Elements of Quantum Field Theory

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Contents

I. INTRODUCTION

A. Why Classical Field Theory?

The interactions between point-particles are described by some kind of forces in non-relativistic mechanics. This picture meet unexpected difficulties when its relativistic extension is sought.

1. These forces can not be instantaneous because simultaneity is not a Lorentz-invariant concept.

2. The relativistic invariance is incompatible with the usual Cauchy-problem setup of classical mechanics. In fact, the world line of a particle, $x^\mu(s)$, parameterized by the proper time satisfies the condition

$$\frac{dx^\mu(s)}{ds}\frac{dx_\mu(s)}{ds} = 1 \tag{1}$$

and the equation

$$\frac{d^2x^\mu(s)}{ds^2}\frac{dx_\mu(s)}{ds} = 0 \tag{2}$$

follows, showing that the four-velocity must be perpendicular to the four-acceleration. But the Cauchy-problem consists of the free prescription of the coordinates and the velocities for the initial conditions which might lead to contradiction to Eq. (2) for a given interaction. The possible way out from this difficulty is to give up the Cauchy-problem rearrangement and allow that the forces acting on the particles on a given instant are given not only on some initial conditions but they depend on the whole past evolution of the system.

3. The algebraic study of the Poisson-bracket structure of relativistic Hamiltonian Mechanics leads to a "no-go theorem", stating that any Lorentz-invariant particle dynamics given in terms of Lagrangians lead to vanishing accelerations, ie. only non-interacting relativistic particles can be described by the usual Lagrangian formalism.

The solution of these problems is the use of the time honored field concept in Physics. They are supposed to play the role of fundamental degrees of freedom in relativistic mechanics.

B. Why Quantum Field Theory?

There are several reasons Quantum Field Theory is required to extend and to complete the formalism of usual non-relativistic Quantum Mechanics.

The conceptual problems, treated by Quantum Field Theory are the following:

1. According to special relativity mass and energy are equivalent. In other words, energy and mass can be converted into each other. Mass is quantized, cf. mass spectrum of elementary particles, and kinetic energy can be used to create particles, particles can annihilate leaving behind energy only in the form of other particles. The number of degrees of freedom becomes a dynamical variable. The Schrödinger equation or its relativistically covariant generalizations can not cope with this phenomenon.

2. The particle number is ill defined for any bound state. This is because interactions consists of particle exchanges and there are infinitely many particles exchanged when a bound state is formed. The interaction between static particles can be recast in terms of a potential acting between the particles. But the bounded particles are in motion and their dynamics is influenced by the retarded effects of the particle exchange. This latter can not be brought into the form of an interaction potential and requires the possibility of changing the particle number. The strength of this many-body effect is proportional to the square of the coupling constant characterizing the interaction and is therefore important for strong interactions.

3. The unique and well defined nature of the state vector is lost in Quantum Mechanics as soon as special relativity, causality in particular, is imposed. But the transition amplitudes remain well defined and obey relativistic transformation rules. Thus a formalism of Quantum Mechanics is sought which is based on amplitudes and states remain hidden. This is achieved in Quantum Field Theory where observables are described in terms of Green functions only.

The 'technical' problems urging us to seek other formalism than those of the usual non-relativistic Quantum Mechanics are:

1. Particles with the same quantum numbers are indistinguishable in Quantum Mechanics and the multi-particle states have to be symmetrized or anti-symmetrized in three dimensions. It is very clumsy and difficult to work with the wave functions of such states because the same information is repeated $N!$ times in an N-particle state. Quantum Field Theory offers a new representation of multi-particle state in which (anti)symmetrization appears in a natural manner.

2. Observables always involve some particles, eg. kinetic energy is a one-body operator, the Coulomb energy is a two-body operator, etc. Their expectation values are usually calculated in the framework of perturbation expansion. Consider an observable for n_O particles. The fundamental interactions are supposed to be simple, involving few particles only. The expectation value of our observable, calculated in the k-th order of the perturbation expansion in a theory where the interactions involve n_i particles, should be made up by considering $n_O + kn_i$ particles altogether. This number is far less than the total number of particles which is usually at least in the order of magnitude of the Avogadro number. How can we organize our formalism to let the gigantic number of spectator particles separate off the dynamics of $n_O + kn_i$ active particles? The answer is provided by the perturbation expansion constructed for Green functions.

Finally, Quantum Field Theory is unique. The states of elementary particles are vectors of the irreducible representations of the appropriate symmetry group, cf. Appendix B. It has been shown that Quantum Field Theory actually produces the most general representation of the kinematic and internal symmetries obtained in terms of local operators in space-time.

C. Basic idea of Quantum Field Theory

Let us start the heuristic introduction of quantum field for non-interacting particles characterized by their dispersion relation, $E = \sqrt{m^2 + \boldsymbol{p}^2}$, in units $\hbar = c = 1$. The energy-momentum of a system of $n(\boldsymbol{p})$ particles with momentum $\boldsymbol{p}$ is

$$E = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) E(\boldsymbol{p}), \qquad \boldsymbol{P} = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) \boldsymbol{p} \tag{3}$$

Harmonic oscillator for each momentum: The only quantum system with similar, equidistant spectrum is the harmonic oscillator. Therefore we introduce a harmonic oscillator for each momentum value. The state of the system is characterized by the occupation number $n(\boldsymbol{p})$. The quantum state can be represented for pedestrians by a set of drawers, one for each value of the momentum and the drawer of momentum $\boldsymbol{p}$ contains $n(\boldsymbol{p})$ balls. The balls are indistinguishable within a drawer by construction because we know their numbers only.

We have canonical variables, $X_{\boldsymbol{p}}$ and $P_{\boldsymbol{q}}$, defined by the canonical commutation relations

$$[X(\boldsymbol{p}), P(\boldsymbol{q})]_\xi = i\delta(\boldsymbol{p} - \boldsymbol{q}), \qquad [X(\boldsymbol{p}), X(\boldsymbol{q})]_\xi = [P(\boldsymbol{p}), P(\boldsymbol{q})]_\xi = 0 \tag{4}$$

where

$$[A, B]_\xi = AB - \xi BA, \tag{5}$$

$\xi = \pm 1$ and the Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3} \left(\frac{P^2(\boldsymbol{p})}{2M(\boldsymbol{p})} + \frac{M(\boldsymbol{p})\omega^2(\boldsymbol{p})}{2} X^2(\boldsymbol{p}) \right) \tag{6}$$

In order to have the desired spectrum we choose

$$\omega(\boldsymbol{p}) = E(\boldsymbol{p}) = \sqrt{m^2 + \boldsymbol{p}^2}, \tag{7}$$

with arbitrary $M(\boldsymbol{p})$.

Quantum field: The operator algebra of harmonic oscillators is simpler in terms of the creation and destruction operators. Let us then introduce the creation and destruction operators

$$\begin{aligned} a(\boldsymbol{p}) &= \frac{M(\boldsymbol{p})\omega(\boldsymbol{p})X(\boldsymbol{p}) + iP(\boldsymbol{p})}{\sqrt{2M(\boldsymbol{p})\omega(\boldsymbol{p})}} \\ a^\dagger(\boldsymbol{p}) &= \frac{M(\boldsymbol{p})\omega(\boldsymbol{p})X(\boldsymbol{p}) - iP(\boldsymbol{p})}{\sqrt{2M(\boldsymbol{p})\omega(\boldsymbol{p})}} \end{aligned} \tag{8}$$

whose commutation relation according to Eq. (4) is

$$\left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right]_\xi = (2\pi)^3 2\omega(\boldsymbol{p})\delta(\boldsymbol{p}-\boldsymbol{q}), \qquad [a(\boldsymbol{p}), a(\boldsymbol{q})]_\xi = \left[a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right]_\xi = 0, \tag{9}$$

where the operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{q})$ were rescaled by the factor $\sqrt{(2\pi)^3 2\omega(\boldsymbol{p})}$ for later convenience.

Our intuition is better in coordinate than momentum space because the local nature of the basic laws of physics render the equations of motions simpler in the former case. Therefore, we introduce a linear superposition of the the momentum dependent, operator valued function $a(\boldsymbol{p})$, a formal Fourier integral to define the quantum field,

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})} a(\boldsymbol{p}) e^{i\boldsymbol{x}\boldsymbol{p}}. \tag{10}$$

D. Fock space

In order to become more familiar with the quantum system introduced above let us explore the structure of the Fock space the field operators act. We shall consider multi-particle states with bosonic and fermionic exchange statistics.

According to the commutation relations (62)-(63) we have an harmonic oscillator for each value of the momentum therefore, the Fock space is the direct product of the Hilbert spaces $\mathcal{H}_{\boldsymbol{p}}$ corresponding to the harmonic oscillator of the momentum $\boldsymbol{p}$,

$$\mathcal{H} = \otimes \prod_{\boldsymbol{p}} \mathcal{H}_{\boldsymbol{p}}. \tag{11}$$

Bosons: Let us consider states with well defined particle number. The simplest state is without any particle, the vacuum,

$$|0\rangle = \otimes \prod_{\boldsymbol{p}} |0\rangle_{\boldsymbol{p}}. \tag{12}$$

A state with a single particle with momentum $\boldsymbol{p}$ is

$$|\boldsymbol{p}\rangle = a^\dagger(\boldsymbol{p})|0\rangle. \tag{13}$$

A one-particle state characterized by the wave function $\Psi_1(\boldsymbol{p})$ in momentum space is

$$|\Psi_1\rangle = \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p})|0\rangle. \tag{14}$$

A two-particle state with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is

$$|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle. \tag{15}$$

Notice that the symmetry of the state with respect to the exchange,

$$|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle = a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle = |\boldsymbol{p}_2, \boldsymbol{p}_1\rangle = a^\dagger(\boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1)|0\rangle \tag{16}$$

imposes the vanishing canonical commutation relations (4) with $\xi = +1$.

A two-particle state with a given wave function is

$$|\Psi_2\rangle = \frac{1}{2} \int \tilde{d\boldsymbol{p}_1} \tilde{d\boldsymbol{p}_2} \Psi_2(\boldsymbol{p}_1, \boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle. \tag{17}$$

Finally, an arbitrary state can be written by means of infinitely many wave functions as

$$|\Psi\rangle = \left[\Psi_0 + \int \tilde{d\boldsymbol{p}} \Psi_1(\boldsymbol{p}) a^\dagger(\boldsymbol{p}) + \frac{1}{2} \int \tilde{d\boldsymbol{p}_1} \tilde{d\boldsymbol{p}_2} \Psi_2(\boldsymbol{p}_1, \boldsymbol{p}_2) a^\dagger(\boldsymbol{p}_1) a^\dagger(\boldsymbol{p}_2)|0\rangle + \cdots\right]|0\rangle. \tag{18}$$

Let us now introduce the particle number operator

$$N = \int \tilde{d\boldsymbol{k}} a^\dagger(\boldsymbol{p}) a(\boldsymbol{p}) \tag{19}$$

which identifies the vacuum by the condition

$$N|0\rangle = 0. \tag{20}$$

One can easily identify the particle number in the states (14) and (17),

$$N|\Psi_n\rangle = n|\Psi_n\rangle \tag{21}$$

by means of the commutation relations (62)-(63).

Fermions: The preceding discussion can easily be extended for fermions. The one and two particle states are

$$\begin{aligned}
|\Psi_1\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}}}[\Psi^p_{1\alpha}(\boldsymbol{p})c^\dagger_\alpha(\boldsymbol{p}) + \Psi^a_{1\alpha}(\boldsymbol{p})d^\dagger_\alpha(\boldsymbol{p})]|0\rangle \\
|\Psi_2\rangle &= \frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}_1}}\frac{d^3p_1}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{p}_1}}[\Psi^{pp}_{2\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2)c^\dagger_\alpha(\boldsymbol{p}_1)c^\dagger_\beta(\boldsymbol{p}_2) + \Psi^{pa}_{2\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2)c^\dagger_\alpha(\boldsymbol{p}_1)d^\dagger_\beta(\boldsymbol{p}_2) \\
&\quad + \Psi^{ap}_{2\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2)d^\dagger_\alpha(\boldsymbol{p}_1)c^\dagger_\beta(\boldsymbol{p}_2) + \Psi^{aa}_{2\alpha\beta}(\boldsymbol{p}_1,\boldsymbol{p}_2)d^\dagger_\alpha(\boldsymbol{p}_1)d^\dagger_\beta(\boldsymbol{p}_2)]|0\rangle
\end{aligned} \tag{22}$$

where the index p and a corresponds to particle or anti-particle state, respectively. The anti-symmetrization of the multi-particle states, eg.

$$|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle = c^\dagger(\boldsymbol{p}_1)c^\dagger(\boldsymbol{p}_2)|0\rangle = -|\boldsymbol{p}_2,\boldsymbol{p}_1\rangle = -c^\dagger(\boldsymbol{p}_2)c^\dagger(\boldsymbol{p}_1)|0\rangle \tag{23}$$

requires the replacement of the canonical commutation relations with canonical anti-commutation relations by setting $\xi = -1$ in Eq. (4).

Dynamics: The exponential function if the Fourier integral (10) is reminiscent of the wave function of a particle with momentum $\boldsymbol{p}$. The generalization for a time-dependent filed would be

$$\phi(t,\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})}a(\boldsymbol{p})e^{-it\omega(\boldsymbol{p})+i\boldsymbol{x}\boldsymbol{p}}. \tag{24}$$

We need harmonic oscillator only for the mass shell $\omega^2(\boldsymbol{p}) - \boldsymbol{p}^2 = m^2$ therefore we impose the condition

$$\int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})}a(\boldsymbol{p})[m^2 - \omega^2(\boldsymbol{p}) + \boldsymbol{p}^2]e^{-it\omega_{\boldsymbol{p}}+i\boldsymbol{x}\boldsymbol{p}} = (\Box + m^2)\phi(t,\boldsymbol{x}) = 0, \tag{25}$$

ie. either the harmonic oscillator is on the mass-shell or its coordinate and momentum are vanishing. The Lorentz invariance leads to an equation of motion, the Klein-Gordon equation in this manner.

The formal similarity of the field (24) with the one-particle wave function is the origin of the historic name, "second quantization", of quantum field theory. Because the representation of multi-particle systems by harmonic oscillator requires quantization, the first one. The result, a "wave function (24) which is operator values appears as if we had gone through a a second quantization process when the number valued wave function is replaced by operator valued function. But the analogy between wave function and the field operator holds for non-interacting particles only. In fact, there is no way to define the state of a single particle in an interactive quantum system since interaction generates entanglement.

The same time dependence can be obtained in a more systematic manner by solving the Heisenberg equation, Eq. (E5) for the quantum field. The Hamiltonian (6), written in terms of the creation and destruction operators,

$$H = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})}\omega(\boldsymbol{p})\left(a^\dagger(t,\boldsymbol{p})a(t,\boldsymbol{p}) + \frac{1}{2}\right) \tag{26}$$

(Since the Hamiltonian of a system of non-interacting particles is time independent we can take the creation and destruction operators in this expression at arbitrary time t.) leads to the equation of motion

$$i\partial_0 a(t,\boldsymbol{p}) = \int \frac{d^3p}{(2\pi)^3 2\omega(\boldsymbol{p})}\omega(\boldsymbol{p})[a(t,\boldsymbol{p}), a^\dagger(t,\boldsymbol{p})a(\boldsymbol{p})]_+. \tag{27}$$

Notice that we have commutator in the Heisenberg equation irrespectively of the exchange statistics of the particles handled by the field $\phi(x)$. The identity $[A,BC]_+ = B[A,C]_+ + [A,B]_+C$ and the canonical commutation relations (9) give the equation of motion

$$i\partial_0 a(t,\boldsymbol{p}) = \omega(\boldsymbol{p})a(t,\boldsymbol{p}), \tag{28}$$

with solution (24).

Though the free particle dynamics can be constructed in such a simple manner, interactions render the previous considerations far more involved. In order to find the possible interactions which are compatible with certain symmetries and work out the time dependence generated by them we turn to the variation formalism, introduced in Appendix A, in the following Chapter.

II. CANONICAL QUANTIZATION

The generalization of the heuristic treatment of noninteracting many-particle system presented in the previous Chapter for interactive particles can be achieved by means of the Lagrangian description.

A. Single particle

Let us start with a single d-dimensional particle, characterized by the Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}})$. The coordinates are $\boldsymbol{x}$, canonical momentum is defined as

$$\boldsymbol{p} = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}} \tag{29}$$

and the kinematic structure is given by imposing the Heisenberg commutation relation

$$[\hat{x}_j, \hat{p}_k] = i\delta_{j,k}, \qquad [\hat{x}_j, \hat{x}_k] = [\hat{p}_j, \hat{p}_k] = 0, \tag{30}$$

for the canonical operator pair. The operators will be equipped by a hat in this short introduction to canonical quantization for the sake of clarity. The well known solution of the canonical commutation relations is

$$\hat{\boldsymbol{x}}\psi(\boldsymbol{x}) = \boldsymbol{x}\psi(\boldsymbol{x}), \qquad \hat{\boldsymbol{p}}\psi(\boldsymbol{x}) = \frac{1}{i}\frac{\partial}{\partial \boldsymbol{x}}\psi(\boldsymbol{x}). \tag{31}$$

The dynamics is generated by the Hamiltonian

$$H = \dot{\boldsymbol{x}}\boldsymbol{p} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{32}$$

expressed in terms of the canonical pair $\boldsymbol{x}, \boldsymbol{p}$ as the Schrödinger equation,

$$i\partial_0|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{33}$$

for the state $|\psi\rangle$. The states are characterized by their wave function, $\psi(\boldsymbol{x}) = \langle \boldsymbol{x}|\psi\rangle$, obtained by projecting them on the coordinate eigenstates $|\boldsymbol{x}\rangle$, satisfying the condition

$$\hat{\boldsymbol{x}}|\boldsymbol{x}\rangle = \boldsymbol{x}|\boldsymbol{x}\rangle. \tag{34}$$

B. Quantum field

A classical, one real component scalar field theory can be obtained from this point particle by imagining it as a system with coordinate whose components are labeled by space points, $x_j \to x_{\boldsymbol{x}}$ which traditionally is denoted by $\phi(\boldsymbol{x})$ in the case of a scalar field. In other words the field theory is a dynamical system specified by the coordinate $\Phi(\boldsymbol{x})$, by a field configuration.

To render the subsequent mathematical expressions well defined one should introduce a lattice in the coordinate space, $\boldsymbol{x} \to a\boldsymbol{n}$, where a is the lattice spacing and $\boldsymbol{n} = (n_1, n_2, n_3)$ is a vector with integer components, $0 \le n_j \le N$. The minimal distance of this theory, a, is called ultraviolet (UV) cutoff. The maximal distance, $L = Na$, the size of the quantization box is usually called the infrared (IR) cutoff. The continuum limit $a \to 0$, performed by keeping the physical content of the theory fixed, is called renormalization. The removal of the IR cutoff, $L \to \infty$ is the thermodynamical limit. But the limiting procedure will be considered later only, we continue now our discussion with a small but finite a and a large but finite L and the coordinate of our system is the field configuration $\phi_{\boldsymbol{n}} = \phi(a\boldsymbol{n})$ which can be considered as a point in the N^3-dimensional coordinate space.

