons are emitted from boxes and detected, with some probability, by a set of detectors. It is assumed that the photons are emitted according to a Poisson distribution with unknown mean value f_j for box j. The mean values g_i of the counts observed by the detectors are supposed to be related to the values f_j through the system of equations

$$g_i = \sum_j p_{ij} f_j ,$$

where the p_{ij} are known transition probabilities. The unknown values f_j are finally taken to be those values which maximize the log-likelihood function belonging to the observed counts in the different detectors.

In our case we may describe the positions of the boxes by x_i, $i = 1(1)m$, satisfying relation (1.2), and assume that, for the measurements g_i,

$$g_i := g(x_i) = \int_0^1 k(x_i,y) f(y)\, dy , \qquad i = 1(1)m . \tag{2.3}$$

Relation (2.3) is a necessary condition for f to be a solution of the integral equation (1.1). From relation (2.3) we obtain

$$\sum_{i=1}^{n} g_i = \sum_{i=1}^{n} \int_0^1 k(x_i,y) f(y)\, dy , \tag{2.4}$$

a relation which will henceforth be taken as a constraint on f.

Let P_i, $i = 1(1)m$, be m independent Poisson-distributed random variables with mathematical expectation $E(P_i) = g_i$. If p_i is a realization of P_i, $i = 1(1)m$, the corresponding likelihood function is given by

$$l_m(f) := \prod_{i=1}^{m} \exp(-g_i) g_i^{p_i} / p_i! , \tag{2.5}$$

where g_i is given by the right-hand side of relation (2.3). Taking the logarithm of both sides of relation (2.5), and leaving out unimportant constants, we obtain the log-likelihood function

$$\tilde{\Lambda}_m(f) := \sum_{i=1}^{m} p_i \ln \left[\int_0^1 k(x_i,y)f(y)\,dy\right] . \tag{2.6}$$

It is well known that maximizing $\tilde{\Lambda}_m$ for all $f \in \mathcal{K}$ which satisfy the constraint (2.4) yields a maximum likelihood estimate of f. The p_i in relation (2.6) are estimates of the values g_i, $i = 1(1)m$. We may therefore replace p_i in relation (2.6) by g_i. As $m \to \infty$ we may replace the sums in relations (2.4) and (2.6) by integrals to obtain formally

$$\Gamma := \int_0^1 g(x)\,dx = \int_0^1 \int_0^1 k(x,y)f(y)\,dy\,dx = \int_0^1 a(y)f(y)\,dy , \tag{2.7}$$

$$\Lambda(f) := \int_0^1 g(x) \ln \left[\int_0^1 k(x,y)f(y)\,dy\right] dx . \tag{2.8}$$

We may consider Λ as the continuous version of the log-likelihood function $\tilde{\Lambda}_m$ in relation (2.6); Λ is a functional of f for fixed k and g which is concave owing to the concavity of the logarithm. Equation (2.7) is a constraint on $f \in \mathcal{K}$. In the sequel we therefore consider the *Constrained Maximization Problem (CMP):* Maximize Λ on $\mathcal{K}_\Gamma$, where

$$\mathcal{K}_\Gamma := \{h \in \mathcal{K} : \int_0^1 a(y)h(y)\,dy = \Gamma\} .$$

Note that $G : \mathcal{K} \to \mathcal{K}_\Gamma$ and $\mathfrak{I} \subset \mathcal{K}_\Gamma$ because of eq. (2.2).

For the CMP the following theorem then holds. It shows the essential connection between the iteration scheme (2.1) and the CMP.

Theorem 1

i) Let $f^* \in \mathfrak{I}$. Then, for all $f \in \mathcal{K}_\Gamma$, the following inequalities hold:

$$\Lambda(f) \leq \Lambda(f^*) \leq \int_0^1 g(x) \ln g(x)\,dx ;$$

i.e. every $f^* \in \mathfrak{I}$ is a global solution of the CMP. In particular, for a solution $f^* \in \mathcal{K}$ of eq. (1.1) we have

$$\Lambda(f^*) = \int_0^1 g(x) \ln g(x)\,dx .$$

ii) For all $f \in \mathcal{K}_\Gamma$ the following inequality holds:

$$\Lambda(f) \leq \Lambda(G(f)) .$$

In particular, if f^* is a solution of the CMP, then

$$\Lambda(f^*) = \Lambda(G(f^*)),$$

which in the case of the uniqueness of f^* implies

$$G(f^*) = f^* .$$

iii) Suppose that Λ is strongly concave on $\mathcal{K}_\Gamma$. Then, a positive $f^* \in \mathcal{K}_\Gamma$ is a solution of the CMP if, and only if, $f^* \in \mathfrak{Z}$ and is positive.