The action written as

$$S = \int dt a^3 \sum_{\boldsymbol{n}} L(\phi_{\boldsymbol{n}}, \nabla\phi_{\boldsymbol{n}}) = \int dt L_t[\phi, \partial\phi] \tag{35}$$

defines the Lagrangian

$$L_t[\phi, \partial\phi] = a^3 \sum_{\boldsymbol{n}} L(\phi_{\boldsymbol{n}}, \nabla\phi_{\boldsymbol{n}}) \tag{36}$$

where the space-derivatives in the Lagrangian are given as

$$\nabla_j \phi(\boldsymbol{x}) = \frac{\phi_{\boldsymbol{n}+\boldsymbol{e}_j} - \phi_{\boldsymbol{n}}}{a}, \tag{37}$$

$\boldsymbol{e}_j$ denoting the unit vector in the direction of j. The canonical momentum is

$$\pi_{\boldsymbol{n}} = \frac{\partial L_t[\phi, \nabla\phi]}{\partial \nabla_0 \phi_{\boldsymbol{n}}} = a^3 \frac{\partial L(\phi(x), \partial\phi(x))}{\partial \partial_0 \phi(a\boldsymbol{n})} \tag{38}$$

The quantization of the lattice regulated theory consists of replacing the canonical coordinates and momenta by operators, $\phi_{\boldsymbol{n}} \to \hat{\phi}_{\boldsymbol{n}}$, $\pi_{\boldsymbol{n}} \to \hat{\pi}_{\boldsymbol{n}}$, and imposing the Heisenberg commutation relation

$$[\hat{\phi}_{\boldsymbol{n}}, \hat{\pi}_{\boldsymbol{n}'}] = i\delta_{\boldsymbol{n},\boldsymbol{n}'}, \qquad [\hat{\phi}_{\boldsymbol{n}}, \hat{\phi}_{\boldsymbol{n}'}] = [\hat{\pi}_{\boldsymbol{n}}, \hat{\pi}_{\boldsymbol{n}'}] = 0. \tag{39}$$

The construction of the momentum for continuous space starts with the definition of the functional derivative $\frac{\delta}{\delta}$, replacing the partial derivative in the definition of the canonical momentum, Eq. (38),

$$\frac{\delta}{\delta\phi(a\boldsymbol{n})} = \lim_{a\to 0} \frac{1}{a^3} \frac{\partial}{\partial\phi_{\boldsymbol{n}}}. \tag{40}$$

The factor $1/a^3$ is needed in this expression to reproduce the relation

$$\frac{\delta}{\delta\phi(\boldsymbol{x})} \int d^3y f(\boldsymbol{y})\phi(\boldsymbol{y}) = \lim_{a\to 0} \frac{1}{a^3} \frac{\partial}{\partial\phi_{\boldsymbol{n}}} a^3 \sum_{\boldsymbol{n}'} f_{\boldsymbol{n}'}\phi_{\boldsymbol{n}'} = f_{\boldsymbol{n}} = f(\boldsymbol{x}). \tag{41}$$

which is reasonable to impose on a functional derivative. The Lagrangian

$$L_t[\phi, \partial\phi] = \int d^3x L(\phi(\boldsymbol{x})), \partial\phi(\boldsymbol{x})) \tag{42}$$

yields the canonical momentum

$$\pi(\boldsymbol{x}) = \frac{\delta L_t[\phi, \nabla\phi]}{\delta\partial_0\phi(\boldsymbol{x})} = \frac{\partial L_t(\phi(x), \partial\phi(x))}{\partial\partial_0\phi(x)}, \tag{43}$$

which satisfies

$$[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = i\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad [\hat{\phi}(\boldsymbol{x}), \hat{\phi}(\boldsymbol{y})] = [\hat{\pi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = 0. \tag{44}$$

The solution of the commutation relation is obtained by analogy with the usual, first quantized quantum mechanical representation of the momentum operator in coordinate space. The coordinate is the field operator $\hat{\phi}(\boldsymbol{x})$ acting on the wave functionals as

$$\hat{\phi}(\boldsymbol{x})\Psi[\phi(\cdot)] = \phi(\boldsymbol{x})\Psi[\phi(\cdot)] \tag{45}$$

where the symbol $\cdot$ stands for an unspecified space location in these equations. The corresponding momentum operator is

$$\hat{\pi}(\boldsymbol{x})\Psi[\phi(\cdot)] = \frac{1}{i}\frac{\delta}{\delta\phi(\boldsymbol{x})}\Psi[\phi(\cdot)]. \tag{46}$$

The wave functional

$$\Psi[\phi(\cdot)] = \langle \phi(\cdot)|\Psi\rangle \tag{47}$$

is defined by means of the field eigenstates

$$\phi(\boldsymbol{x})|\Phi(\cdot)\rangle = \Phi(\boldsymbol{x})|\Phi(\cdot)\rangle. \tag{48}$$

The time evolution is generated by the functional Schrödinger equation

$$i\partial_0|\Psi\rangle = \hat{H}|\Psi\rangle \tag{49}$$

with the Hamiltonian

$$H = \int d^d x[\partial_0\phi(\boldsymbol{x})\pi(\boldsymbol{x}) - L(\phi(\boldsymbol{x})), \partial\phi(\boldsymbol{x}))]. \tag{50}$$

For example, the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - U(\phi) \tag{51}$$

yields

$$\pi(\boldsymbol{x}) = \partial_0\phi(\boldsymbol{x})) \tag{52}$$

and

$$H = \int d^d x \left[-\frac{1}{2}\frac{\delta^2}{\delta\phi(\boldsymbol{x})\delta\phi(\boldsymbol{x})} + \frac{1}{2}(\boldsymbol{\partial}\phi(\boldsymbol{x}))^2 + \frac{m^2}{2}\phi^2(\boldsymbol{x}) + U(\phi(\boldsymbol{x}))\right]. \tag{53}$$

Another, equivalent way of obtaining the Hamiltonian is to use the time-component of the energy-momentum tensor (A52).

The calculation of the spectrum and the eigenstates of this Hamiltonian is exceeds our mathematical capabilities for $U(\phi) \neq 0$ but becomes easy in the absence of interaction, $U(\phi) = 0$, cf. Appendix I. To prepare the way for the perturbative treatment of the interactions we shall follow another, more algebraic construction of the quantum dynamics of non-interacting particles which is equivalent with the procedure outlined above.

C. Explicit solution for free neutral, scalar particle

Instead of working with the functional formalism, generating the dynamics by the functional Schrödinger equation we construct here directly the space-time dependent field operator as a Fourier integral,

$$\phi(t, \boldsymbol{x}) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)a(k)e^{-ik\cdot x}. \tag{54}$$

The desired dynamics is obtained by imposing the equation of motion $(\Box + m^2)\phi(x) = 0$. The operator valued Fourier transform $a(k)$ on the mass shell is the free parameter of the solution. The hermitian nature of the field operator, $\phi(x) = \phi^\dagger(x)$, requires $a(-k) = a^\dagger(k)$. Not all components of the energy-momentum vector k are independent and it is useful to express the energy in terms of the momentum,

$$k^0 = \omega_{\boldsymbol{k}} = +\sqrt{m^2 + \boldsymbol{k}^2}. \tag{55}$$

The positive and negative energy modes play different role in the dynamics thus it is advantageous to separate them from the very beginning. For this end we introduce the notation $a(\boldsymbol{k}) = a(k)$ where (55) is used to express k^0 on the right hand side and write

$$\phi(t, \boldsymbol{x}) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\Theta(k^0)[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}]_{|k^0=\omega_{\boldsymbol{k}}}. \tag{56}$$

One should bear in mind that the convention (55) is used always for the operator valued Fourier integrals. One integrates over k^0,

$$\phi(x) = \int d\tilde{\boldsymbol{k}}[a(\boldsymbol{k})e^{-ik\cdot x} + a^\dagger(\boldsymbol{k})e^{ik\cdot x}]. \tag{57}$$

The integration measure

$$d\tilde{\boldsymbol{k}} = \frac{d^4k}{(2\pi)^4}2\pi\delta(k^2 - m^2)\Theta(k^0) = \frac{d^3k}{(2\pi)^3 2\omega_{\boldsymbol{k}}} \tag{58}$$

is invariant under the proper Lorentz group thus the operator $a(\boldsymbol{k})$ is scalar.

The operator $\phi(x)$ introduced right now differs from the one mentioned in Eq. (48) in having explicit time dependence. It is easy to check that the Heisenberg equation of motion, introduced in Appendix E 1, generates the same time dependence for the field operator as seen in Eq. (56).

Let us now look at the canonical commutation relation. The Lagrangian (51) gives the canonical momentum

$$\pi(x) = -i\int d\tilde{k}\omega_{\boldsymbol{k}}[a(\boldsymbol{k})e^{-ik\cdot x} - a^\dagger(\boldsymbol{k})e^{ik\cdot x}] \tag{59}$$

and the equal time canonical commutation relation reads

$$\begin{aligned}
[\phi(x), \pi(y)]_{|x^0=y^0} &= -i\int d\tilde{\boldsymbol{k}}d\tilde{\boldsymbol{\ell}}\omega_{\boldsymbol{\ell}}[a(\boldsymbol{k})e^{-ikx} + a^\dagger(\boldsymbol{k})e^{ikx}, a(\boldsymbol{\ell})e^{-i\ell y} - a^\dagger(\boldsymbol{\ell})e^{i\ell y}]_{|x^0=y^0} \\
&= -i\int d\tilde{\boldsymbol{k}}d\tilde{\boldsymbol{\ell}}\omega_{\boldsymbol{\ell}}\Big\{\underbrace{[a(\boldsymbol{k}), a(\boldsymbol{\ell})]}_{0}e^{-i(kx+\ell y)} - \underbrace{[a^\dagger(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})]}_{0}e^{i(kx+\ell y)} \\
&\quad + \underbrace{[a^\dagger(\boldsymbol{k}), a(\boldsymbol{\ell})]}_{-(2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell})}e^{i(kx-\ell y)} - \underbrace{[a(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})]}_{(2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k}-\boldsymbol{\ell})}e^{-i(kx-\ell y)}\Big\}_{|x^0=y^0}
\end{aligned} \tag{60}$$

which is supposed to be

$$i\delta(\boldsymbol{x} - \boldsymbol{y}) = i\int \frac{d^3k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{61}$$

This is satisfied only if the commutation relations

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})] = (2\pi)^3 2\omega_{\boldsymbol{k}}\delta(\boldsymbol{k} - \boldsymbol{\ell}), \tag{62}$$

and

$$[a(\boldsymbol{k}), a(\boldsymbol{\ell})] = [a^\dagger(\boldsymbol{k}, a^\dagger(\boldsymbol{\ell})] = 0 \tag{63}$$

are imposed. These commutation relations show that we can indeed interpret $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$ as the annihilation and creation operator of excitations of an harmonic oscillator corresponding to each allowed value of the spatial momentum of a free particle.

Note that the coefficient of the Dirac-delta on the right hand side of the non-vanishing commutation relation must always be identical with the inverse of the integral measure used in the Fourier integral of the field variable. Such a different, relativistically invariant normalization of the creation and destruction operators in Eq. (57) as opposed to Eq. (24) which explains the different normalizations of the right hand sides of the non-trivial canonical commutation relations in Eqs. (9) and (62).

We find further support of this interpretation of these operators by writing the energy-momentum vector in terms of $a(\boldsymbol{k})$ and $a^\dagger(\boldsymbol{k})$. Straightforward calculation leads to the result

$$\begin{aligned}
P^\mu &= \int d^3x T^{0\mu} \\
&= \int d^3x\left[\partial_0\phi\partial^\mu\phi - g^{0\mu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)\right] \\
&= \frac{1}{2}\int d\tilde{\boldsymbol{k}}k^\mu[a(\boldsymbol{k})a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k})a(\boldsymbol{k})],
\end{aligned} \tag{64}$$

c.f. Appendix **??**.

The commutation relation (63) can be used to obtain finally the form

$$P^\mu = \int d\tilde{\boldsymbol{k}} k^\mu a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + g^{0\mu} V \int \frac{d^3k}{(2\pi)^3} \frac{\omega_{\boldsymbol{k}}}{2}, \tag{65}$$

where the volume $V/(2\pi)^3$ is actually the value of the momentum space Dirac-delta on the right hand side of (63) at $\boldsymbol{k} = \boldsymbol{\ell}$. It is easy to understand the first term on the right hand side, it is the sum of the energy-momentum for the particles in different momentum sector of the the theory. The momentum is always additive, the additivity of the energy is valid for non-interacting particles only.

The second term represents the sum of energy of the zero-point fluctuations of the harmonic oscillators. This expression is ultraviolet (large momentum) divergent but being a number we can subtract from the energy without changing the dynamics (The only interaction which is coupled to the energy is gravitation which is ignored here.) One can formally make this subtraction by introducing the normal ordered product $A \to :A:$. The normal ordered product of the creation and annihilation operators is the product of the same operators written in an order where the annihilation operators precede the creation operators, eg. $:a(\boldsymbol{p}_1)a(\boldsymbol{p}_1) := a(\boldsymbol{p}_2)a(\boldsymbol{p}_1)$, $:a(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2) := a^\dagger(\boldsymbol{p}_2)a(\boldsymbol{p}_1)$, etc. and define $P^\mu \to :P^\mu:$.

D. Free charged scalar particle

Most of the particles participate in interactions where the particle number is conserved. The simplest way to impose this conservation law is the introduce a continuous symmetry. In case of a single conserved charge thee natural symmetry group is one dimensional, $G = U(1)$. The corresponding field variable $\phi(x) = \frac{1}{\sqrt{2}}[\phi_1(x) + i\phi_2(x)]$ is complex and the symmetry transformation is $\phi(x) \to e^{i\Phi}\phi(x)$. The simplest way of constructing an $U(1)$ invariant Lagrangian without self-interaction is to add up two identical free Lagrangians for the two real field,

$$\begin{aligned} L &= \sum_{a=1,2} \left[\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{2} \phi_a^2 \right] \\ &= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi. \end{aligned} \tag{66}$$

Self-interaction can be introduced by the potential $L \to L - U(\phi^\dagger\phi)$. We can treat either the real fields ϕ_1 and ϕ_2 or the complex fields ϕ and $\phi^\dagger$ as independent variables. The latter choice is preferred because it leads to simpler expressions.

The canonical momentum of the free system (66) is

$$\pi(x) = \frac{\partial L}{\partial \partial_0 \phi} = \partial_0 \phi^\dagger, \qquad \pi^\dagger(x) = \frac{\partial L}{\partial \partial_0 \phi^\dagger} = \partial_0 \phi \tag{67}$$

and the non-vanishing canonical commutation relations

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{y})] = [\phi^\dagger(\boldsymbol{x}), \pi^\dagger(\boldsymbol{y})] = i\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{68}$$

The plane-wave representation yields

$$\phi(t, \boldsymbol{x}) = \int d\tilde{\boldsymbol{k}} [a(\boldsymbol{k}) e^{-ik\cdot x} + b^\dagger(\boldsymbol{k}) e^{ik\cdot x}] \tag{69}$$

with the non-vanishing commutators

$$[a(\boldsymbol{k}), a^\dagger(\boldsymbol{\ell})] = [b(\boldsymbol{k}), b^\dagger(\boldsymbol{\ell})] = (2\pi)^3 2\omega_{\boldsymbol{k}} \delta(\boldsymbol{k} - \boldsymbol{\ell}). \tag{70}$$

We have two kinds of particles, both characterized by the same dispersion relation $\omega_{\boldsymbol{k}}$. In fact, the energy-momentum is

$$\begin{aligned} P^\mu &= \int d^3x T^{0\mu} \\ &= \int d^3x : \left[\partial_0 \phi^\dagger \partial^\mu \phi + \partial^\mu \phi^\dagger \partial_0 \phi - g^{0\mu} (\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi) \right] : \\ &= \frac{1}{2} \int d\tilde{\boldsymbol{k}} k^\mu : [a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) + a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + b(\boldsymbol{k}) b^\dagger(\boldsymbol{k}) + b^\dagger(\boldsymbol{k}) b(\boldsymbol{k})] : \\ &= \int d\tilde{\boldsymbol{k}} k^\mu [a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + b^\dagger(\boldsymbol{k}) b(\boldsymbol{k})]. \end{aligned} \tag{71}$$

The difference between these particles can be read off from the charge

$$Q(t) = \int d^3x j^0(t, \boldsymbol{x}) \int d\tilde{\boldsymbol{k}}[a^\dagger(\boldsymbol{k})a(\boldsymbol{k}) - b^\dagger(\boldsymbol{k})b(\boldsymbol{k})] \tag{72}$$

belonging to the Noether current

$$j_\mu = i : \phi^\dagger \overset{\leftrightarrow}{\partial}_\mu \phi : \tag{73}$$

where $f(x)\overset{\leftrightarrow}{\partial}_\mu g(x) = f(x)\partial_\mu g(x) - (\partial_\mu f(x))g(x)$. One particle is anti-particle of the other and has opposite conserved internal quantum numbers. The current corresponding to the global $U(1)$ phase symmetry is identified with the electromagnetic current.

E. Neutral vector field

The dynamics of a real vector field $A_\mu(x)$ is usually governed by the Maxwell Lagrangian,

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{74}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The special feature of this Lagrangian is that it contains no $\partial_0 A_0$, the temporal component A_0 has no dynamics. This is the result of gauge invariance, the condition that the Lagrangian and physical quantities must remain invariant under the transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Phi(x). \tag{75}$$

This circumstance leads to serious problems which can be settled by involved constructions, the separation physical and non-physical sectors in the theory. A quantity is called physical and belong the physical sector if it is invariant under gauge transformation. The best way to separate these sectors is the use of the projectors into transverse $A_{T\mu}(x) = T_{\mu\nu}A^\nu(x)$ and longitudinal $A_{L\mu}(x) = L_{\mu\nu}A^\nu(x)$, components of the vector field with

$$T_{\mu\nu} = g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}, \qquad L_{\mu\nu} = \frac{\partial_\mu\partial_\nu}{\Box} \tag{76}$$

which become

$$T_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \qquad L_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \tag{77}$$

in Fourier space. Gauge transformations influence the longitudinal components only and the aforementioned problem of the dynamics of $A_0(x)$ can be avoided by adding a so called gauge fixing term to the Lagrangian which is now written as

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{2}(\partial^\mu A_\mu)^2 \tag{78}$$

The longitudinal and transverse components decouple in the dynamics and follow independent time evolution therefore the gauge fixing term does not modify the dynamics in the physical sector. The equation of motion is

$$\Box A_\mu - (1-\lambda)\partial_\mu\partial_\nu A^\nu = 0 \tag{79}$$

and the generalized momenta are given by

$$\Pi^\mu = \frac{\delta L}{\delta\partial_0 A_\mu} = F^{\mu 0} - \lambda g^{\mu 0}\partial^\nu A_\nu. \tag{80}$$

The canonical quantization is the replacement of the classical field variables by operators satisfying the canonical commutation relations

$$[A_\mu(t, \boldsymbol{x}), \pi_\nu(t, \boldsymbol{y})] = -ig_{\mu\nu}\delta(\boldsymbol{x} - \boldsymbol{y}). \tag{81}$$

The left hand side is a tensor and the only tensor at our disposal which may appear on the right hand side is the metric tensor $g_{\mu\nu}$. Another complication of the dynamics of vector fields is that the metrics is non-definite, ie. there

will be components of the vector field which have the wrong sign in their canonical commutation relation. The overall sign in Eq. (81) is chosen in such a manner that most of the components have the right sign. This leaves A_0 with the wrong sign which problem is usually fixed by the Gupta-Bleuer quantization performed in Fock-space equipped with indefinite scalar product. We do not follow this line of thought here and remark only that the plane-wave decomposition of the vector field is

$$A_\mu(x) = \int d\tilde{\boldsymbol{k}} \sum_{\lambda=0}^{3} [a_\lambda(\boldsymbol{k})\epsilon_{\lambda\mu}(\boldsymbol{k})e^{-ik\cdot x} + a_\lambda^\dagger(\boldsymbol{k})\epsilon_{\lambda\mu}(\boldsymbol{k})e^{ik\cdot x}], \tag{82}$$

where the four polarization vector, $\epsilon_{\lambda\mu}(\boldsymbol{k})$ are usually defined in the following manner. Let us introduce a time-like unit vector n^μ, $n^\mu n_\mu = 1$ and set $\epsilon_{0\mu}(\boldsymbol{k}) = n^\mu$ for the time-like polarization state. The longitudinal polarization is $\epsilon_{3\mu}(\boldsymbol{k}) = (\omega_{\boldsymbol{k}}, \boldsymbol{k})$ and the transverse states are defined by two orthogonal directions $\epsilon_{\lambda\mu}(\boldsymbol{k})$, $\lambda = 1, 2$ which are orthogonal to $\epsilon_{0\mu}(\boldsymbol{k})$ and $\epsilon_{3\mu}(\boldsymbol{k})$, too. One arrives finally at the condition

$$\epsilon_\lambda^\mu(\boldsymbol{k})\epsilon^*_{\lambda'\mu}(\boldsymbol{k}) = g_{\lambda\lambda'}. \tag{83}$$

The non-vanishing canonical commutation relations for the creation and annihilation operators are

$$[a_\lambda(\boldsymbol{k}), a_{\lambda'}^\dagger(\boldsymbol{\ell})] = -g_{\lambda\lambda'} 2\omega_{\boldsymbol{k}} \delta(\boldsymbol{k} - \boldsymbol{\ell}). \tag{84}$$

F. Charged fermion

The Lagrangian for charged, non-interacting fermions is

$$L =\to \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi} \to \psi\bar{\psi}[i\partial_\mu\gamma^\mu - m]\psi. \tag{85}$$

The corresponding energy-momentum tensor is

$$T^{\mu\nu} = \partial^\nu\bar{\psi}\frac{\partial L}{\partial\partial_\mu\bar{\psi}} + \frac{\partial L}{\partial\partial_\mu\psi}\partial^\nu\psi - g^{\mu\nu}L = \frac{i}{2}[\bar{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\bar{\psi}\gamma^\mu\psi] \tag{86}$$

and the Noether current reads as

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{87}$$

The plane-wave representation of the solution of the Dirac equation is

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} \sum_{\alpha=1,2} [c_\alpha(\boldsymbol{k})u^{(\alpha)}(\boldsymbol{k})e^{-ik\cdot x} + d_\alpha^\dagger(\boldsymbol{k})v^{(\alpha)}(\boldsymbol{k})(\boldsymbol{k})e^{ik\cdot x}] \tag{88}$$

where the bi-spinors $u^{(\alpha)}(\boldsymbol{k})$ and $v^{(\alpha)}(\boldsymbol{k})$ are given by Eqs. (D66). The energy-momentum vector in terms of the operators $c_\alpha(\boldsymbol{k})$ and $d_\alpha(\boldsymbol{k})$ is

$$\begin{aligned} P^\mu &= \int d^3x T^{0\mu} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}}k^\mu \sum_{\alpha=1,2} [c_\alpha^\dagger(\boldsymbol{k})c_\alpha(\boldsymbol{k}) - d_\alpha(\boldsymbol{k})d_\alpha^\dagger(\boldsymbol{k})] \end{aligned} \tag{89}$$

the negative sign on the right hand side coming from that of the normalization of v in Eq. (D67). The canonical commutation relations, such as Eqs. (62)-(63) would yield unbounded Hamiltonian from below and the system had no ground state due to the negative sign mentioned above. To avoid this problem on imposes canonical anti-commutation relations,

$$\begin{aligned} (2\pi)^3 2\omega_{\boldsymbol{k}}\delta_{\alpha,\beta}\delta(\boldsymbol{k} - \boldsymbol{\ell}) &= \{c_\alpha(\boldsymbol{k}), c_\beta^\dagger(\boldsymbol{\ell})\} = \{d_\alpha(\boldsymbol{k}), d_\beta^\dagger(\boldsymbol{\ell})\}, \\ 0 &= \{c_\alpha(\boldsymbol{k}), c_\beta(\boldsymbol{\ell})\} = \{d_\alpha(\boldsymbol{k}), d_\beta(\boldsymbol{\ell})\} \end{aligned} \tag{90}$$

for fermion fields.

The energy-momentum vector after the subtraction of the divergent contribution of the zero-point fluctuations, realized by the use of the normal ordered product,

$$P^\mu = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}}k^\mu \sum_{\alpha=1,2}[c^\dagger_\alpha(\boldsymbol{k})c_\alpha(\boldsymbol{k}) + d^\dagger_\alpha(\boldsymbol{k})d_\alpha(\boldsymbol{k})]. \tag{91}$$

The definition of the normal ordering is the same as for bosonic operators except a sign factor corresponding to each exchange, eg. $:c_\alpha(\boldsymbol{p}_1)c_\beta(\boldsymbol{p}_1) := -c_\beta(\boldsymbol{p}_2)c_\alpha(\boldsymbol{p}_1)$, $:c_\alpha(\boldsymbol{p}_1)c^\dagger_\beta(\boldsymbol{p}_2) := -c^\dagger_\beta(\boldsymbol{p}_2)c_\alpha(\boldsymbol{p}_1)$.

The Noether current corresponding to the $\psi(x) \to e^{i\Phi}\psi(x)$ $U(1)$ symmetry, identified by the electric current, is

$$Q = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{\boldsymbol{k}}} \sum_{\alpha=1,2}[c^\dagger_\alpha(\boldsymbol{k})c_\alpha(\boldsymbol{k}) - d^\dagger_\alpha(\boldsymbol{k})d_\alpha(\boldsymbol{k})]. \tag{92}$$

III. SCATTERING PROCESSES

Neither the coordinate $\boldsymbol{x}$ nor the momentum $\boldsymbol{p}$ can be defined experimentally with arbitrary precision in high energy physics. The fine resolution in space finds the vacuum-polarization pairs around a point-like charge, the e^+e^- pairs in the case of an electron e^-. The equivalence of fermionic states leads to the indistinguishability of the original valence electron and the electron of the polarized pairs and introduces a finite charge distribution around even point-like charges. The dispersion relation $E = \sqrt{m^2 + \boldsymbol{p}^2}$ becomes $E = |\boldsymbol{p}|$ at high energy and the uncertainty relation $\Delta pt \approx \Delta Et \approx \hbar$ introduces the error $\Delta p \approx \hbar/t$ whcich diverges because $t \to 0$ for fast processes. But scattering cross sections remain measurable and serve as basic observables at high energy.

A. Asymptotic states

Let us suppose that the interactions between the particles making up the beam and the target is short ranged and therefore can be neglected before and after the collision. One can then introduce non-interacting asymptotic states, $|i\rangle$ and $|f\rangle$ for the description of the the system before and after the collision, respectively. These states should contain well separated wave-packets, characterized by their average momenta, because any overlap between the packets induces interactions. The asymptotic in and our spaces are unitary equivalent when any bound state formation between the particles is ignored. The time evolution gives a unitary map of the asymptotic in space into the asymptotic out space, in particular $|i\rangle \to |f\rangle = S|i\rangle$ where

$$S = T[e^{-i\int_{-\infty}^{\infty} dtH_i(t)}] \tag{93}$$

in the interaction representation. Let us denote the free field operators belonging to the asymptotic in and out particles by $\phi_i(x)$ and $\phi_o(x)$, respectively. One expects then that the true, interacting field operator interpolates between the asymptotic fields,

$$\lim_{t\to\pm\infty}\phi(t,\boldsymbol{x}) = \phi^{\,o}_{i}(t,\boldsymbol{x}). \tag{94}$$

But the problem is that the asymptotic states and fields are well defined for well separated wave-packets only,

$$\lim_{t\to\pm\infty}\langle f|\phi(t,\boldsymbol{x})|i\rangle = \langle f|\phi^{\,o}_{i}(t,\boldsymbol{x})|i\rangle. \tag{95}$$

Such a weak relations are not sufficient to establish a strong, operator equation like Eq. (94) which will be violated for products of operators which are evaluated by inserting the resolution of the identity as a summation over all basis states. A formal consideration, namely that the omission of the interaction from the equations of motion leaves behind linear equations whose solutions are well defined up to multiplicative constant, suggests that the relation

$$\phi^{\,o}_{i}(t,\boldsymbol{x}) = \sqrt{Z^{\,o}_{i}}\phi(x) \tag{96}$$

can be used in weak equations, ie. in matrix elements between well separated wave-packets. Time inversion imposes the constraint $Z_i = Z_o = Z$.

B. Cross section

Let us consider an inelastic scattering process of two scalar particles going into n particles, $1,2 \to 1,\ldots,n$. The initial state will be given by

$$|i\rangle = \int d\tilde{\boldsymbol{p}}_1 d\tilde{\boldsymbol{p}}_2 \psi_1(\boldsymbol{p}_1)\psi_2(\boldsymbol{p}_1)|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle \tag{97}$$

and the transition probability $W_{f\leftarrow i} = |\langle f|i(t=\infty)\rangle|^2$ is given in terms of the scattering amplitude

$$\langle f|i(t=\infty)\rangle = \langle f|S|i\rangle. \tag{98}$$

We are not interested in trivial forward scattering and separate off a trivial energy-momentum conserving Dirac-delta by writing

$$\langle f|S|i\rangle = i\langle f|T|i\rangle = \int d\tilde{\boldsymbol{p}}_1 d\tilde{\boldsymbol{p}}_2 \psi_1(\boldsymbol{p}_1)\psi_2(\boldsymbol{p}_1) i(2\pi)^4\delta(p_f - p_1 - p_2)\langle f|\mathcal{T}|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle \tag{99}$$

where $S = \mathbb{1} + iT$ and the matrix elements of $\mathcal{T}$ are obtained from those of T by separating off a singular function of the four-momenta, the Dirac delta standing for the energy-momentum conservation. Therefore, we find

$$\begin{aligned} W_{f\leftarrow i} \;=\; & \int d\tilde{\boldsymbol{p}}_1 d\tilde{\boldsymbol{p}}_2 d\tilde{\boldsymbol{q}}_1 d\tilde{\boldsymbol{q}}_2 \psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1)(2\pi)^4\delta(p_f - p_1 - p_2)(2\pi)^4\delta(q_1 + q_2 - p_1 - p_2) \\ & \times\langle f|\mathcal{T}|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle^*\langle f|\mathcal{T}|\boldsymbol{q}_1,\boldsymbol{q}_2\rangle. \end{aligned} \tag{100}$$

For sufficiently monochromatic (but still non-overlapping!) wave-packets we have $\langle f|\mathcal{T}|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle \approx \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle$ where $\bar{p}_j$ is the average momentum of the j-th particle and we simplify the transition probability as

$$\begin{aligned} W_{f\leftarrow i} \;=\; & \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle^*\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle \int d\tilde{\boldsymbol{p}}_1 d\tilde{\boldsymbol{p}}_2 d\tilde{\boldsymbol{q}}_1 d\tilde{\boldsymbol{q}}_2 \psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1) \\ & \times(2\pi)^4\delta(p_f - p_1 - p_2)(2\pi)^4\delta(q_1 + q_2 - p_1 - p_2) \\ =\; & \langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle^*\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle \int d^4x d\tilde{\boldsymbol{p}}_1 d\tilde{\boldsymbol{p}}_2 d\tilde{\boldsymbol{q}}_1 d\tilde{\boldsymbol{q}}_2 \psi_1^*(\boldsymbol{p}_1)\psi_2^*(\boldsymbol{p}_1)\psi_1(\boldsymbol{q}_1)\psi_2(\boldsymbol{q}_1) \\ & \times(2\pi)^4\delta(p_f - p_1 - p_2)e^{-ix\cdot(q_1+q_2-p_1-p_2)} \\ =\; & |\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2(2\pi)^4\delta(p_f - \bar{p}_1 - \bar{p}_2)\int d^4x |\psi_1(\boldsymbol{x})|^2|\psi_2(\boldsymbol{x})|^2 \end{aligned} \tag{101}$$

where

$$f(x) = \int d\tilde{\boldsymbol{p}} e^{-ix\cdot p} f(p)_{|p^0=\omega_{\boldsymbol{q}}}. \tag{102}$$

Eq. (101) tells that the transition probability density in the space-time is

$$\frac{dW_{f\leftarrow i}}{dV\,dt} = (2\pi)^4\delta(p_f - \bar{p}_1 - \bar{p}_2)|\psi_1(\boldsymbol{x})|^2|\psi_2(\boldsymbol{x})|^2|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2. \tag{103}$$

In the next step we factorize off the characteristic quantities of the beam and the target. Let us suppose that particle 1 makes up the beam and particle 2 is the target, at rest in the laboratory frame. For sufficiently monochromatic state we have the flux

$$i\psi^*(x)\overset{\leftrightarrow}{\partial}_\mu\psi(x) \approx 2\bar{p}_\mu|\psi(x)|^2. \tag{104}$$

giving the target particle density

$$\frac{dn_2}{dV} = 2\bar{p}_2^0|\psi_2(x)|^2, \qquad \bar{p}_2^0 = m_2 \tag{105}$$

and the incident flux of the beam

$$\boldsymbol{j}_b = \underbrace{\frac{\bar{\boldsymbol{p}}_1}{\bar{p}_1^0}}_{\text{velocity}} \cdot \underbrace{2\bar{p}_1^0|\psi_1(x)|^2}_{\text{density}} = 2\bar{\boldsymbol{p}}_1|\psi_1(x)|^2. \tag{106}$$

The normalized transition probability (103) defines the Lorentz invariant differential cross section $d\sigma$ by

$$\frac{dW_{f\leftarrow i}}{d^3xdt} = \frac{dn_2}{d^3x}|\boldsymbol{j}_b|d\sigma, \tag{107}$$

in particular

$$\begin{aligned} d\sigma &= (2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)\frac{|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2}{4m_2|\bar{\boldsymbol{p}}_1|} \\ &= (2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)\frac{|\langle f|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2}{4\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}}. \end{aligned} \tag{108}$$

As an example let us suppose that the final state contains n scalar particles with momenta constrained in the region $\Delta\subset\mathbb{R}^{3n}$,

$$d\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2-m_1^2m_2^2}}\int_\Delta d\tilde{\boldsymbol{q}}_1\cdots d\tilde{\boldsymbol{q}}_n(2\pi)^4\delta(p_f-\bar{p}_1-\bar{p}_2)|\langle \boldsymbol{q}_1\cdots\boldsymbol{q}_n|\mathcal{T}|\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{p}}_2\rangle|^2. \tag{109}$$

The general lesson of this particular example is that the experimentally accessible cross sections are expressed in terms of trivial kinematic factors characterizing the detectors and the transition amplitude magnitude square. The latter is usually obtained in the framework of the perturbation expansion.

IV. PERTURBATION EXPANSION

The only general method to calculate transition amplitudes is based on perturbation expansion. The perturbation series is usually written in terms of the Green-functions.

A. Green functions

Let us write the Hamiltonian as $H = H_0 + H_i$ where the perturbation, H_i, is a space of the interaction energy density, $H_i(t,\boldsymbol{x})$, in the interaction representation. For example, in the case of neutral, self-interacting scalar particles, described by the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 \tag{110}$$

the interaction Hamiltonian is

$$H_i = \frac{g}{4!}\int d^3x\phi^4(t,\boldsymbol{x}). \tag{111}$$

The initial and final states are created from the vacuum by the application of the asymptotic in and out field operators, therefore we write $|i\rangle = \phi_i(y_1)\cdots\phi_i(y_{m_i})|0\rangle$, $|f\rangle = \phi_f(z_1)\cdots\phi_f(z_{m_f})|0\rangle$ where $y_1^0,\ldots,y_{m_i}^0\to-\infty$ and $z_1^0,\ldots,z_{m_f}^0\to\infty$. The transition amplitude is

$$\begin{aligned} \mathcal{A} &= \langle 0|\phi_f(z_1)\cdots\phi_f(z_{m_f})T[e^{-i\int d^4xH_i(x)}]\phi_i(y_1)\cdots\phi_i(y_{m_i})|0\rangle_0 \\ &= Z^{-\frac{1}{2}(n_i+n_f)}\langle 0|T[\phi(z_1)\cdots\phi(z_{m_f})e^{-i\int d^4xH_i(x)}\phi(y_1)\cdots\phi(y_{m_i})]|0\rangle_0, \end{aligned} \tag{112}$$

where the definition of the time ordered product allows us to put the asymptotic fields under the time ordering. Notice the index 0, reminding that the expectation value refers to free field operators (as always in the interaction representations). The perturbation for these Green function consists of the expansion of the S-matrix,

$$\begin{aligned} \mathcal{A} &= \langle 0|T[\phi_f(z_1)\cdots\phi_f(z_{m_f})e^{-\frac{ig}{4!}\int d^4x\phi^4(x)}\phi_f(y_1)\cdots\phi_f(y_{m_i})]|0\rangle_0 \\ &= \sum_{n=0}^\infty\frac{1}{n!}\left(\frac{-ig}{4!}\right)^n\int dx_1\cdots dx_n\langle 0|T[\phi_f(z_1)\cdots\phi_f(z_{m_f})\phi^4(x_1)\cdots\phi^4(x_n)\phi_f(y_1)\cdots\phi_f(y_{m_i})]|0\rangle_0, \end{aligned} \tag{113}$$

and the result is that the transition amplitude becomes a sum of vacuum expectation values of the time ordered product of the free, elementary field operators.

The n-th order Green function, briefly the free n-point function of a free, hermitian field operator $\phi(x)$ is the vacuum expectation value

$$i^{n/2}G_0(x_1,\ldots,x_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle_0. \tag{114}$$

The index 0 is to reminder us to take the expectation value for non-interacting particles. For conserved charge the field operator is non-hermitian and we define

$$i^{n}G_0(x_1,\ldots,x_m,y_1,\ldots,y_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_m)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]|0\rangle_0. \tag{115}$$

The dynamics, and in particular the vacuum state $|0\rangle$ and the Green functions, remain invariant under the phase transformation $\psi(x)\to e^{i\Phi}\psi(x)$, $\psi^\dagger(x)\to e^{-i\Phi}\psi^\dagger(x)$ therefore $G=0$ for $m\neq n$ and we have the definition

$$i^{n/2}G_0(x_1,\ldots,x_n,y_1,\ldots,y_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_n)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]|0\rangle_0 \tag{116}$$

for the n-point function which describes the propagation of n particles or anti-particles. We start at the vacuum at the right of the matrix element with no excitations. Since a particle (anti-particle) is created by ϕ ($\phi^\dagger$) and is removed by $\phi^\dagger$ (ϕ) we need the same number of ϕ as $\phi^\dagger$ in the expectation value in order to have non-vanishing result. This is another argument for the vanishing of Green functions with $n\neq m$.