Remark: It can be shown that Λ is strongly concave on $\mathcal{K}_\Gamma$ if, and only if, eq. (1.1) has a unique solution in C[0,1].

For the proofs of Theorem 1, see Theorems 1 to 4 in ref. [8]. The results of Theorem 1 clearly show that it is worth while to investigate further the properties of the iterative method. For instance, the property (ii) implies that each iteration step of the scheme (2.1) leads to a Λ-value which is at least as large as the one of the previous step. In particular, Theorem 1 shows that the iteration procedure (2.1) refers primarily to the CMP, and refers to the integral equation (1.1) only if there exists a solution in $\mathcal{K}$. In the case where Λ is strongly concave on $\mathcal{K}_\Gamma$, a positive solution of the CMP is characterized as a positive element of $\mathfrak{Z}$. The set of functions $\mathfrak{Z}$ reflects the well-known Kuhn-Tucker conditions in non-linear programming which are used in refs. [6] and [7], too. In the next two sections we shall investigate the convergence and some other properties of the iterative method (2.1).

3. CONVERGENCE PROPERTIES OF THE ITERATIVE METHOD

In order to investigate convergence of the iteration scheme (2.1), one would like to be able to show that the operator G is contracting. The usual method of showing this consists in proving that its Fréchet derivative at a fixed point of G with respect to any norm is bounded by a number less than 1. But a non-trivial analysis in ref. [8] (corollary of Theorem 6) shows that this is not possible in general. Nevertheless, certain convergence properties of the iteration scheme may be proved, and are given below.

Theorem 2: Let $f_{n+1} = G(f_n)$, $n \in \mathbb{N}_0$, $f_0 \in \mathcal{K}$ and positive. Then,
i) $\Lambda(f_n)$ converges monotonically to some real number as $n \to \infty$,
ii) $\int_0^1 f_n(y)\,[1 - T(f_n)(y)]^2\,dy$ converges to zero as $n \to \infty$.
In particular, $T(f_n)$ converges uniformly to 1 on each subinterval of [0,1] on which all f_n are uniformly bounded away from zero.

For the proof, see ref. [8] (Theorem 7 and its corollary). Point (i) of Theorem 2 states a property which is clearly advantageous for the maximization of Λ on $\mathcal{K}_\Gamma$. As far as the significance of point (ii) is concerned, note that we have

$$\int_0^1 f(y)\,[1 - T(f(y))]^2\,dy = 0$$

for all $f \in \mathfrak{Z}$. Furthermore, point (ii) allows f_n to converge to zero on certain subintervals of [0,1] as $n \to \infty$.

The following theorem confirms what has been said in connection with point (iii) of Theorem 1.

Theorem 3: For positive $f_0 \in \mathcal{K}$, let $f_{n+1} = G(f_n)$ converge to $f^* \in C[0,1]$ with respect to $L_1[0,1]$ as $n \to \infty$. Then $f^* \in \mathfrak{Z}$, which implies that f^* is the solution of the CMP.

The question of whether the iterative scheme converges is still open. The main obstacle to the proof of convergence is the fact that, contrary to the discrete case which is investigated in the next section, the relative compactness of the sequence f_n with respect to the commonly used function spaces is not available.

4. THE DISCRETE CASE

In order to perform the iterations (2.1) numerically, we discretize by using the same quadrature formula for all integrals in (2.1). We obtain the following discretized form of the iteration scheme (2.1):

$$
\begin{aligned}
&\phi^{n+1} = G_m(\phi^n)\,,\\
&\phi^n := (\phi_i^n)_{i=1(1)m} \in \mathcal{K}_m := \{\phi \in \mathbb{R}^m \setminus \{0\} : \phi_i \geq 0\,, \qquad i = 1(1)m\}, \qquad n \in \mathbb{N}_0\,,\\
&G_m(\phi) := [G_{m,i}(\phi)]_{i=1(1)m}\,, \qquad \phi := (\phi_i)_{i=1(1)m} \in \mathcal{K}_m\,, \qquad (4.1)\\
&G_{m,i}(\phi) := \phi_i T_{m,i}(\phi)\,, \qquad T_{m,i}(\phi) := \frac{1}{a_i}\sum_{j=1}^{m} c_j k_{ji} g_i \Big/ \sum_{s=1}^{m} c_s k_{js}\phi_s\\
&a_i := \sum_{j=1}^{m} c_j k_{ji}\,, \qquad g_j := g(z_j)\,, \qquad k_{ji} := k(z_j, z_i)\,, \qquad i,j = 1(1)m\,.
\end{aligned}
$$

Note that n is the iteration index and m the discretization index. By construction, we have $\phi^n \in \mathcal{K}_{\Gamma_m}$, $n \in \mathbb{N}$, for $\phi^0 \in \mathcal{K}$, where

$$
\mathcal{K}_{\Gamma_m} := \{\phi \in \mathcal{K}_m : \sum_{i=1}^{m} c_i a_i \phi_i = \Gamma_m\}\,, \qquad \Gamma_m := \sum_{i=1}^{m} c_i g_i\,.
$$

By using the quadrature formula to discretize the functional Λ defined in eq. (2.8) we obtain the approximation

$$
\Lambda_m(\phi) := \sum_{i=1}^{m} c_i g_i \ln\Big(\sum_{j=1}^{m} c_j k_{ij}\phi_j\Big)\,, \qquad \phi \in \mathcal{K}_{\Gamma_m}\,.
$$

In practical applications, as mentioned in the Introduction, it is obvious that k_{ij} and g_i are to be replaced by $\tilde{k}_i(z_j)$ and $\tilde{g}_i$, respectively, $i,j = 1(1)m$. In this form the iterative scheme (4.1) can be interpreted as an iteration procedure for the system of linear equations (1.4). All statements in sections 2 and 3 concerning the continuous case can be carried over to the discrete case, but we do not formulate them explicitly. The following convergence statement holds.

Theorem 4: For every positive $\phi^0 \in \mathcal{K}_m$ the sequence ϕ^n defined by eq. (4.1) converges to $\phi^* \in \mathcal{K}_{\Gamma_m}$, where ϕ^* maximizes the functional Λ_m on $\mathcal{K}_{\Gamma_m}$.

Contrary to the continuous case, even global convergence can be shown here with respect to all positive elements of $\mathcal{K}_m$. The statement of maximization in this theorem follows immediately by

carrying over Theorem 3 to the iteration (4.1). The proof of the first part of Theorem 4 is obtained by writing the iterative scheme (4.1) as the following equivalent iteration procedure

$$\lambda_i^{n+1} = \lambda_i^n \sum_{j=1}^{m} n_j^* p_{ij} \Big/ \sum_{r=1}^{m} p_{rj}\lambda_r^n$$

$$\lambda_i^n := \frac{c_i a_i \phi_i^n}{\Gamma_m}, \qquad n_j^* := \frac{c_j g_j}{\Gamma_m}, \qquad p_{ij} := \frac{c_j k_{ji}}{a_i}, \qquad i,j = 1(1)m, \qquad n \in \mathbb{N}_0,$$

for which convergence is proved in ref. [7] for all positive starting values λ_i^0, $i = 1(1)m$. With respect to the completeness of the assumptions of the convergence theorem in ref. [7], the addendum in ref. [8] should also be taken into account. Note that the following equations hold:

$$\sum_{j=1}^{m} n_j^* = 1, \qquad \sum_{j=1}^{m} p_{ij} = 1, \qquad i = 1(1)m.$$

Furthermore, the iterates λ_i^n, $n \in \mathbb{N}$, automatically satisfy the constraint

$$\sum_{i=1}^{m} \lambda_i^n = 1$$

because of $\phi^n \in \mathcal{K}_{\Gamma_m}$.