B. Propagator

First we start with the Green function with $n=1$, called the propagator,

$$\begin{aligned} iG_0(x,x') &= \langle 0|T[\phi(x)\phi^\dagger(x')]|0\rangle \\ &= \int \tilde{d}k\tilde{d}q\langle 0|T[(a(\boldsymbol{k})e^{-ik\cdot x}+b^\dagger(\boldsymbol{k})e^{ik\cdot x})(a^\dagger(\boldsymbol{q})e^{iq\cdot x'}+b(\boldsymbol{q})e^{-iq\cdot x'})]|0\rangle \\ &= \int \tilde{d}k\tilde{d}q[\Theta(t-t')\langle 0|a(\boldsymbol{k})a^\dagger(\boldsymbol{q})|0\rangle e^{iq\cdot x'-ik\cdot x}+\Theta(t'-t)\langle 0|b(\boldsymbol{q})b^\dagger(\boldsymbol{k})|0\rangle e^{ik\cdot x-iq\cdot x'}] \end{aligned} \tag{117}$$

where the properties $a(\boldsymbol{p})|0\rangle = b(\boldsymbol{p})|0\rangle = 0$ of the vacuum were used in the last equation. The canonical commutation relation gives finally

$$iG_0(x,x') = \int \tilde{d}k[\Theta(t-t')e^{-ik\cdot(x-x')}+\Theta(t'-t)e^{ik\cdot(x-x')}]. \tag{118}$$

What happened in the equations can be said in words in the following manner: Depending on the order of the time argument either $\phi\approx a+b^\dagger$ or $\phi^\dagger\approx b+a^\dagger$ acts first on the vacuum and an anti-particle or a particle is created. This object must be removed by the action of the second operator in order to arrive at the vacuum and yield a non-vanishing contribution. The two-point functions $G(x,y)$ therefore describes the propagation (transition amplitude) of a particle or anti-particle for $x^0>y^0$ or $x^0<y^0$, respectively.

There is another, more elegant and general procedure to obtain a simple expression for the propagator. For this end we start with an apparently more complicated problem, the proof that the operator $T[\phi(x)\phi^\dagger(y)]$ satisfies the equation

$$(\Box_x+m^2)T[\phi(x)\phi^\dagger(y)] = -i\delta(x-y)\mathbb{1} \tag{119}$$

off the mass-shell. The left hand side is an operator in two senses: It is an operator acting in the Fock space and can be viewed as another operator which acts on functions, defined on the space-time. The right hand side states that both operators are proportional to the identity. The equation (119) can not be valid within the null-space of the Klein-Gordon operator. (The null-space of a linear operator consists of vectors which are mapped into the zero vector.) The null-space of the equation of motion is the mass-shell, it contains just the free particle plane waves in our case. The identity for space-time functions, $\delta(x-y)$, must be replaced by zero on the right hand side hence the properties of the operator $T[\phi(x)\phi^\dagger(y)]$ can not be read off from eq. (119) within the null-space.

By taking the expectation value of this equation between the vacuum we get

$$(\Box_x+m^2)\langle 0|T[\phi(x)\phi^\dagger(y)]|0\rangle = (\Box_x+m^2)iG(x,y) = -i\delta(x-y) \tag{120}$$

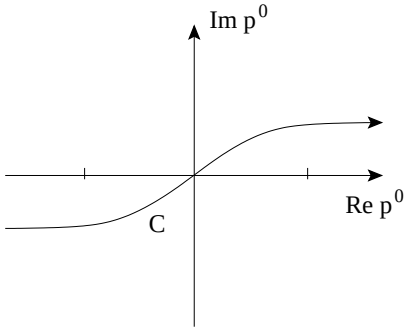


FIG. 1: Poles of the free propagator and the integration contour, corresponding to the Feynman propagator, Eq. (117).

namely the propagator is the inverse of the Klein-Gordon operator, a genuine Green function in the sense of the mathematical theory of linear differential equations. There are different Green-functions, depending on their definition on the mass-shell. We introduce below a single Green-function only. It will be the causal Green-function, needed for the construction of transition amplitudes.

The proof of (119) starts with the definition of the time ordered product,

$$(\Box_x + m^2)T[\phi(x)\phi^\dagger(y)] = (\Box_x + m^2)[\Theta(x^0 - y^0)\phi(x)\phi^\dagger(y) + \Theta(y^0 - x^0)\phi^\dagger(y)\phi(x)] \tag{121}$$

where contribution not involving time derivative can be brought into the time ordering,

$$(\Box_x + m^2)T[\phi(x)\phi^\dagger(y)] = T[(-\Delta_{\boldsymbol{x}} + m^2)\phi(x)\phi^\dagger(y)] + \partial^2_{x^0}[\Theta(x^0 - y^0)\phi(x)\phi^\dagger(y) + \Theta(y^0 - x^0)\phi^\dagger(y)\phi(x)]. \tag{122}$$

The two time derivatives give

$$\begin{aligned}(\Box_x + m^2)T[\phi(x)\phi^\dagger(y)] &= T[(-\Delta_{\boldsymbol{x}} + m^2)\phi(x)\phi^\dagger(y)] + \partial_{x^0}\{T[\partial_0\phi(x)\phi^\dagger(y)] + \delta(x^0 - y^0)[\phi(x)\phi^\dagger(y) - \phi^\dagger(y)\phi(x)]\} \\ &= T[(-\Delta_{\boldsymbol{x}} + m^2)\phi(x)\phi^\dagger(y)] + \partial_{x^0}\{T[\partial_0\phi(x)\phi^\dagger(y)] + \delta(x^0 - y^0)[\phi(x), \phi^\dagger(y)]\}. \end{aligned} \tag{123}$$

The equal time commutator in the last line, being taken for coordinates only without momenta, is vanishing, thus

$$\begin{aligned}(\Box_x + m^2)T[\phi(x)\phi^\dagger(y)] &= T[(-\Delta_{\boldsymbol{x}} + m^2)\phi(x)\phi^\dagger(y)] + \partial_{x^0}T[\partial_0\phi(x)\phi^\dagger(y)] \\ &= T[(-\Delta_{\boldsymbol{x}} + m^2)\phi(x)\phi^\dagger(y)] + T[\partial_0^2\phi(x)\phi^\dagger(y)] + \delta(x^0 - y^0)[\partial_0\phi(x), \phi^\dagger(y)]. \end{aligned} \tag{124}$$

The equal time commutator is now for the coordinate $\phi^\dagger$ and it canonical momentum $\pi^\dagger = \partial L/\partial\partial_0\phi^\dagger = \partial_0\phi$ and assumes the value $-i\delta(\boldsymbol{x}-\boldsymbol{y})\mathbb{1}$, yielding Eq. (119). Another lesson this equation has is that the operator $T[\phi(x)\phi^\dagger(y)]$ is c-number times the identity operator of the Fock-space outside of the null-space of the Klein-Gordon operator $\Box+m^2$.

The solution of Eq. (120) can easily be obtained in the Fourier transformation because the Klein-Gordon operator is diagonal in the energy-momentum space,

$$\begin{aligned}G(x,y) &= -(\Box + m^2)^{-1} \\ &= \int \frac{dp}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2}. \end{aligned} \tag{125}$$

This expression for the inverse if ill defined unfortunately because the denominator is vanishing just in the physically most important regime, for plane waves satisfying the equation of motion!

The usual strategy, followed in the mathematical literature is to define the Fourier integral by deforming slightly the integration contour in such a manner the the singularities are avoided. For a given momentum $\boldsymbol{p}$ there are poles at $p^0 = \pm\omega_{\boldsymbol{p}}$, depicted in Fig. 1. There are four possible way to go around two poles. What is the physical content of this procedure?

This question is rather involved because one hesitates to accept that an infinitesimal change of the contour of integration may lead to finite, observable physical differences in the result. But this is not the first such a situation, the spontaneous magnetization of a ferromagnet, spontaneous symmetry breaking in general, has similar features. In fact, the infinitesimal, asymmetrical fluctuations break the symmetry of the ground state of the ferromagnet whose

instability amplifies this effect to a finite, observable phenomenon. One is faced with the issue of the invariance with respect to the time reversal, $t \to -t$. The Lagrangian of the known fundamental interactions (except to weak interaction, ignored at the moment) show this discrete symmetry. But this is not what we see in reality where the excited states of an isolated composite system, such as an atom, relaxes to its ground state and never become excited due to its coupling to a radiation field. What happens in infinitely large volume is that any energy passed over the field variables gets radiated out in the Universe and the chance of its recollection at the place of the atom is vanishing.

We want to describe such a physics, involving this slight breakdown of the time reversal invariance and we have to put this feature in already at the level of the free system, before the interactions are introduced. This need is indicated by the singularity in defining the free propagator. The way we resolve this problem captures a certain pattern of symmetry breaking because the time reversal always involves a complex conjugation which distinguishes among the different ways of avoiding the poles.

The different integration contours correspond to adding different infinitesimal imaginary part to the one-particle energy, $\omega_{\boldsymbol{p}} \to \omega_{\boldsymbol{p}} \pm i\epsilon$, with $\epsilon \to 0^+$ which is equivalent to the infinitesimal imaginary shift of the rest mass, $m^2 \to m^2 \pm i\epsilon$.

What we want to achieve is the slight suppression of the excited states. In view of the phase factor e^{-iEt} generated by the time evolution for an eigenstate of the Hamiltonian we needs $E \to E - i\epsilon$ for the excited states, $E > 0$. The two-point function (117) propagates the anti-particle like excitations backward in time therefore, the negative energy single particle states should be suppressed for $t \to -\infty$ which requires $E \to E + i\epsilon$ for $E < 0$. The Feynman-contour, C in Fig. 1, is called after his inventor and corresponds to the shift $m^2 \to m^2 - i\epsilon$. The correct expression of the Feynman propagator is

$$G(x,y) = \int \frac{dp}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{126}$$

and one should bear in mind that whenever we carry out the energy-momentum integral the energy integral must be made first leaving the momentum integration at later stage.

For completeness we give the propagator for higher spin particles, too. The Lagrangian

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}A^\mu A_\mu - \frac{\lambda}{2}(\partial^\mu A_\mu)^2 \tag{127}$$

describes a massive spin 1 particle and the corresponding free propagator is

$$\begin{aligned} iG_{\mu\nu}(x,y) &= \langle 0|T[A_\mu(x)A_\nu(y)]|0\rangle \\ &= -i\int \frac{dp}{(2\pi)^4} e^{-ip\cdot(x-y)} \left(\frac{g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}}{p^2 - m^2 + i\epsilon} + \frac{\frac{p_\mu p_\nu}{m^2}}{p^2 - \frac{m^2}{\lambda^2} + i\epsilon} \right). \end{aligned} \tag{128}$$

The propagator of spin 1/2 fermions, described by the Lagrangian

$$L = \bar\psi[i\partial\!\!\!/ - m]\psi \tag{129}$$

is

$$\begin{aligned} iG_{\alpha\beta}(x,y) &= \langle 0|T[\psi_\alpha(x)\bar\psi_\beta(y)]|0\rangle \\ &= i\int \frac{dp}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p\!\!\!/ - m + i\epsilon} \\ &= i\int \frac{dp}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{p\!\!\!/ + m}{p^2 - m^2 + i\epsilon}. \end{aligned} \tag{130}$$

C. Weak form of Wick theorem

We have seen the explicit construction of the two-point function. The higher order Green functions can be obtained by means of the Wick theorem.

Let us consider the vacuum expectation value

$$A = \langle 0|a(\boldsymbol{q}_1)a(\boldsymbol{q}_2)a^\dagger(\boldsymbol{p}_1)a^\dagger(\boldsymbol{p}_2)|0\rangle \tag{131}$$

for the sake of a simple example. The canonical commutation relations yield

$$\begin{aligned} A &= \langle 0|a(\boldsymbol{q}_1)[a^\dagger(\boldsymbol{p}_1)a(\boldsymbol{q}_2) + (2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_2)]a^\dagger(\boldsymbol{p}_2)|0\rangle \\ &= (2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_2)(2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{q}_1-\boldsymbol{p}_2) + (2\pi)^3 2\omega_{\boldsymbol{p}_1}\delta(\boldsymbol{p}_1-\boldsymbol{q}_1)(2\pi)^3 2\omega_{\boldsymbol{p}_2}\delta(\boldsymbol{q}_1-\boldsymbol{p}_2). \end{aligned} \tag{132}$$

This quantity is non-vanishing only when each particle, created by the an operators $a^\dagger$, is removed by an operator a. There are two possible pairing possible between the creation and destruction operators, corresponding to the two contributions in the last line. The two particles are noninteracting therefore their contribution to the matrix element factorizes.

This simple example shows that the factorization of the Green-function into the products of propagators is achieved, as for any other identity for expectation values in the vacuum, by means of the canonical commutation relations. In order to deal with time-dependent fields one introduces the pairing or contraction, $\overbrace{AB}$, of two operators, A and B which satisfy the commutation relation

$$[A,B]_\xi = c(A,B)\mathbb{1}, \tag{133}$$

with some c-number function, $c(A,B)$. We seek the pairing with the following two properties:

1. $\langle 0|\overbrace{AB}|0\rangle = \langle 0|T[AB]|0\rangle$,
2. $\overbrace{AB} = f(A,B)\mathbb{1}$, $f(A,B)$ being a c-number.

The choice $\overbrace{AB} = T[AB]$ violates condition 2. for the free particle plane waves as noted after Eq. (119). To recover point 2 without loosing point 1 we define

$$\overbrace{AB} = T[AB] - :AB: . \tag{134}$$

The property 1 is obviously satisfied. If the order of the operators A and B is the same after the time and the normal ordering the we have point 2, as well, with $f(A,B)=0$. If the order is different then there is a relative sign, ξ, between the two contributions and the commutator (133) can be used to prove $f(A,B) = \pm c(A,B)$. Note that the contraction is non-vanishing for canonically conjugated pairs only.

The complete field operator in the space-time $\phi(x)$, usually consists of the sum of annihilation and creation operators, corresponding to the positive and negative energy parts, $\phi(x) = A(x) + B^\dagger(x)$, as in Eq. (69) in the case of a scalar field. The pairing is additive and non-vanishing between creation and annihilation operators of the same particle only, e.g.

$$\begin{aligned} \overbrace{\phi(x)\phi^\dagger(y)} &= \overbrace{[A(x)+B^\dagger(x)][A^\dagger(y)+B(y)]} \\ &= \overbrace{A(x)A^\dagger(y)} + \overbrace{B^\dagger(x)B(y)}, \end{aligned} \tag{135}$$

Let us consider the Green function

$$i^2 G(x_1,x_2,y_1,y_2) = \langle 0|T[\phi(x_1)\phi(x_2)\phi^\dagger(y_1)\phi^\dagger(y_2)]|0\rangle \tag{136}$$

as an example. The first application of the definition (134) yields the result

$$\begin{aligned} i^2(x_1,x_2,y_1,y_2) &= \langle 0|T[\phi(x_1)[\overbrace{\phi(x_2)\phi^\dagger(y_1)} + :\phi(x_2)\phi^\dagger(y_1):]\phi^\dagger(y_2)]|0\rangle \\ &= \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle + \langle 0|T[\phi(x_1):\phi(x_2)\phi^\dagger(y_1):\phi^\dagger(y_2)]|0\rangle. \end{aligned} \tag{137}$$

Note that the time ordering remains valid between the operators inside and outside of the normal ordering in the last term, which can be written as

$$\langle 0|T[\phi(x_1)(\xi A^\dagger(y_1)A(x_2) + A(x_2)B(y_1) + B^\dagger(x_2)A^\dagger(y_1) + B^\dagger(x_2)B(y_1))\phi^\dagger(y_2)]|0\rangle. \tag{138}$$

Since canonically conjugate pairs contribute only this expectation value assumes the form

$$\begin{aligned} &\xi\langle 0|\overbrace{A(x_1)A^\dagger(y_1)}|0\rangle\langle 0|\overbrace{A(x_2)A^\dagger(y_2)}|0\rangle + \xi\langle 0|\overbrace{B^\dagger(x_1)B(y_1)}|0\rangle\langle 0|\overbrace{A(x_2)A^\dagger(y_2)}|0\rangle \\ &+\xi\langle 0|\overbrace{A(x_1)A^\dagger(y_1)}|0\rangle\langle 0|\overbrace{B^\dagger(x_2)B(y_2)}|0\rangle + \xi\langle 0|\overbrace{B^\dagger(x_1)B(y_1)}|0\rangle\langle 0|\overbrace{B^\dagger(x_2)B(y_2)}|0\rangle \end{aligned} \tag{139}$$

after using the definition (134) again. We have finally

$$i^2 G(x_1,x_2,y_1,y_2) = \langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_2)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_1)}|0\rangle + \xi\langle 0|\overbrace{\phi(x_1)\phi^\dagger(y_1)}|0\rangle\langle 0|\overbrace{\phi(x_2)\phi^\dagger(y_2)}|0\rangle. \tag{140}$$

The generalization of this argument for n-point functions is straightforward though somehow laborious. One possibility is to establish a recursive proof. Another way is to rely on the generator functional, as described in Appendix G. The third, shortest, but most formal way to proceed is to use the path integral representation of transition amplitude, the direction not followed in this notes.

The key qualitative observation to find the expression for a general free Green function is is that the harmonic oscillators, corresponding to different momentum sectors are non-interacting. Therefore, the contributions to the Green functions factorizes into the product of expectation values in different sectors, the pairings. Let us consider the for instance the Green function

$$i^n G_0(x_1,\ldots,x_n,y_1,\ldots,y_n) = \langle 0|T[\phi(x_1)\cdots\phi(x_n)\phi^\dagger(y_1)\cdots\phi^\dagger(y_n)]|0\rangle_0 \tag{141}$$

and the point among x_1,...,x_n,y_1,...,y_n with the earliest time coordinate. A particle or an anti-particle is created there which must be removed by another operator before reaching the bra $\langle 0|$ for $t=\infty$. Such an elementary operation is repeated at $n-1$ other points. The contribution to the Green function will be the sum of different possible choices of the pairs of space-time events, corresponding to the insertion and extraction of an elementary excitation. Once the pairs have been chosen the transition amplitude of the elementary excitations are multiplied together. The result is the weak form of Wick theorem, stating that the $2n$-point function is the sum over all possible pairing of the an operator ϕ with a $\phi^\dagger$,

$$i^n G_0(x_1,\ldots,x_n,y_1,\ldots,y_n) = \sum_{\pi\in S_n} \xi^{\sigma_\pi} iG_0(x_1-y_{\pi(1)})\cdots iG_0(x_n-y_{\pi(n)}), \tag{142}$$

where the summation over S_n, the permutation of n objects and $\sigma_\pi = 0,1$ is the order of the permutation π. For real scalar field we have

$$iG(x_1,\ldots,x_{2n}) = \frac{i^n}{n!2^n}\sum_{\pi\in S_{2n}} G(x_{\pi(1)}-x_{\pi(2)})\cdots G(x_{\pi(2n-1)}-x_{\pi(2n)}), \tag{143}$$

where denominator follows from the observation that there are $n!$ possible order of the n pairs of space-time points yielding identical contributions and the factor 2^n arises from the double counting of each pairs.

D. Examples

It is the easiest to demonstrate the higher orders in the framework of a simpler scalar model described by the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4 \tag{144}$$

and look into the propagator of a particle in $\mathcal{O}\left(g^2\right)$ in perturbation expansion,

$$\begin{aligned}
\langle 0|T[\phi(y)S\phi(x)]|0\rangle &= \langle 0|T[\phi(y)e^{-\frac{ig}{4!}\int dz\phi^4(z)}\phi(x)]|0\rangle \\
&= \sum_{n=0}^{\infty}\left(\frac{-ig}{4!}\right)^n\frac{1}{n!}\int dz_1\cdots dz_n\langle 0|T[\phi(y)\phi^4(z_1)\cdots\phi^4(z_n)\phi(x)]|0\rangle \\
&= \langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle \\
&\quad + \frac{(-ig)^2}{2(4!)^2}\int dz_1dz_2\langle 0|T[\phi(y)\phi^4(z_1)\phi^4(z_2)\phi(x)]|0\rangle + \mathcal{O}\left(g^3\right).
\end{aligned} \tag{145}$$

The Feynman graph of $\mathcal{O}\left(g^0\right)$, depicted in Fig. 2 consists of a single line connecting the initial and final particles and its contribution is

$$\langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(0)} = \langle 0|T[\phi(y)\phi(x)]|0\rangle = iG(x,y). \tag{146}$$

FIG. 2: The Feynman graphs $\mathcal{O}\left(g^0\right)$ contributing to the propagator (145).

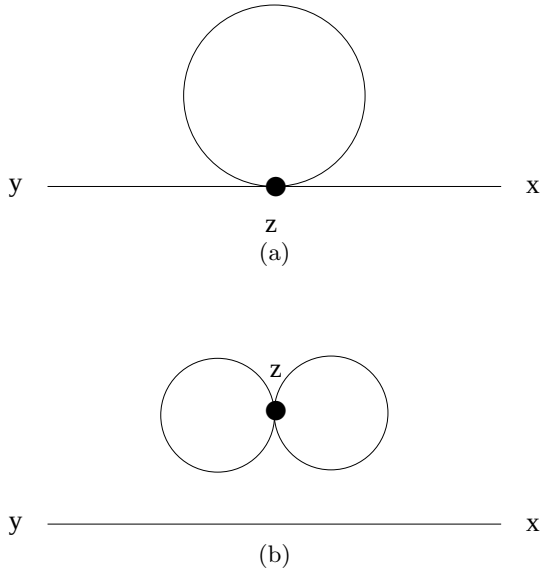


FIG. 3: The Feynman graphs $\mathcal{O}\left(g\right)$ contributing to the propagator (145).

The $\mathcal{O}\left(g\right)$ graphs, shown in Fig. 3, give

$$\begin{aligned}\langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(1)} &= -\frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle \\ &= \underbrace{-\frac{ig}{2}\int dz iG(y,z)iG(z,z)iG(z,x)}_{(a)}\underbrace{-\frac{ig}{8}\int dz iG(y,x)iG(z,z)iG(z,z)}_{(b)}\end{aligned} \tag{147}$$

where the factor 4! in the denominator is eliminated in the second line by the number of possible joining of the internal propagator line to the vertex at z and the other factors in the denominators arise due to the double counting of graphs by exchanging end points of an internal line or exchanging different internal lines attached to the same vertex. Graph (a) gives in momentum space

$$\begin{aligned}G_a^{(1)}(p) &= \int dy e^{-ip(y-x)}\langle 0|T[\phi(y)S\phi(x)]|0\rangle_a^{(1)} \\ &= -\frac{g}{2}\int dy e^{-ip(y-x)}\int dz\frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4}G(p_1)G(p_2)G(p_3)e^{ip_1(y-z)+ip_2(z-z)+ip_3(z-x)} \\ &= -\frac{g}{2}\int dy e^{-ip(y-x)}\int \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4}(2\pi)^4\delta(p_1-p_3)G(p_1)G(p_2)G(p_3)e^{ip_1y-ip_3x} \\ &= -\frac{g}{2}G^2(p)\int\frac{dp_2}{(2\pi)^4}G(p_2) \\ &= \frac{g}{2}\frac{1}{(p^2-m^2+i\epsilon)^2}\int\frac{dp_2}{(2\pi)^4}\frac{1}{p_2^2-m^2+i\epsilon}\end{aligned} \tag{148}$$

Graph (b), being disconnected, is canceled by the normalization of the vacuum-to-vacuum transition amplitude in the

denominator

$$iG_{\rm int}(y,x) = \frac{\langle 0|T[\phi(y)S\phi(x)]|0\rangle}{\langle 0|S|0\rangle}. \tag{149}$$

In fact, the normalized propagator is

$$\begin{aligned} iG^{(1)}_{\rm int}(y,x) &= \frac{\langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle}{\langle 0|0\rangle - \frac{ig}{4!}\int dz\langle 0|\phi^4(z)|0\rangle} + \mathcal{O}\left(g^2\right) \\ &= \langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle + \frac{ig}{4!}\langle 0|T[\phi(y)\phi(x)]|0\rangle \int dz\langle 0|\phi^4(z)|0\rangle + \mathcal{O}\left(g^2\right) \\ &= \langle 0|T[\phi(y)\phi(x)]|0\rangle - \frac{ig}{4!}\int dz\langle 0|T[\phi(y)\phi^4(z)\phi(x)]|0\rangle_a + \mathcal{O}\left(g^2\right). \end{aligned} \tag{150}$$

This is a special case of a general theorem, asserting the cancellation of disconnected diagrams in calculating well normalized Green-functions.

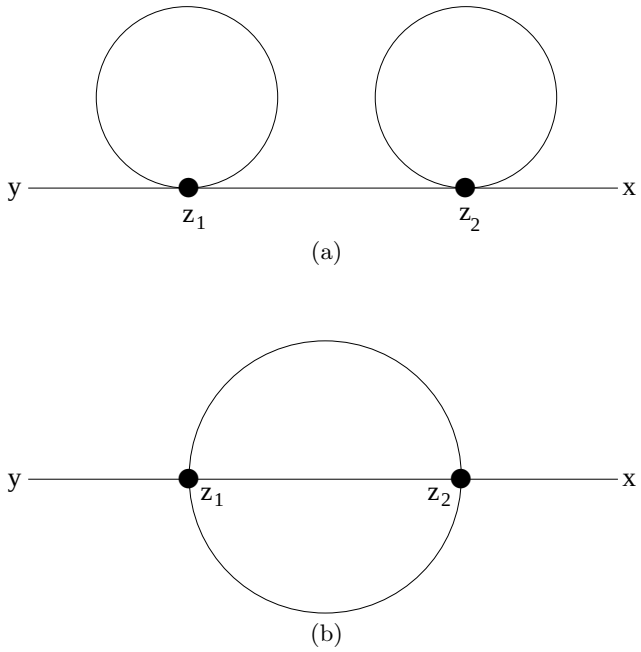


FIG. 4: The Feynman graphs $\mathcal{O}\left(g^2\right)$ contributing to the propagator (145).

The connected $\mathcal{O}\left(g^2\right)$ graphs, shown in Fig. 4, give

$$\begin{aligned} \langle 0|T[\phi(y)S\phi(x)]|0\rangle^{(2)} &= \frac{(-ig)^2}{2(4!)^2}\int dz_1dz_2\langle 0|T[\phi(y)\phi^4(z_1)\phi^4(z_2)\phi(x)]|0\rangle \\ &= \underbrace{\frac{(-ig)^2}{4}\int dz_1dz_2 iG(y-z_1)iG(z_1-z_1)iG(z_1-z_2)iG(z_2-z_2)iG(z_2-x)}_{(a)} \\ &\quad \underbrace{+\frac{(-ig)^2}{3!}\int dz_1dz_2 iG(y-z_1)[iG(z_1-z_1)]^3iG(z_2-x),}_{(b)} \end{aligned} \tag{151}$$

and the momentum space expressions are

$$\begin{aligned} iG_a^{(2)}(p) &= \int dy e^{-ip(y-x)} \langle 0|T[\phi(y)S\phi(x)]|0\rangle_a^{(2)} \\ &= -\frac{ig^2}{4}\int dy e^{-ip(y-x)} \int dz_1 dz_2 \frac{dp_1}{(2\pi)^4}\cdots\frac{dp_5}{(2\pi)^4} G(p_1)G(p_2)G(p_3)G(p_4)G(p_5) \\ &\quad \times e^{ip_1(y-z_1)+ip_2(z_1-z_1)+ip_3(z_1-z_2)+ip_2(z_2-z_2)+ip_3(z_2-x)} \\ &= -\frac{ig}{4}G^3(p)\left[\int \frac{dp_2}{(2\pi)^4}G(p_2)\right]^2 \end{aligned} \tag{152}$$

and

$$\begin{aligned} iG_b^{(2)}(p) &= \int dy e^{-ip(y-x)} \langle 0|T[\phi(y)S\phi(x)]|0\rangle_b^{(2)} \\ &= -\frac{ig^2}{4}\int dy e^{-ip(y-x)} \int dz_1 dz_2 \frac{dp_1}{(2\pi)^4}\cdots\frac{dp_5}{(2\pi)^4} G(p_1)G(p_2)G(p_3)G(p_4)G(p_5) \\ &\quad \times e^{ip_1(y-z_1)+ip_2(z_1-z_2)+ip_3(z_1-z_2)+ip_2(z_1-z_2)+ip_3(z_2-x)} \\ &= -\frac{ig}{4}G^2(p)\int \frac{dp_2}{(2\pi)^4}\frac{dp_3}{(2\pi)^4}G(p_2)G(p_3)G(p-p_2-p_3) \end{aligned} \tag{153}$$

Let us consider now QED as a more realistic theory. The total Lagrangian is

$$L = \bar{\psi}[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{154}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The quadratic part of the Lagrangian gives rise to the free, non-interacting time evolution and the cubic part represent the minimal coupling interaction with

$$H_i = e\int d^3x \bar{\psi}\gamma^\mu\psi A_\mu. \tag{155}$$

The $e^-e^- \to e^-e^-$ elastic scattering amplitude can be written as

$$\begin{aligned} \mathcal{A} &= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)S\psi_{i\alpha_1}(y_1)\psi_{f\alpha_2}(y_2)|0\rangle_0 \\ &= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)T[e^{-ie\int dx\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)}]\psi_{i\alpha_1}(y_1)\psi_{f\alpha_2}(y_2)|0\rangle_0 \\ &= \sum_{n=0}^{\infty}\frac{(-ie)^n}{n!}\int dx_1\cdots dx_n \\ &\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)A_{\mu_1}(x_1)\cdots\bar{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)A_{\mu_n}(x_n)\psi_{\alpha_1}(y_1)\psi_{\alpha_2}(y_2)]|0\rangle_0. \end{aligned} \tag{156}$$

The vacuum expectation value is taken for non-interacting fields therefore the electron and photon sectors factorizes,

$$\begin{aligned} \mathcal{A} &= \sum_{n=0}^{\infty}\frac{(-ie)^n}{n!}\int dx_1\cdots dx_n\langle 0|T[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)]|0\rangle_0 \\ &\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\cdots\bar{\psi}(x_n)\gamma^{\mu_n}\psi(x_n)\bar{\psi}_{\alpha_1}(y_1)\bar{\psi}_{\alpha_2}(y_2)]|0\rangle_0 \\ &= \langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar{\psi}_{\alpha_1}(y_1)\bar{\psi}_{\alpha_2}(y_2)]|0\rangle_0 \\ &\quad +\frac{(-ie)^2}{2!}\int dx_1 dx_2\langle 0|T[A_{\mu_1}(x_1)A_{\mu_2}(x_2)]|0\rangle_0 \\ &\quad \times\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar{\psi}(x_1)\gamma^{\mu_1}\psi(x_1)\bar{\psi}(x_2)\gamma^{\mu_2}\psi(x_2)\bar{\psi}_{\alpha_1}(y_1)\bar{\psi}_{\alpha_2}(y_2)]|0\rangle_0 + \mathcal{O}\left(e^4\right). \end{aligned} \tag{157}$$

The leading order, $\mathcal{O}\left(e^2\right)$ connected graph, the one-photon exchange shown in Fig. 5, corresponds to the contributions

$$\begin{aligned} \mathcal{A}^{(1)} &= -e^2\int dx_1 dx_2 iD_{\mu_1\mu_2}(x_1-x_2)iG_{\beta_1\kappa_1}(z_1-x_1)\gamma^{\mu_1}_{\kappa_1\rho_1}iG_{\rho_1\alpha_1}(x_1-y_1)iG_{\beta_1\kappa_1}(z_2-x_2)\gamma^{\mu_2}_{\kappa_2\rho_2}iG_{\rho_2\alpha_2}(x_2-y_2) \\ &\quad +e^2\int dx_1 dx_2 iD_{\mu_1\mu_2}(x_1-x_2)iG_{\beta_1\kappa_1}(z_1-x_1)\gamma^{\mu_1}_{\kappa_1\rho_1}iG_{\rho_1\alpha_1}(x_1-y_2)iG_{\beta_1\kappa_1}(z_2-x_2)\gamma^{\mu_2}_{\kappa_2\rho_2}iG_{\rho_2\alpha_2}(x_2-y_1) \end{aligned} \tag{158}$$

The vertex where a photon line ends at an electron line is shown in an enlarged manner in Fig. 6.

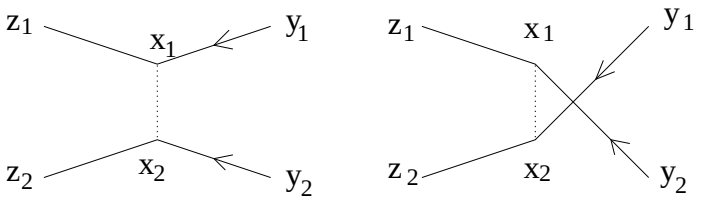


FIG. 5: One-photon exchange between two electrons.

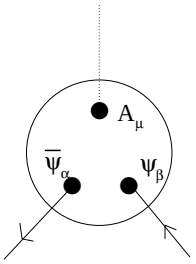


FIG. 6: The electron-photon vertex in bubble notation.

E. Feynman rules

We present briefly the general rules of constructing the contributions to the perturbation series. Let us suppose that we are interested in the transition amplitude between the initial state of n_{i-} electrons created by $\bar{\psi}_{\alpha_j^{-i}}(y_j^{-i})$, n_{i+} positrons created by $\psi_{\alpha_j^{+i}}(y_j^{+i})$ and n_{i0} photons created by $A_{\mu_j^{i0}}(y_j^{i0})$ and the final state consisting of n_{f-} electrons created by $\bar{\psi}_{\alpha_j^{-f}}(y_j^{-f})$, n_{f+} positrons created by $\psi_{\alpha_j^{+f}}(y_j^{+f})$ and n_{f0} photons created by $A_{\mu_j^{f0}}(y_j^{f0})$. The $\mathcal{O}\left(e^{\ell}\right)$ contributions to this amplitudes can be constructed in the following manner.

First we make the list of different graphs by placing the symbols on the paper, representing the space-time in such a manner that the final electron and positron states are denoted by the exchanged symbols, ψ and $\bar{\psi}$, respectively. Next we place ℓ vertices, bubbles of Fig. 6 on the paper. For such a set of symbols we can construct a number a Feynman graphs, consisting the distribution of oriented links starting at a $\bar{\psi}$ symbol and ending at a ψ and non-oriented lines connecting pairs of A's. Each different distribution of the lines denotes a different graph.

For a given graph we can construct the corresponding mathematical expression by assigning an electron propagator $iG_{alpha\beta}(x,y)$ to each oriented line pointing from a $\bar{\psi}_\beta(y)$ to $\psi_\alpha(x)$, a photon propagator $D_{\mu\nu}(x,y)$ to each non-oriented line between the symbols $A_\mu(x)$ and $A_\nu(y)$ and finally a factor $ie\gamma^\mu_{\alpha\beta}$ to each vertex where the symbols A_μ, $\bar{\psi}_\alpha$ and ψ_β are found. We integrate the product of these expression over the space-time locations of vertices and divide the whole integral by the order of the symmetry group of the graph. This symmetry group consists of permutations of the internal lines which leave the graph invariant. The order of the symmetry group is the number of its elements.

The Feynman rules, the assignment of a diagram to each contribution to the scattering amplitude within the framework of the perturbation expansion, will be demonstrated in the framework of the nuclear field theoretical model of Eq. (A23). Let us consider the nucleon-nucleon inelastic scattering amplitude,

$$\begin{aligned}
\mathcal{A} &= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)S\bar{\psi}_{i\alpha_1}(y_1)\bar{\psi}_{f\alpha_2}(y_2)|0\rangle_0 \\
&= \langle 0|\psi_{f\beta_1}(z_1)\psi_{f\beta_2}(z_2)T[e^{ig\int dx\bar{\psi}(x)\psi(x)\sigma(x)}]\bar{\psi}_{i\alpha_1}(y_1)\bar{\psi}_{f\alpha_2}(y_2)|0\rangle_0 \qquad (159)\\
&= Z_\psi^{-2}\sum_{n=0}^{\infty}\frac{(ig)^n}{n!}\int dx_1\cdots dx_n\langle 0|T[\psi_{\beta_1}(z_1)\psi_{\beta_2}(z_2)\bar{\psi}(x_1)\psi(x_1)\sigma(x_1)\cdots\bar{\psi}(x_n)\psi(x_n)\sigma(x_n)\bar{\psi}_{\alpha_1}(y_1)\bar{\psi}_{\alpha_2}(y_2)]|0\rangle_0.
\end{aligned}$$

The Feynman rules are slightly different in momentum space. One starts by making the Fourier-transform of the external leg dependence, after that introduces a four-momentum variable for each line, replaces the propagators with their momentum space expression, introduces a momentum conserving factor $((2\pi)^4\delta^{(4)}(P)$ where P is the total momentum arriving into the vertex and integrates over all momentum variables.

V. GAUGE THEORIES

The discussion of gauge theories in this chapter is restricted to classical field theory, no issues of quantization will be considered.

A. Local symmetries

It has been realized only in the fifties by C. N. Yang and R. L. Mills that the global symmetries of physics are contradicting to the spirit of special relativity. Let us consider for example a global internal symmetry, represented by the transformation

$$\psi(x) \to \psi^{\omega}(x) = \omega\psi(x) \tag{160}$$

acting an the multi-component field variable $\phi(x)$ with ω being an element of the symmetry group, typically $\omega \in G = O(n)$ (real fields) or $\omega \in G = U(n)$ (complex fields). The symmetry is global because we have to apply the same change of base in the internal space anywhere and anytime in the Universe in order to keep the dynamics unchanged. Is the rigid application of the symmetry transformation really necessary in our world where special relativity holds? One can not exchange information between two locations in the space-time separated by space-like interval, $\Delta t^2 - \Delta \boldsymbol{x}^2 < 0$ according to special relativity. How can then be a problem in using different bases in describing the physics at space-like-separated regions? The symmetry transformations which seem to be in harmony with special relativity should concern change of bases in locations which can exchange physical signals.

The suggestion of Yang and Mills is to give up any correlations among bases used at different space-time locations and to us local symmetries,

$$\psi(x) \to \psi^{\omega}(x) = \omega(x)\psi(x). \tag{161}$$

This is another extreme possibility, opposite to the global transformations. It creates obvious problems if applied to space-time regions with time-like separation, $\Delta t^2 - \Delta \boldsymbol{x}^2 > 0$ which can exchange signals. We shall briefly come back to this problem, the issue of gauge-fixing later.

Let us, for the sake of example, consider a world with up and down quarks and strong interactions only. The definition of a quark being up or down is a convention used by physicists to construct models and communicate the results of their work. Physicists at different laboratories may use different definitions, called in general conventions below. Experimental physicists need no conventions for their work since measurements are performed without making any reference to internal spaces and bases. But they need conventions, as well, to compare their findings with model predictions. In this imaginary world the physics is the same whatever type of convention is used. We assume now that this physics is invariant under local $SU(2)$ isospin transformations, $\omega(x) \in SU(2)$, and inquire about the consequence of this postulate. We shall follow gauging, the modification of a Lagrangian from a global symmetry group, called the gauge group G to the same symmetry group but in realized in a local manner. The resulting complete symmetry group of the theory, $\mathcal{G} = \otimes \prod_x G_x$ is gigantic and has infinite dimensions. We shall see that the price of such an upgrade of the symmetry is the introduction of a symmetry group valued vector field, the gauge field.