Note that the limiting point ϕ^* depends on the starting point ϕ^0 if Λ_m has no unique maximum point on $\mathcal{K}_{\Gamma_m}$. Furthermore, if both the discrete constrained maximization problem and the discrete version of eq. (1.1),

$$\sum_{j=1}^{m} c_j k_{ij} \phi_j = g_i, \qquad i = 1(1)m,$$

have a unique solution in $\mathcal{K}_m$, then ϕ^* is a solution of this system of equations. The proof follows from the discrete version of Theorem 1 (i). The speed of convergence is characterized by the spectral radius $\varrho := \varrho[G_m'(\phi^*)]$, where

$$G_m'(\phi^*) := [\frac{\partial}{\partial \phi_j} G_{m,i}(\phi^*)]_{i,j=1(1)m}$$

is the Jacobian of G_m at the limiting point ϕ^* defined in Theorem 4; ϱ is also called rate of convergence. In ref. [8] (Lemma 3) it is shown that

$$G_m'(\phi^*) = \tilde{I}_m - L_m,$$

where

$$\tilde{I}_m := \operatorname{diag}[t_i^{(m)}]_{i=1(1)m}, \qquad t_i^{(m)} := T_{m,i}(\phi^*) \in (0,1], \qquad i = 1(1)m,$$

$$L_m := D_m C_m^T B_m C_m, \qquad C_m := (c_j k_{ij})_{i,j=1(1)m},$$

$$D_m := \operatorname{diag}(\phi_i^*/a_i)_{i=1(1)m}, \qquad B_m := \operatorname{diag}\left[c_i g_i / \left(\textstyle\sum_{r=1}^{m} c_r k_{ir} \phi_r^*\right)^2\right]_{i=1(1)m}.$$

In order to determine ϱ we require only the submatrix $L_{m,r}$ of $G'_m(\phi^*)$ which is derived from L_m by leaving out the i^{th} row and the i^{th} column for all $i \in A := \{j : \phi_j^* = 0\}$. The smallest eigenvalue $\lambda_1^{(m)}$ of $L_{m,r}$ plays a special role. In ref. [8] (Theorem 12) the following result is given:

Theorem 5: For the spectral radius $\varrho[G'_m(\phi^*)]$ we have

$$\varrho[G'_m(\phi^*)] = \max_{i \in A}[1 - \lambda_1^{(m)}, t_i^{(m)}],$$

where $\lambda_1^{(m)}, t_i^{(m)} \in [0,1]$.

Note that the rate of convergence ϱ is less than 1 if, and only if, $L_{m,r}$ is regular and $t_i^{(m)} < 1$ for $i \in A$. This is true, for instance, if C_m is regular and ϕ^* is positive. If ϱ is less than 1, the iteration (4.1) always converges to ϕ^*, as defined in Theorem 4, for every starting point close enough to ϕ^* (see ref. [9], pp. 300–301). In this case, ϕ^* must be an isolated maximum point of Λ_m, and therefore the only one, because the set of maximum points of Λ_m is convex and each of them is a fixed point of G_m.

It is expected that $1-\lambda_1^{(m)}$ will tend to 1 as $m \to \infty$ under certain assumptions. For a special model problem investigated in ref. [8] (section 6) we have been able to prove, by using an appropriate quadrature formula, that the rate of convergence is

$$\varrho[G'_m(\phi^*)] = 1 - \frac{1}{m^2}.$$

This means that the rate of convergence tends quadratically to unity as the step size m^{-1} decreases. This is a big disadvantage for the application of the iterative scheme (4.1). On the other hand, it should be emphasized that according to Theorem 4 the iteration procedure (4.1) is globally convergent to a maximum point ϕ^* even when the rate of convergence is unity.

5. NUMERICAL EXAMPLE

In order to illustrate the iterative method (4.1) we apply it to an example taken from ref. [4]. Other numerical tests of the method may be found in refs. [2, 6–8]. The calculations were done on an IBM 3090 computer in double precision, i.e. with a precision accuracy of approximately 15 digits.

Let f be the density of a variable $y \in [0,1]$. The measurement of y is performed in the following way. The acceptance probability (efficiency) is

$$e(y) := 1 - 2\left(y - \frac{1}{2}\right)^2, \qquad y \in [0,1].$$

The value y is transformed to

$$y_{tr}(y) := 2y\left(1 - \frac{y}{10}\right), \qquad y \in [0,1].$$

Finally, y_{tr} is measured with the Gaussian resolution function

$$r(x,y) := \exp\left(-[x - y_{tr}(y)]^2/2\sigma^2\right), \qquad x,y \in [0,1], \qquad \sigma = 0.1.$$

The different measurement steps then yield the density

$$g(x) := \int_0^1 k(x,y)f(y)\,dy\,, \qquad x \in [0,1]\,,$$

where $k(x,y) := r(x,y)e(y)$.