Let us start with a theory defined by the Lagrangian $L(\phi, \partial\phi)$ with global symmetry, $\omega \in G$. The Lagrangian has ultra-local terms, involving the field variable $\phi(x)$ at strictly the same space-time point, such as the mass term $\frac{1}{2}m^2\phi_a(x)\phi_a(x)$ or a local potential $U(\phi_a(x)\phi_a(x))$. There is no difference between global and local symmetry transformations as far as these terms are concerned. But pieces of the Lagrangian involving space-time derivative of the field are actually detecting the variation of the field on the space-time and are not strictly local. What is important from the point of view of the symmetry is that the transformation rule

$$\partial_\mu\phi(x) \to \partial_\mu\phi^{\omega}(x) = \partial_\mu\omega\phi(x) = \omega\partial_\mu\phi(x) \tag{162}$$

of the global symmetry transformation is modified for local symmetry briefly gauge transformations,

$$\partial_\mu\phi(x) \to \partial_\mu\phi^{\omega}(x) = \partial_\mu\omega(x)\phi(x) = \omega(x)\partial_\mu\phi(x) + (\partial_\mu\omega(x))\phi(x), \tag{163}$$

the trouble maker being the last term. It arises because the derivative compares the field values at neighboring points,

$$\partial_\mu\phi(x) = \lim_{\epsilon\to 0}\frac{\phi(x+\epsilon n_\mu) - \phi(x)}{\epsilon} \tag{164}$$

and this term represents the contribution due to the different conventions in different points. This contribution should not be there if by difference of the field variables we mean "physical" difference. We should transform the

field variables into the same convention before subtraction. The expressing of the field at y into the convention of x, $\phi(y) \to \omega(x \leftarrow y)\phi(y)$, is a change of basis again. We are interested in this transformation for space-time points within each others vicinity when, the continuous dependence on the space-time coordinate assumed, this transformation is close to the identity. The possible moves of y into a neighboring x are characterized by an infinitesimal vector $\Delta x^\mu = x^\mu - y^\mu$ and the corresponding change of base

$$W(y \leftarrow x) = \mathbb{1} - \Delta x^\mu A_\mu(x) + \mathcal{O}\left(\Delta^2 x\right) \tag{165}$$

is given in terms of four generators of the gauge group, $A_\mu(x)$ corresponding to the possible linearly independent moves of the point x. The use of a basis τ^a, $a = 1, \ldots, N$ for the Lie-algebra (generators) of an N-dimensional gauge group allows us to write

$$A_\mu(x) = A^a_\mu(x)\tau^a \tag{166}$$

for real gauge groups and

$$A_\mu(x) = A^a_\mu(x) i\tau^a \tag{167}$$

for complex gauge groups. The generators are antisymmetric or hermitian matrices for real or complex gauge groups, respectively. In the case of the gauge group $SU(N)$ a natural choice of basis is which satisfies the normalization condition

$$\text{tr } \tau^a\tau^b = \frac{1}{2}\delta^{ab}. \tag{168}$$

and the commutation relations

$$\left[\tau^a, \tau^b\right] = i f^{abc}\tau^c \tag{169}$$

where the structure constants f^{abc} is symmetric with respect to cyclic permutations.

The measurable quantities are obviously independent of the choice of basis for the field variable therefore they must be gauge invariant.

B. Geometry and dynamics

The space-time points are called events because they express that something happened "here and now". We need an internal space for the more complete characterization of the events, for the specification of what kind of particles has been seen "here and now". Thus the manifold carrying these information is $\mathcal{M} = \mathbb{R}^4 \otimes I$ where the first factor denotes the external space and the second one stands for the internal space, eg. $I = \mathbb{R}$ for a real scalar field and $i = \mathbb{R}^n$ for an n-component real field, etc. This manifold remains unchanged when a global symmetry is implemented. But local, gauge symmetries requires the further extension of the manifold because the internal space points are given in terms of a locally freely changeable basis. In order to preserve the possibility of comparing internal space points, particle types, residing at different space-time locations we have to keep track how conventions change in the space-time. Thus our manifold is $\mathcal{M} = \mathbb{R}^4 \otimes G \otimes I$, G being the symmetry group acting locally helping to identify the symmetry transformation $\omega(x) \in G$ for each space-time point which brings the convention at x into the same 'reference convention', defined for any observation. The matter, represented as particles or points, p, in the third factor $p \in I$ of $\mathcal{M}$ is distributed in the space-time, $\mathbb{R}^4$, and the second factor G is needed to find the physical interpretation of the point p in terms of generally accepted and used names. We shall see that the interactions influence the geometry of the first two factors in $\mathcal{M}$, namely general relativity, gravitation, and interactions described by Yang-Mills theories *are* equivalent with the appearance of curvature in the external space $\mathbb{R}^4$ and $\mathbb{R}^4 \otimes G$, respectively.

The similarity between differential geometry and gauge theory will be demonstrated below by the introduction of few common key constituents, covariant derivatives, parallel transport and the curvature tensor.

C. Covariant derivative

Once we have an expression for the compensation needed to bring the field around a space-time point into the convention at the same point we can define the covariant derivative

$$D_\mu = \partial_\mu + A_\mu \tag{170}$$

as the derivative of the field $\phi(x)$ computed always in the convention at x by

$$\begin{aligned} D_\mu\phi(x) &= \lim_{\epsilon\to 0}\frac{[1+\epsilon n\cdot A(x+\epsilon n)]\phi(x+\epsilon n_\mu)-\phi(x)}{\epsilon} \\ &= (\partial_\mu + A_\mu)\phi(x). \end{aligned} \tag{171}$$

The gauge field which appears in the definition of the covariant derivative is sometime called compensating field since its role is to compensate out the contributions of the inhomogeneous conventions from the derivative of a physical field.

Let us now find out the transformation rule for the gauge field $A_\mu(x)$ during the gauge transformation

$$\psi(x)\to\psi^\omega(x)=\omega(x)\psi(x). \tag{172}$$

The covariant derivative is the derivative of the field computed in fixed convention therefore $D_\mu\phi(x)$ transforms in the same way,

$$D_\mu\psi(x)=(\partial_\mu+A_\mu(x))\psi(x)\to D^\omega_\mu\psi^\omega(x)=(\partial_\mu+A^\omega_\mu(x))\psi^\omega(x)=\omega(x)D_\mu\psi(x)=\omega(x)(\partial_\mu+A_\mu(x))\psi(x), \tag{173}$$

yielding

$$\omega(\partial_\mu+A_\mu)\psi=(\partial_\mu+A^\omega_\mu)\psi^\omega=(\partial_\mu\omega)\psi+\omega\partial_\mu\psi+A^\omega_\mu\omega\psi \tag{174}$$

and

$$\omega A_\mu=\partial_\mu\omega+A^\omega_\mu\omega. \tag{175}$$

Let us use the space-time derivative of the identity $\omega(x)\omega^\dagger(x)=1$,

$$0=(\partial_\mu\omega)\omega^\dagger+\omega\partial_\mu\omega^\dagger, \tag{176}$$

to write

$$A_\mu\to A^\omega_\mu=\omega(\partial_\mu+A_\mu)\omega^\dagger. \tag{177}$$

The transformation rule (173) gives the rule of replacing the partial derivative with covariant derivative in the Lagrangian,

$$L(\phi,\partial\phi)\to L(\phi,D\phi)=L(\phi,(\partial+iA)\phi), \tag{178}$$

as the rule of gauging. The interaction induced in this manner between the particle described by the field ϕ and the gauge field is called minimal coupling.

Notice that relation between gauging and the construction of the Noether current. The new coordinates introduced in Chapter A 2, related to the global symmetries transformations, are actually local, gauge transformations. Furthermore, the term $\mathcal{O}(A)$ of the Lagrangian

$$\begin{aligned} L(\phi,(\partial+iA)\phi) &= L(\phi,\partial\phi)+\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}iA^a_\mu\tau^a_P\phi+\mathcal{O}\left(A^2\right) \\ &= L(\phi,\partial\phi)-J^{\mu a}A_\mu+\mathcal{O}\left(A^2\right), \end{aligned} \tag{179}$$

is the scalar product of the Noether current given by Eq. (A40) and the gauge field of the symmetry in question.

D. Parallel transport

Since two internal space vectors residing at two different space-time locations can not be compared in their natural bases we need a definition what physically equivalent internal space vectors mean at different space-time points. This is achieved by the parallel transport,a generalization of the construction of the covariant derivative.

Let us consider a continuously derivable path $\gamma^\mu:[0,1]\to\mathbb{R}^4$ in the space-time with $\gamma^\mu(0)=x^\mu_i$ and $\gamma^\mu(1)=x^\mu_f$ as initial and final points, respectively and a field $\phi(x)$ defined on this path. We would like to characterize the situation that the values of this field long our path, $\phi(\gamma(s))$ are physically equivalent, despite the possible dependence of the

components of $\phi(\gamma(s))$ on s when expressed in terms of local bases. Suppose that we have a physical method to check the equivalence of the field along the path. The resulting field $\phi(\gamma(s))$ satisfies the equation

$$\phi(y) = W_\gamma(y,x)\phi(x) \tag{180}$$

where $W_\gamma(y,x)$ is a symmetry (basis) transformation which naturally depends on the choice of the points x and y and a somehow surprising manner will also depend on the path γ, too.

We have just assumed that the function $W_\gamma(y,x)$ is well defined say by experimental physicists. Since this function compensate the change of conventions along the path it should closely be related to the compensating field $A_\mu(x)$ introduced in defining the covariant derivative. In fact, the compensation introduced in Eqs. (165) and (171) are parallel transports for infinitesimal distances. The covariant derivative gives the change of a numerical components of a field after the elimination of the contributions due to the changing conventions. The condition of parallel transport is thus

$$\frac{d\gamma^\mu}{ds} D_\mu \phi(\gamma(s)) = 0. \tag{181}$$

It shows that the space-time dependence is just canceling the convention dependence and can be written as an equation for the parallel transport transformation

$$\frac{d\gamma^\mu}{d\tau} D_{y^\mu} W_\gamma(y,x) = 0 \tag{182}$$

according to Eq. (180). This equation looks like the Schrödinger equation, Eq. (E10), with time dependent Hamiltonian, $S(s) = \frac{d\gamma^\mu}{d\tau} D_{y^\mu}$ and its solution will be obtained, accordingly, by the generalization of the time ordered product (E11). The path ordered product of non-commuting objects defined along the path γ is defined as

$$T[A(s_A)B(s_B)] = \Theta(s_A - s_B)A(s_A)B(s_B) + \Theta(s_B - s_A)B(s_B)A(s_A) \tag{183}$$

and the desired parallel transport,

$$W_\gamma(y,x) = P\left[e^{-\int_x^y d\gamma^\mu A_\mu(\gamma)}\right] = P\left[e^{-\int_0^1 ds \frac{d\gamma^\mu(s)}{ds} A_\mu(\gamma(s))}\right], \tag{184}$$

satisfies Eq. (182) what can be seen by repeating the steps (E16)-(E16). A more illuminating argument is the following: Let us write first the integral in the exponent in Eq. (184) as

$$\int_0^1 ds \frac{d\gamma^\mu(s)}{ds} A_\mu(\gamma(s)) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^N \frac{d\gamma^\mu(s_j)}{ds} A_\mu(\gamma(s_j)) \tag{185}$$

where $s_j = j/N$. The exponential function of an operator is defined by its Taylor-series,

$$e^{-\frac{1}{N}\sum_{j=1}^N \frac{d\gamma^\mu(s_j)}{ds} A_\mu(\gamma(s_j))} = 1 - \frac{1}{N}\sum_{j=1}^N \frac{d\gamma^\mu(s_j)}{ds} A_\mu(\gamma(s_j)) + \frac{1}{N^2} \sum_{j_1,j_2=1}^N \frac{d\gamma^\mu(s_{j_1})}{ds} \frac{d\gamma^\mu(s_{j_2})}{ds} A_\mu(\gamma(s_{j_1}))A_\mu(\gamma(s_{j_2})) + \cdots \tag{186}$$

and the path ordering applies term-by-term,

$$W_\gamma(y,x) = \lim_{N\to\infty} \left[1 - \frac{1}{N}\sum_{j=1}^N \frac{d\gamma^\mu(s_j)}{ds} A_\mu(\gamma(s_j)) + \frac{1}{N^2} \sum_{j_1,j_2=1}^N \frac{d\gamma^\mu(s_{j_1})}{ds} \frac{d\gamma^\mu(s_{j_2})}{ds} P[A_\mu(\gamma(s_{j_1}))A_\mu(\gamma(s_{j_2}))] + \cdots\right] \tag{187}$$

We would have

$$W_\gamma(y,x) = \lim_{N\to\infty} \prod_{j=1}^N e^{-\frac{1}{N}\frac{d\gamma^\mu(s_j)}{ds} A_\mu(\gamma(s_j))} \tag{188}$$

according to the well known rule $e^a e^b = e^{a+b}$, valid for numbers, without paying attention to the non-commutativity of the objects occurring in the product. But the path ordering places the contributions corresponding to higher j more to the left in the products and we find

$$W_\gamma(y,x) = \lim_{N\to\infty} e^{-\frac{1}{N}\frac{d\gamma^\mu(s_N)}{ds} A_\mu(\gamma(s_N))} \cdots e^{-\frac{1}{N}\frac{d\gamma^\mu(s_1)}{d\tau} A_\mu(\gamma(s_1))} \tag{189}$$

by repeating the same resummation as in Eq. (188). The path ordering succeeded in factorizing the dependence on the N-th division point to the integral at the very left of the product. The final step is the calculation of the partial derivative of W along the path,

$$\begin{aligned}
\frac{1}{N}\frac{d\gamma^\mu}{ds}\partial_{y^\mu}W_\gamma(y,x) &= W_\gamma(y,x) - W_\gamma\left(y-\frac{1}{N}\frac{d\gamma^\mu(s_N)}{ds},x\right)\\
&= \left[e^{-\frac{1}{N}\frac{d\gamma^\mu(s_N)}{ds}A_\mu(\gamma(s_N))}-\mathbb{1}\right]e^{-\frac{1}{N}\frac{d\gamma^\mu(s_{N-1})}{ds}A_\mu(\gamma(s_{N-1}))}\cdots e^{-\frac{1}{N}\frac{d\gamma^\mu(s_1)}{d\tau}A_\mu(\gamma(s_1))}\\
&\approx -\frac{1}{N}\frac{d\gamma^\mu(s_N)}{ds}A_\mu(\gamma(s_N))W_\gamma\left(y-\frac{1}{N}\frac{d\gamma^\mu(s_N)}{ds},x\right)\\
&\approx -\frac{1}{N}\frac{d\gamma^\mu(1)}{ds}A_\mu(\gamma(1))W_\gamma(y,x),
\end{aligned} \tag{190}$$

which yields Eq. (182).

The parallel transport transformation, $W_\gamma(y,x)$, satisfies a simple multiplication rule. Let us consider two joined paths, γ_1 and γ_2, with $\gamma_1(1)=\gamma_2(0)$. Their sum, $\gamma_1+\gamma_2$, is the path obtained by following first γ_1 and after γ_2,

$$\gamma_1+\gamma_2(s) = \begin{cases}\gamma_1(2s) & 0\le s\le\frac{1}{2}\\ \gamma_2(2s-1) & \frac{1}{2}\le s\le 1\end{cases}. \tag{191}$$

The path ordering gives

$$W_{\gamma_1+\gamma_2}(\gamma_2(1),\gamma_1(0)) = W_{\gamma_2}(\gamma_2(1),\gamma_2(0))W_{\gamma_1}(\gamma_1(1),\gamma_1(0)). \tag{192}$$

E. Wilson-loop

The use of the parallel transport in gauge theory is the construction of the Wilson loops, a complete set of gauge invariant observables. They are called complete because their knowledge allows us to reconstruct the gauge field up to a gauge transformation.

We start by the gauge transformation properties of the parallel transport,

$$\begin{aligned}
\omega(y)\phi(y) &= W^\omega_\gamma(y,x)\omega(x)\phi(x)\\
W^\omega_\gamma(y,x) &= \omega(y)W_\gamma(y,x)\omega^\dagger(x),
\end{aligned} \tag{193}$$

indicating that the parallel transformation corresponding to a closed loop changes by a similarity transformation,

$$W^\omega_\gamma(x,x) = \omega(x)W_\gamma(x,x)\omega^\dagger(x) \tag{194}$$

and its trace,

$$w[\gamma] = \mathrm{tr}W_\gamma(x,x) \tag{195}$$

called Wilson-loop, is gauge invariant.

The physical interpretation of a Wilson-loop is that its vacuum expectation value gives the amplitude of the creation and annihilation of a particle-anti particle pair in the vacuum. Let us construct the initial state by placing a fermion corresponding to the fundamental representation of the group $SU(n)$ at the location $B=(t_i,\boldsymbol{x}_p)$ in the space time and its anti-particle at $A=(t_i,\boldsymbol{x}_{ap})$. We consider a rectangular Wilson for simplicity where the charges are at rest. This can be achieved by assuming that their mass is very large and their dynamics in space can be ignored. Therefore the final state contains a particle at $C=(t_f,\boldsymbol{x}_p)$ and an anti-particle at $D=(f_f,\boldsymbol{x}_{ap})$, cf. Fig. 7. The Lagrangian

$$\begin{aligned}
L &= \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu+iA_\mu)\psi-(\partial_\mu\bar\psi-\bar\psi iA_\mu)\gamma^\mu\psi]-m\bar\psi\psi\\
&= \frac{i}{2}[\bar\psi\gamma^\mu\partial_\mu\psi-\partial_\mu\bar\psi\gamma^\mu\psi-m\bar\psi\psi]-\bar\psi\gamma^\mu A_\mu\psi
\end{aligned} \tag{196}$$

yields the Hamiltonian

$$H(t) = \int d^3x[m\bar\psi(\boldsymbol{x})\psi(\boldsymbol{x})+A^a_\mu(t,\boldsymbol{x})\bar\psi(\boldsymbol{x})\gamma^\mu\tau^a_P\psi(\boldsymbol{x})] \tag{197}$$

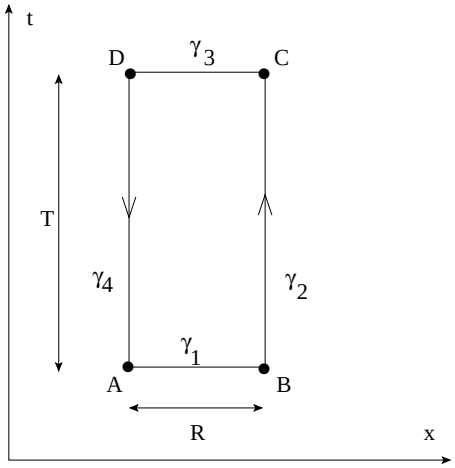


FIG. 7: Wilson-loop corresponding to the creation and annihilation of a pair of particle-anti particle.

in the limit $m \to \infty$ and the time evolution operator

$$U = T[e^{-\int_{-\infty}^{\infty} dt H(t)}] \tag{198}$$

in the Schrödinger representation.

The canonical anti-commutation relation

$$\{\psi(t,\boldsymbol{x}), \psi^\dagger(t,\boldsymbol{y})\} = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{199}$$

shows that the fermion field operator can be used to create or destruct a particle or anti-particle at a given space location. Let us introduce the states $|j\rangle_p = \psi_\alpha^{j\dagger}(\boldsymbol{x}_p)\chi_\alpha|0\rangle$ and $|j\rangle_{ap} = \chi_\alpha^\dagger\psi_\alpha^j(\boldsymbol{x}_{ap})|0\rangle$ corresponding to the j-th basis vector of the fundamental representation where the internal index j and Dirac index α are shown explicitly on the field operator $\psi_\alpha^j(x)$ and χ_α is a Dirac bi-spinor corresponding to a positive energy solution of the free Dirac-equation. The pair must be in a neutral, ie. singlet state which is

$$\begin{aligned}
|\psi\rangle_i &= \sum_{j=1}^{n} |j\rangle_p \otimes |j\rangle_{ap} \\
&= \sum_{j,k=1}^{n} (W_{\gamma_1}(B,A))_{j,k}|j\rangle_p \otimes |k\rangle_{ap} \\
|\psi\rangle_f &= \sum_{j,k=1}^{n} (W_{\gamma_3}(C,D))_{j,k}|j\rangle_p \otimes |k\rangle_{ap}
\end{aligned} \tag{200}$$

where the horizontal paths of the Wilson-loop are used to construct the state by means of conventions defined at the same space-time points, in other words, the structure, showed in the first line is given in the conventions of the point B of the particle.