In order to see how the iteration procedure works, we assume that f is essentially a sum of three Breit–Wigner distributions. This means that

$$f(y) := \sum_{k=1}^{3} b_k \frac{d_k^2}{(y-s_k)^2 + d_k^2}, \qquad y \in [0,1]\,,$$

with parameters given by

k	b_k	s_k	d_k
1	1.0	0.2	1.0
2	10.0	0.4	0.1
3	5.0	0.75	0.1

For the numerical treatment of this example the extended Simpson's rule is used. In our case, we choose the g_i in eq. (4.1) as

$$g_i := \tilde{g}_i := \sum_{j=1}^{m} c_j k_{ij} f(z_j)\,, \qquad i = 1(1)m\,,$$

since f is known. The weights c_j and the knots z_j are those of the extended Simpson's rule. The iteration procedure (4.1) yields the following results, as shown in table 1 and fig. 1, where

$$\delta_n(x_i) := f(x_i) - \phi_i^n\,, \qquad i = 1(1)m\,,$$

$$e_n := 100 \max_{1 \le i \le m} \frac{|\delta_n(x_i)|}{f(x_i)}\,, \qquad n \in \mathbb{N}_0\,.$$

We confine ourselves to the case $m = 51$ and the starting values $\phi_i^0 = 1$, $i = 1(1)m$. The dependence of the iterative method on the starting values has been illustrated by some examples in ref. [8]. On the other hand, statistical reasons suggest the choice of constant starting values.

We can see from table 1 that, in the beginning, the iterations yield large improvements, a fact which has also been observed for other examples in ref. [8]. However, the speed of convergence is rather slow, which is in agreement with the consequences of Theorem 5. Figure 1 indicates that the errors e_n can only be reduced to a certain level. Note the oscillations of the errors e_n. The minimum error for the 1200 iterations considered in fig. 1 is $e_{504} = 0.47$. Given the fact that the right-hand side of the integral equation (1.1) is generally affected by measurement errors in practical problems, the determination of the solution with high precision is not usually necessary. Therefore, a few iterations can yield a satisfactory result. Nevertheless, it should be kept in mind that the iterative procedure (4.1) is globally convergent.

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Table 1

f and δ_n, n = 1, 2, 3, 6, 50, 100

x	f	δ_1	δ_2	δ_3	δ_6	δ_{50}	δ_{100}
0.0	1.637	−0.315	−0.069	−0.073	−0.077	−0.034	−0.013
0.04	1.789	−0.259	0.000	−0.003	−0.029	0.001	−0.001
0.08	1.984	−0.245	0.043	0.050	0.001	0.014	0.003
0.12	2.248	−0.298	0.041	0.075	0.024	0.003	0.006
0.16	2.617	−0.431	−0.024	0.063	0.059	−0.019	−0.003
0.20	3.160	−0.631	−0.177	−0.023	0.077	−0.012	−0.016
0.24	3.993	−0.822	−0.422	−0.222	0.001	0.033	0.009
0.28	5.309	−0.764	−0.621	−0.470	−0.206	0.033	0.034
0.32	7.340	−0.027	−0.360	−0.411	−0.328	−0.073	−0.032
0.36	9.904	1.536	0.703	0.374	0.077	−0.048	−0.046
0.40	11.339	2.574	1.545	1.089	0.560	0.137	0.080
0.44	10.037	1.578	0.804	0.501	0.153	−0.009	−0.007
0.48	7.628	−0.010	−0.272	−0.279	−0.247	−0.094	−0.044
0.52	5.800	−0.857	−0.665	−0.448	−0.160	−0.004	0.003
0.56	4.779	−1.067	−0.651	−0.354	−0.001	0.057	0.020
0.60	4.401	−0.979	−0.563	−0.301	−0.004	0.040	0.018
0.64	4.580	−0.668	−0.417	−0.273	−0.119	−0.037	−0.011
0.68	5.300	−0.003	−0.011	−0.042	−0.108	−0.081	−0.045
0.72	6.264	0.925	0.682	0.483	0.203	0.040	0.023
0.76	6.428	1.241	0.917	0.657	0.328	0.097	0.055
0.80	5.324	0.537	0.330	0.150	−0.028	−0.054	−0.045
0.84	3.963	−0.237	−0.209	−0.242	−0.220	−0.056	−0.023
0.88	2.959	−0.598	−0.346	−0.258	−0.127	0.021	0.025
0.92	2.301	−0.677	−0.279	−0.133	0.007	0.034	0.010
0.96	1.867	−0.662	−0.196	−0.036	0.067	−0.001	−0.011
1.00	1.570	−0.652	−0.165	−0.010	0.048	−0.037	0.006