The time evolution operator for an infinitesimal time Δt is

$$\begin{aligned}
\langle j|e^{-i\Delta t H(t)}|k\rangle_p &= \langle j|[1 - i\Delta t \int d^3x [m\bar\psi(t,\boldsymbol{x})\psi(t,\boldsymbol{x}) + A_\mu^a(t,\boldsymbol{x})\bar\psi(t,\boldsymbol{x})\gamma^\mu\tau_P^a\psi(t,\boldsymbol{x})]|k\rangle_p \\
&= \delta_{j,k} - i\Delta t\langle j|m\bar\psi(t,\boldsymbol{x}_p)\psi(t,\boldsymbol{x}_p) + A_\mu^a(t,\boldsymbol{x}_p)\bar\psi(t,\boldsymbol{x}_p)\gamma^\mu\tau_P^a\psi(t,\boldsymbol{x}_p)]|k\rangle_p \\
&= \delta_{j,k}(1 - i\Delta t m) - i\Delta t A_0^a(t,\boldsymbol{x}_p)(\tau_P^a)_{j,k}
\end{aligned} \tag{201}$$

up to terms $\mathcal{O}\left(\Delta t^2\right)$. The multiplication of expressions like this in the chronological product yields the time evolution operator for the internal space dynamics of the particle and the antiparticle

$$\begin{aligned}
U_p &= T[e^{-i\int_{t_i}^{t_f} dt[m + A_0(t,\boldsymbol{x}_p)]}], \\
U_{ap} &= T[e^{-i\int_{t_i}^{t_f} dt[m - A_0^*(t,\boldsymbol{x}_{ap})]}] \\
&= (T[e^{-i\int_{t_i}^{t_f} dt[-m + A_0(t,\boldsymbol{x}_{ap})]}])^*,
\end{aligned} \tag{202}$$

respectively. Finally we can put everything together for the transition amplitude,

$$\begin{aligned}
\mathcal{A}_p &= \langle\psi_f|U|\psi_i\rangle \\
&= \sum_{jj'kk'=1}^{n} (W_{\gamma_3}(C,D))^*_{j',k'}\langle j'|T[e^{-i\int_{t_i}^{t_f} dt[m+A_0(t,\boldsymbol{x}_p)]}]|j\rangle_p (W_{\gamma_1}(B,A))_{j,k}\langle k'|(T[e^{-i\int_{t_i}^{t_f} dt[-m+A_0(t,\boldsymbol{x}_{ap})]}])^*|k\rangle_{ap} \\
&= \sum_{jj'kk'=1}^{n} W^\dagger_{\gamma_3}(C,D))_{k',j'}\langle j'|T[e^{-i\int_{t_i}^{t_f} dt[m+A_0(t,\boldsymbol{x}_p)]}]|j\rangle_p (W_{\gamma_1}(B,A))_{j,k}\langle k|(T[e^{-i\int_{t_i}^{t_f} dt[-m+A_0(t,\boldsymbol{x}_{ap})]}])^\dagger|k'\rangle_{ap} \\
&= e^{-2mTi}\mathrm{tr}W^\dagger_{\gamma_3}(C,D))W_{\gamma_2}(C,B))W_{\gamma_1}(B,A)W^\dagger_{\gamma_4}(D,A) \\
&= e^{-2mTi}w[\gamma_4+\gamma_3+\gamma_2+\gamma_1]
\end{aligned} \tag{203}$$

which is indeed the Wilson-loop up to a trivial self energy factor.

The Wilson-loop can be used to read off the potential between static charges. The creation and annihilation of the charges induce transient phenomena for the dynamics of the gauge field, to be specified below, which are in a good approximation independent of the time-span of the propagation of the charges. If these phenomena could be neglected then we could identify the amplitude with $e^{-2mTi--iTV(\boldsymbol{x}-\boldsymbol{y})}$ where $V(\boldsymbol{x})$ is the interaction energy of a static particle-anti particle singlet pair inserted in the vacuum. By letting the $t_f - t_i \to \infty$ the transient effects, being $\mathcal{O}\left((t_f-t_i)^0\right)$, become negligible compared to the contribution of the particle-anti particle pair energy, an $\mathcal{O}(t_f - t_i)$ contribution, and we have

$$2m + V(\boldsymbol{x}_p - \boldsymbol{x}_{ap}) = \lim_{T\to\infty}\frac{i}{T}\ln\langle 0|w[\gamma_4+\gamma_3+\gamma_2+\gamma_1]|0\rangle \tag{204}$$

where the expectation value is taken for the gauge field vacuum.

F. Curvature or field strength tensor

The Wilson-loop plays important role in geometry, it defines the curvature tensor. Let us consider the parallel transport of a field (or vector in the language of differential geometry) along a rectangle defined by the edges x, $x+u$, $x+u+v$ and $x+v$ in space-time, u and v being infinitesimal, non-parallel vectors. The change of the field during the parallel transport is infinitesimal, as well, $\phi \to \phi + \delta\phi$ and should be linear in u, v and ϕ itself. Therefore one expects the relation

$$\delta\phi^a = R^a_{\mu\nu b}u^\mu v^\nu \phi^b. \tag{205}$$

We use here covariant and contravariant internal indices in discussing the curvature tensor because the internal space is identical with the external one in differential geometry and general relativity. For non-gravitational gauge theory on should simply disregard the position of the internal indices. The coefficients $R^a_{\mu\nu b}$ form a tensor, called curvature tensor in differential geometry because they appear in a covariant equation. According to the definition of the parallel transport, Eq. (184),

$$\begin{aligned}
R^a_{\mu\nu b}u^\mu v^\nu &\approx (e^{v\cdot G(x)}e^{u\cdot G(x+v)}e^{-v\cdot G(x+u)}e^{-u\cdot G(x)})^a_b \\
&\approx \left[\left(\mathbb{1}+v\cdot G(x)+\frac{1}{2}[v\cdot G(x)]^2\right)\left(\mathbb{1}+u\cdot G(x+v)+\frac{1}{2}[u\cdot G(x+v)]^2\right)\right. \\
&\quad \left.\times\left(\mathbb{1}-v\cdot G(x+u)+\frac{1}{2}[v\cdot G(x+u)]^2\right)\left(\mathbb{1}-u\cdot G(x)+\frac{1}{2}[u\cdot G(x)]^2\right)\right]^a_b \\
&\approx (\mathbb{1}-iu^\mu v^\nu F_{\mu\nu})^a_b
\end{aligned} \tag{206}$$

The field strength tensor, alias curvature tensor, is

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{207}$$

It is a generator valued field,

$$F_{\mu\nu} = F^a_{\mu\nu}\tau^a_P, \tag{208}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{209}$$

The curvature measures the non-triviality of the parallel transport along an infinitesimal closed loop. In a similar manner, the gauge field strength is the measure of the non-triviality of the parallel transport of a state along a closed path. Another way saying this is that curvature or field strength is non-vanishing in a region if the parallel transport is path dependent there. In fact, let us break a closed loop γ into two parts, $\gamma = \gamma_1 + \gamma_2$, like in Fig. 7 and notice that the conditions $W_{\gamma_1+\gamma_2}(A, A) = W_{\gamma_1}(B, A)W_{\gamma_2}(A, B) = 1$ and $W_{\gamma_1}(B, A) = W^{-1}_{\gamma_2}(A, B) = W_{\gamma_2^{-1}}(B, A)$ are equivalent.

The transformation rule for the parallel transport on a closed path,

$$\mathbb{1} - u^\mu v^\nu F_{\mu\nu}(x) \to \omega(x)[\mathbb{1} - u^\mu v^\nu F_{\mu\nu}(x)]\omega^\dagger(x) \tag{210}$$

gives the rule of transformation

$$F_{\mu\nu}(x) \to \omega(x) F_{\mu\nu}(x)\omega^\dagger(x). \tag{211}$$

G. Gauge field dynamics

The gauging, the upgrade of a global symmetry to a local one brings in a generator valued vector field. We are accustomed to the fact that fields corresponds to particles. Therefore the gauging of a symmetry suggests the presence of spin 1 bosons in the system. The dynamics of these particle can not come from the Lagrangian (178) because of the lack of the velocities $\partial_0 A_\mu$ in it. The simplest solution is the add a new term to the Lagrangian $L \to L + L_A$ where L_A satisfies the following conditions:

1. It should be quadratic in the velocities, $L_A = \mathcal{O}\left(D^2\right)$.
2. It should be Lorentz invariant.
3. It should be gauge invariant.

The unique solution of these constraint on a space-time with trivial topology is the Yang-Mills Lagrangian,

$$L_{YM} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu})^2 = -\frac{1}{4g^2}(F^a_{\mu\nu})^2, \tag{212}$$

which is fixed up to the coupling constant g. It is advantageous to use the notation $A_\mu \to gA_\mu$ in perturbation expansion, giving

$$L_{YM} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}, \tag{213}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{214}$$

The non-vanishing field-strength is always the indication of the non-triviality of the parallel transport in that region.

Notice that the conditions evoked above restricts the free parameters of the Lagrangian for a single coupling constant for such set of the generators which have non-vanishing commutation relations among each others. When the gauge group is a direct product of two subgroups then the generators taken form different subgroups commute and each subgroup has independent coupling constant.

H. Fundamental interactions

It is a remarkable feature of Nature that all known interactions belong to the class of gauge theories.

Electrodynamics: The electromagnetic interaction is defined as the interaction realized by the gauge field which enters in physics by the gauging of the $G_{QED} = U(1)$ global phase symmetry of Quantum Mechanics. This latter stands for the invariance of the observables under the transformation $\psi(x) \to e^{-i\Phi}\psi(x)$ of the wave function. The

upgrading of this symmetry to gauge symmetry leads to the introduction of a single vector field (the group $U(1)$ is one dimensional) $iA_\mu(x)$ (the only generator of $U(1)$ is $\tau = i$) and the gauging rule $\partial_\mu \to \partial_\mu + i\frac{e}{c}A_\mu$, $\hat{\boldsymbol{p}} \to \hat{\boldsymbol{p}} + \frac{\hbar e}{c}\boldsymbol{A}$

$$L = \bar{\psi}(iD_\mu\gamma^\mu + m)\psi - \frac{1}{4e^2}F_{\mu\nu}^2. \tag{215}$$

The traditional notation is $A_\mu \to eA_\mu$:

$$L \to \bar{\psi}(i\partial_\mu\gamma^\mu + m - eA_\mu\gamma^\mu)\psi - \frac{1}{4}F_{\mu\nu}^2. \tag{216}$$

The parametrization $\omega = e^{-i\Phi}$ leads to the gauge transformation rules

$$\psi(x) \to e^{-i\Phi(x)}\psi(x), \qquad A_\mu \to A_\mu + \partial_\mu\Phi \tag{217}$$

Weak interaction: Let us start with the the chiral left spinors $(u_L(x), d_L(x))$ for the up and down quarks. The strong interaction has a global isospin $SU(2)$ symmetry which mixes the left and right chiral spinors in the same manner. The weak interaction is not parity invariant therefore it influences the left and right fermions differently. Let us take isospin transformations acting on the left chiral quark spinors only,

$$\psi_L(x) = \begin{pmatrix} u_L(x) \\ d_L(x) \end{pmatrix} \to \omega\psi_L(x) = e^{i\Phi^a\sigma^a}\begin{pmatrix} u_L(x) \\ d_L(x) \end{pmatrix}, \qquad u_R(x) \to u_R(x), \qquad d_R(x) \to d_R(x), \tag{218}$$

σ^s denoting the Pauli matrices, the generators of the $SU(2)$ group. The gauge theory, based on this global symmetry contains three gauge fields, $A_\mu(x) = A_\mu^a(x)\frac{\sigma^a}{2i}$, corresponding to the generators of the group $SU(2)_L$, the Pauli matrices. To complete the transformations on the quark isospin doublet we introduce an $U(1)_Y$ phase transformation operating on both spinors,

$$u_R(x) \to e^{-iy_u\Phi}u_R(x), \qquad d_R(x) \to e^{-iy_d\Phi}d_R(x), \qquad u_L(x) \to e^{-iy_u\Phi}u_L(x), \qquad d_L(x) \to e^{-iy_d\Phi}d_L(x). \tag{219}$$

The gauging of this $U_Y(1)$ global symmetry leads to an additional gauge field. These four gauge fields, arising from the local symmetry $G_{EW} = SU(2)_L \otimes U(1)_Y$ describe the unified electro-weak interactions in the Georgi-Glashow-Salam model.

Strong interaction: The hadron states constructed in the quark model show a three-fold degeneracy in color space. This implies an $G_S = SU(3)$ symmetry group,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \to \omega \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix}, \tag{220}$$

$\omega \in SU_S(3)$ which generates 8(=dim$SU(3)$) gauge fields. These, coupled to the Noether currents

$$j_\mu^a(x) = \bar{\psi}(x)\gamma^\mu\tau^a\psi(x) \tag{221}$$

by minimal coupling represent the strong interactions. The Lagrangian is

$$\begin{aligned} L &= \sum_f \bar{\psi}_f(iD_\mu\gamma^\mu + m_f)\psi_f - \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu})^2 \\ &= \sum_f \bar{\psi}_f\left(i\partial_\mu\gamma^\mu + m_f + igA_\mu^a\frac{\tau^a}{2i}\gamma^\mu\right)\psi_f - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c)^2 \end{aligned} \tag{222}$$

where f is the flavor index and τ^a are the Gell-Mann matrices.

Gravity: The equivalence principle states that the gravitational forces can locally be eliminated by a suitable chosen coordinate system. Thus we can introduce a local coordinate system around a given point of space-time where flat space and Lorentz symmetry is recovered at the point in question. This principle leaves a Lorentz transformation free since this preserves the flat Minkowski space-time structure at the given point. One is thus lead to a gauge transformation

$$\phi(x) \to S(\Lambda)\phi(\Lambda^{-1}\cdot x) \tag{223}$$

with the Lorentz group as gauge group. By imposing continuity in space-time we restrict the gauge group to the proper Lorentz group.

This gauge symmetry is introduced by the equivalence principle, the distinguished feature of the gravitational interaction therefor one suspects that the dynamics arising in a gauge theory based on this symmetry is General Relativity. One can actually reproduce Einstein equation of General Relativity as the Euler-Lagrange equation of such a gauge theory.

VI. SPONTANEOUS SYMMETRY BREAKING

The usual meaning of symmetry in physics is the existence of transformations of the degrees of freedom which can be carried out at any time. Namely, the performance of a symmetry transformation on the old initial conditions generates new initial conditions and the time evolution starting from these is identical with the time evolution obtained by applying the symmetry transformation for the state of the system (as a point in phase space in classical mechanics and a state vector in quantum mechanics) at any time. This statement refers to the dynamics, the Hamiltonian should be invariant under the symmetry transformations.

There is another, frequently used context for symmetry when on talks about the symmetry of a given trajectory in classical mechanics or state in quantum mechanics. For example the circular motion around a point-like mass is spherically symmetric. But this is a specially simple motion, the spherically symmetric dynamics naturally allows more complicated, non-symmetrical orbitals. In a similar manner, some of the eigenstates of a symmetrical Hamiltonian may be symmetrical, others not. For example the electron eigenstates of the non-relativistic hydrogen atom with non-vanishing angular momentum are not symmetric despite the spherical symmetry of the Coulomb field of the proton.

The symmetry of the dynamics is not obvious at a glance of the motion of the system, what we see first is the symmetry of the trajectory or the state. Not all states are equally important, the lowest lying state is more characteristic of the typical behavior of the system than an excited state. A symmetry is called to be realized in the Wigner mode or is unbroken if not only the Hamiltonian but the gourd state is symmetric. The symmetry is broken dynamically or spontaneously if the ground state of a symmetrical Hamiltonian is asymmetric.

Let us write the operator realizing a symmetry transformation as $U = e^{iQ}$ in the quantum case. The symmetry of the Hamiltonian is expressed by the condition $[Q, H] = [U, H] = 0$, the symmetry of the ground state $|0\rangle$ is reflected in the equation $Q|0\rangle = 0$ or $U|0\rangle = |0\rangle$. We have therefore

1. Symmetry in the Wigner mode: $[Q, H] = [U, H] = 0$, $Q|0\rangle = 0$ and $U|0\rangle = |0\rangle$ or
2. Symmetry spontaneously broken mode: $[Q, H] = [U, H] = 0$ and $Q|0\rangle \neq 0$, $U|0\rangle \neq |0\rangle$.

A. Discrete symmetries

A simple example for the spontaneously broken space inversion symmetry is the following though experiment. Let us take a flexible plastic ruler which is approximately straight. Both the dynamics and the shape are reflection invariant. Push the two end of the ruler towards each other by our hands in a symmetrical manner, ie. with open palm, perpendicular to the ruler. The ruler develops a curved shape, bent by the pressure exerted by our palm at the end. The ground state is not symmetrical anymore thought the Hamiltonian rests reflection symmetric. What happened is that the symmetrical state becomes an unstable equilibrium when the pressure is applied and a weak quantum or thermal fluctuation makes the system to choose a stable but asymmetric position. In other words, originally the ground state is non-degenerate (straight ruler) but the pressure makes it degenerate and the system has to choose on of the several degenerate states, neither of them being singlet alone.

A more elaborate example is offered by the ferromagnet. The order parameter, the local magnetization in a given direction, is a scalar field $\phi(\boldsymbol{x})$ and the free energy is supposed to stay invariant under the change of sign, $\phi(\boldsymbol{x}) \to -\phi(\boldsymbol{x})$. The simplest non-trivial local free energy functional displaying this symmetry is

$$F[\phi] = \int d^3x \left[\frac{1}{2}(\nabla\phi(\boldsymbol{x}))^2 + U(\phi(\boldsymbol{x}))\right], \qquad U(\phi) = \frac{r}{2}\phi^2 + \frac{g}{4}\phi^4, \qquad g > 0 \tag{224}$$

in the spirit of the Landau-Ginzburg double expansion, mentioned at Eq. (A70). The inhomogeneities cost free energy, therefore the lowest free energy configuration must be homogeneous, $\phi(\boldsymbol{x}) = \Phi$. The value of $\Phi = \Phi_0 = \langle\phi(\boldsymbol{x})\rangle$ is chosen by minimizing the potential $U(\phi)$. For $r > 0$ $\Phi_0 = 0$ and for $r < 0$

$$\Phi_0 = \sqrt{\frac{-r}{g}}. \tag{225}$$

One may therefore write $T - T_c \approx r$ in the vicinity of the phase transition.