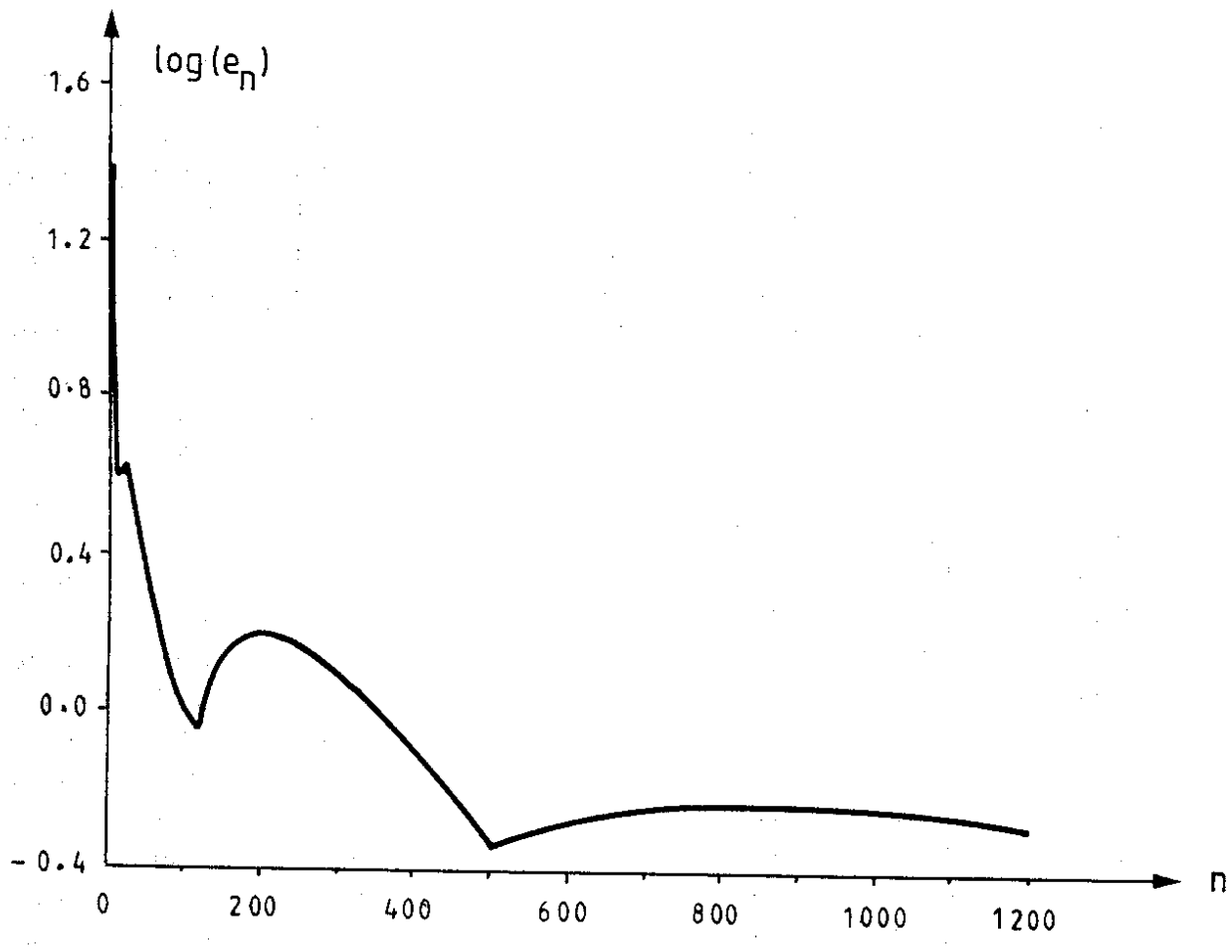

Fig. 1 $\log(e_n)$, $n = 1, 2, \ldots, 1200$.