The previous example can be generalized for particles where the symmetry $G = Z_2$, realized as $\phi(x) \to -\phi(x)$, is broken spontaneously. The Lagrangian is

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4}\phi^4, \tag{226}$$

and the corresponding Hamiltonian reads as

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4}\phi^4\right] \tag{227}$$

with $m^2 < 0$ where the canonical momentum,

$$\pi(x) = \frac{\partial L}{\partial\partial_0\phi(x)} = \partial_0\phi(x) \tag{228}$$

is represented by the operator

$$\pi(x) = \frac{1}{i}\frac{\delta}{\delta\phi(x)}. \tag{229}$$

It is plausible that the field expectation value is space-time independent in the ground state due to the positive definite nature of the second term in the Hamiltonian. Let us split the field into the sum of its expectation value and fluctuations,

$$\phi(x) = \phi_0 + \chi(x) \tag{230}$$

with

$$\phi_0 = \langle 0|\phi(\boldsymbol{x})|0\rangle. \tag{231}$$

The value of ϕ_0 can be obtained approximately by ignoring the dynamics of the inhomogeneous modes. In this case on has simply to minimize the local potential to find

$$\phi_0 \approx \sqrt{\frac{-m^2}{g}}. \tag{232}$$

The dynamics of the fluctuations around this homogeneous expectation value is characterized by the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}(\partial\chi)^2 - \frac{m^2}{2}(\phi_0+\chi)^2 - \frac{g}{4}(\phi_0+\chi)^4 \\ &= \frac{1}{2}(\partial\chi)^2 - \phi_0(\underbrace{m^2 + g\phi_0^2}_{=0})\chi - \underbrace{\left(\frac{m^2}{2} + \frac{3g}{2}\phi_0^2\right)}_{=-2m^2=\frac{1}{2}m_H^2}\chi^2 - \frac{g}{4}(4\phi_0\chi^3 + \chi^4) - \frac{m^2}{2}\phi_0 - \frac{g}{4}\phi_0^4. \end{aligned} \tag{233}$$

The non-vanishing expectation value for the $\boldsymbol{p} = 0$, homogeneous mode corresponds coherent states. One can simplest see this by considering the harmonic oscillator of the particles with vanishing momentum. The harmonic oscillator potential which leads to non-vanishing coordinate expectation value is a shifted one therefore the Hamiltonian is

$$H_0 = \frac{P_0^2}{2M_0} + \frac{M_0\omega_0^2}{2}(X_0 - \phi_0)^2. \tag{234}$$

Its ground state is

$$|0\rangle' = e^{-i\phi_0 P_0}|0\rangle \tag{235}$$

where $|0\rangle$ is the ground state of the Hamiltonian with unshifted potential because the shift is made by the operator $e^{-\phi_0 P_0}$,

$$e^{i\phi_0 P_0} f(X_0) e^{-i\phi_0 P_0} = f(X_0 + \phi_0). \tag{236}$$

A special case of the Baker-Cambell-Hausdorff formula,

$$e^{A+B} = e^A a^B e^{-\frac{1}{2}[A,B]}, \tag{237}$$

valid for operators A, B whose commutator, $[A,B]$ commutes with them, shows that the shifted ground state,

$$|0\rangle' = e^{-\sqrt{\frac{M_0\omega_0}{2}}\phi_0(a_0-a_0^\dagger)}|0\rangle = e^{\sqrt{\frac{M_0\omega_0}{2}}\phi_0 a_0^\dagger} e^{-\sqrt{\frac{M_0\omega_0}{2}}\phi_0 a_0} e^{-\frac{M_0\omega_0}{2}\phi_0^2}|0\rangle = e^{\sqrt{\frac{M_0\omega_0}{2}}\phi_0 a_0^\dagger} e^{-\frac{M_0\omega_0}{2}\phi_0^2}|0\rangle, \tag{238}$$

is indeed a coherent state with ill defined particle number.

Another, more systematical approach is based on variation method where the ground state wave functional is sought in the Gaussian form,

$$\Psi_0[\Phi,\phi] = e^{-\frac{1}{2}\int d^3x d^3y [\phi(\boldsymbol{x})-\Phi] K(\boldsymbol{x},\boldsymbol{y}) [\phi(\boldsymbol{x})-\Phi]}. \tag{239}$$

by generalizing the wave function (I1) for $\langle 0|\phi(x)|0\rangle \neq 0$. The previous argument corresponds to ignoring the interaction for the particle modes with non-vanishing momentum. The spontaneous symmetry breaking occurs for $m^2 < 0$ and the fluctuations around this vacuum consist of particles with mass $m_H = 2\sqrt{-m^2}$ and

$$K = \sqrt{-\Delta - 4m^2} \tag{240}$$

according to Eq. (I4).

The particle number density,

$$n = \int \frac{d^3p}{(2\pi)^3} \langle 0|a^\dagger(\boldsymbol{p})a(\boldsymbol{p})|0\rangle \tag{241}$$

has finite contribution coming from the homogeneous mode, $\boldsymbol{p} = 0$. One can see this by writing the wave functional in momentum space,

$$\Psi_0[\Phi,\phi] = e^{-\frac{V}{2}\int \frac{d^3p}{(2\pi)^3} [\tilde\phi(-\boldsymbol{p})-\Phi]\sqrt{m_H^2+\boldsymbol{p}^2}[\tilde\phi(\boldsymbol{p})-\Phi]}, \tag{242}$$

with

$$\tilde\phi(\boldsymbol{p}) = \int d^3x \phi(\boldsymbol{x}) e^{-i\boldsymbol{p}\boldsymbol{x}} \tag{243}$$

and noting that this is a ground state of the appropriate harmonic oscillator for each momentum sector except for $\boldsymbol{p} = 0$ where the ground state is shifted, $\tilde\phi(0) \to \tilde\phi(0) + \Phi$. The lesson is that the vacuum with spontaneous symmetry breaking contains a homogeneous (Bose-Einstein) condensate.

As an application of spontaneously symmetry breaking let us consider the Yukawa-model for a massless fermion and scalar particle,

$$L = \bar\psi i\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) - g\phi\bar\psi\psi, \tag{244}$$

where the potential is symmetric with respect to the change of sign, $U(-\phi) = U(\phi)$ and its minimum at $\phi = \pm\phi_0 \neq 0$ is non-degenerate. As a result, there is a condensate of the scalar particle in the vacuum with say $\langle 0|\phi(x)|0\rangle = \phi_0$ (we have to choose one of the degenerate vacua) and the quantum fluctuations are described by the field $\chi(x)$ where $\phi(x) = \phi_0 + \chi(x)$. The vacuum expectation value of the scalar field is close to the minimum of the potential for weakly coupled particles, when $g \ll 1$ and the higher than second order derivatives of the potential are small. The Lagrangian for the fluctuating fields is

$$L = \bar\psi[i\partial\!\!\!/ - g\phi_0]\psi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}U''(\phi_0)\chi^2 - g\chi\bar\psi\psi - U(\phi_0) + \mathcal{O}\left(\chi^3\right), \tag{245}$$

where the contributions $\mathcal{O}\left(\chi^3\right)$ stand for the self interactions of the the scalar particles above the condensate. We have a scalar particle in this vacuum with mass $m_H = \sqrt{U''(\phi_0)}$ and a massive fermion with mass $M = g\phi_0$. The condensate, generated by the spontaneous breakdown of the formal symmetry $\phi \to -\phi$ of the theory induces a mass for particles which are massless in the naive, unstable vacuum without condensate.

The importance of the mass generation outlined above is that it is a soft process. A phenomenon might be called hard or soft if it is present at arbitrary high energy or at low energies only, respectively. The spontaneous symmetry

breaking is a soft phenomenon because it characterizes the vacuum and the excitations at low energies. In fact, at high energy the potential energy is negligible compared with the kinetic energy and any effect of the potential energy is weak. Such difference become important because quantum field theoretical models are in contradiction with continuous space-time and we have to introduce a minimal distance for the proper treatment. The renormalization of a model is the limit where this minimal distance tends to zero. Models which give converging observables in this limit are called renormalizable. There are models, namely gauge theories, where renormalizability excludes certain particles from having non-vanishing mass. But it might happen, as in the case of weak interactions, where phenomenology strongly suggests that there are in fact massive particles which should be massless to render their model renormalizable. How could we have the best from both worlds: on the on hand, keeping the renormalizability in order to be able to ignore the minimal distance in the theory and the other, having mass for certain particles to reproduce observed phenomena? Soft processes come to our rescue in such problems. When the mass is generated by a soft process then the massive behavior of the particle is obvious at low energy but the high energy behavior remains massless.

B. Adiabatic approximation

The symmetry of the vacuum state can not depend on the parameters of the Lagrangian in a continuous manner, thus the spontaneous symmetry breaking signals a singular point in the observables computed as functions of the parameters of the theory. It must keep in mind that such a critical points, phase transitions in the physical jargon, are mathematically possible for infinite systems only, which contains infinitely many degrees of freedom. The fact that even our Universe has a finite size, suggesting that no phase transition could be observed, raises some problems to settle before spontaneous symmetry breaking is to be used in model building.

Let us start with a little theorem, that the ground state of the one-dimensional Hamiltonian,

$$H = \frac{p^2}{2m} + U(x) \tag{246}$$

where $U(x)$ is a bounded function, acting on the space of single-component wave functions, is non-degenerate. This can be shown by variational method, the lowering of the expectation value of the energy for any test function which is vanishing at some values of the coordinate. As a result, the ground state wave function which can be chosen to be real for this Hamiltonian, represented by a real operator, must be non-vanishing for any x. Due to the continuity of the wave-function ($U(x)$ is bounded) the ground state wave function has a definite sign. Therefore it must be non-degenerate (degenerate eigenvectors of a hermitian operator are either degenerate or orthogonal and there are no orthogonal functions with fixed sign). Therefore there is no spontaneous symmetry breaking in this system. A rather trivial generalization of this theorem shows the absence of spontaneous symmetry breaking in non-relativistic Quantum Mechanics for finite number of spinless particles. In case of non-vanishing spin the wave function has several components and the theorem fails, cf. atoms or nuclei with spin $S > 0$, where the ground state has $2S + 1$-fold degeneracy. One of von Neumann's theorems, the unitary equivalence of the realizations of the Heisenberg commutation relations for finite number of spinless non-relativistic particles, supports the same conclusion.

Let us increase the number of particles and consider the typical time scales of the dynamics. The time scale of the microscopical, elementary processes is in a good approximation independent of the size of the system. But certain collective modes might slow down and become classical as the number of degrees of freedom becomes large. The translation and rotation of rigid bodies will serve as an example in the following order of magnitude estimate to understand the dynamical origin of the breakdown of translation and rotation symmetries in macroscopic bodies. The symmetry of the dynamics with respect to translations and rotations assures that the total momentum $\boldsymbol{P}$ and angular momentum $\boldsymbol{L}$ are preserved and their dynamics can be described by a simple model Hamiltonian

$$H_{coll} = \frac{1}{2} P_j (M^{-1})_{jk} P_k + \frac{1}{2} L_j (\Theta^{-1})_{jk} L_k \tag{247}$$

where the elements of the matrices M and Θ are 3×3 matrices whose matrix elements have the order of magnitude $M^{-1}_{jk} \approx M^{-1}$ and $\Theta_{jk} \approx \Theta = MR^2$ for a rigid body of size R. We place the macroscopic body into a heat bath of temperature T, expressed in Kelvin. Let us see first the when can we consider these collective motions classical. For this end we estimate the average level splitting compared to the excitation energy,

$$\frac{\Delta E_{transl}}{E_{transl}} \approx \frac{\Delta \frac{P^2}{M}}{k_B T} \approx \Delta P \frac{P}{M k_B T} \approx \frac{\hbar}{R} \frac{P}{M k_B T} \tag{248}$$

and we find approximately the same ration for rotations,

$$\frac{\Delta E_{rot}}{E_{rot}} \approx \frac{\Delta \frac{L^2}{\Theta}}{k_B T} \approx \Delta L \frac{L}{\Theta k_B T} \approx \hbar \frac{PR}{MR^2 k_B T} = \frac{\hbar}{R} \frac{P}{M k_B T} \approx \frac{10^{-27}}{R} \frac{P}{M 10^{-16} T} = 10^{-11} \frac{P}{RMT} \tag{249}$$

where the finite constants were ignored in these order of magnitude estimates. The final expression is given in CGS units.

Thus the motion is semiclassical for macroscopic bodies even at as low temperature as the cosmic background radiation. The characteristic time scale of these collective motions are

$$\begin{aligned} t_{transl} &= \frac{R}{v_{transl}} = \frac{RM}{P} \\ t_{rot} &= \frac{1}{\Omega} = \frac{\Theta}{L} \approx \frac{MR^2}{PR} = \frac{RM}{P} \approx \frac{RM}{\sqrt{k_B T M}} = R\sqrt{\frac{M}{k_B T}} \approx 10^8 R\sqrt{\frac{M}{T}}. \end{aligned} \tag{250}$$

This characteristic time scale is extremely large compared to the time scales of microscopic motion therefore the collective coordinates, such as the position and the orientation of the body can be in a very good approximation be considered as constants in studying the internal dynamics of the system. The applicability of such an adiabatic approximation is the dynamical origin of the symmetry broken solution of quantum field theoretical models even if they are applied for macroscopically large but finite systems despite the general theorem about the absence of such phenomenon in finite systems.

C. Continuous symmetries

The dynamics of the symmetry breaking becomes considerable more involved if the symmetry broken by a condensate is continuous. Let us consider the breakdown of the symmetry $G = U(1)$ in a scalar model with complex field, defined by the Lagrangian

$$L = \partial\phi^*\partial\phi - m^2\phi^*\phi - \frac{g}{2}(\phi^*\phi)^2. \tag{251}$$

The existence of the conserved Noether-current indicates that the particle number is conserved in this model. The canonical momenta,

$$\pi = \frac{\partial L}{\partial\partial_0\phi} = \partial_0\phi^*, \qquad \pi^* = \frac{\partial L}{\partial\partial_0\phi^*} = \partial_0\phi \tag{252}$$

lead to the Hamiltonian

$$\begin{aligned} H &= \int d^3x \left[\pi\partial_0\phi + \pi^*\partial_0\phi^* - L\right] \\ &= \int d^3x \left[\pi^*\pi + \nabla\phi^*\nabla\phi + m^2\phi^*\phi + \frac{g}{2}(\phi^*\phi)^2\right]. \end{aligned} \tag{253}$$

The repetition of the free quantum-fluctuation approximation of the real scalar field case above yields the vacuum expectation value

$$\langle\phi(x)\rangle = \phi_0 = e^{i\theta}\sqrt{\frac{-m^2}{g}} \tag{254}$$

and the dynamics of the quantum fluctuations is determined by writing the field variable as

$$\phi(x) = \rho(x)e^{i\Theta(x)}, \qquad \rho(x) \geq 0 \tag{255}$$

which gives the Lagrangian

$$L = (\partial\rho)^2 + \rho^2(\partial\Theta)^2 - m^2\rho^2 - \frac{g}{2}\rho^4. \tag{256}$$

A further shift, $\rho(x) = \rho_0 + \chi(x)$, with $\rho_0 = \sqrt{\frac{-m^2}{g}}$ introduces the fluctuations $\chi(x)$ governed by the Lagrangian

$$L = (\partial\chi)^2 + 2m^2\chi^2 - \frac{g}{2}(4\rho_0\chi^3 + \chi^4) + \rho_0^2(\partial\Theta)^2 - m^2\rho_0^2 - \frac{g}{2}\rho_0^4 \tag{257}$$

We have a massless particle described by $\theta = \rho_0\Theta$, called the Goldstone particle and a massive particle with $m_H^2 = -4m^2$.

What happened here is a special case of a general theorem: The breakdown of each direction of a continuous symmetry group generates a massless particle mode. To demonstrate this theorem in a less obvious case let us consider a model for an n-components scalar field $\phi^a(x)$ with a continuous symmetry group corresponding to the infinitesimal transformations $\phi(x) \to \phi(x) + \delta\phi(x)$ with $\delta\phi(x) = \epsilon^\alpha \tau^\alpha \phi(x)$ involving the generators $\{\tau^\alpha, \alpha = 1 \ldots, N\}$. The Lagrangian is assumed to be

$$L = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) \tag{258}$$

and the vacuum should contain a condensate,

$$\langle 0|\phi^a(x)|0\rangle = \phi_0^a \tag{259}$$

which satisfies the equation

$$\frac{\partial U}{\partial \phi^a}_{|\phi=\phi_0} = 0 \tag{260}$$

in the approximation employed above. A direction in the symmetry group is called unbroken if its generators reserve the vacuum,

$$\epsilon_s^\alpha \tau^\alpha \phi_0 = 0. \tag{261}$$

The symmetry is called broken in directions where the generators change the vacuum,

$$\epsilon_b^\alpha \tau^\alpha \phi_0 \neq 0. \tag{262}$$

The mass square for the different particles in the model are the eigenvalues of the mass matrix

$$M_{ab}^2 = \frac{\partial^2 U(\phi)}{\partial \phi^a \partial \phi^b}_{|\phi=\phi_0} \tag{263}$$

because the particles correspond to the normal modes of the quadratic part of the action. The original, formal symmetry of the model implies

$$\delta U = \partial_a U(\phi) \epsilon^\alpha (\tau^\alpha)^{ab} \phi^b = 0 \tag{264}$$

for arbitrary ϕ. The derivative of this equation with respect to ϕ^b is

$$0 = \partial_d \partial_a U(\phi) \epsilon^\alpha (\tau^\alpha)^{ab} \phi^b + \partial_a U(\phi) \epsilon^\alpha (\tau^\alpha)^{ad} \tag{265}$$

let us now set $\phi = \phi_0$ when the second term is vanishing and we find that either a generator belongs to an unbroken symmetry direction or there is a vanishing eigenvalue of the mass matrix M^2.

What happens is that the quantum fluctuations of the field experiences no restoring force to the equilibrium position in the directions of broken symmetries coming from the local potential in the action due to its the degeneracy in these directions, cf. Eq. (264). The energy of these modes comes from the kinetic energy only, from the space-time derivatives in the action. This contribution is arbitrarily small for symmetry broken fluctuations which are slowly varying in the space-time. The relativistic invariance assures that all excitations of the theory are organized into particle modes. The particle whose energy starts with arbitrary small values is massless. This is the Goldstone boson.

D. Higgs mechanism

The lesson of the spontaneous breakdown of continuous symmetries is that the restoring force for the fluctuations in the symmetry broken direction comes from the kinetic energy only. What happens when the symmetry in question becomes a local symmetry? First of all, local symmetries controlling finite number of degrees of freedom (in a theory with a large but finite UV cutoff) can never be broken dynamically. All what one can expect is that the global part of the gauge invariance is broken. Such a spontaneous breakdown of continuous gauge symmetry produces a surprising phenomenon because this weak restoring force acting an the Goldstone-modes disappears. In fact, the local symmetry is just the statement that fluctuations in the symmetry directions do not change the action, ie. the restoring force of the equilibrium position is exactly vanishing. One can go further and realize that these modes simply decouple form

the rest of the system because the gauge transformations, change of conventions, do not influence the gauge invariant, physical sector of the theory. Thus we expect that in gauge theories the Goldstone-modes decouple from the physical sector. We note that the condensate arising from the spontaneous symmetry breaking may generate mass to particles, coupled to the condensed ones. One finds that when the particles are coupled to the gauge field by minimal coupling then the condensate of the particles participating in the spontaneous symmetry breaking generates mass for the gauge boson corresponding to the broken directions. The final count is that massless Goldstone-modes disappear from the physical sector but the gauge field components corresponding to these internal symmetry directions acquire mass. The massless spin 1 particle has two helicity states, the massive particle has three. Therefore the degree of freedom represented by the Goldstone-bosons get eaten up by the gauge boson. This is the Higgs-mechanism.

Instead of a general proof let us consider first the simplest case, scalar QED, an Abelian gauge theory with a scalar particle. The Lagrangian is chosen to be

$$\begin{aligned} L &= -\frac{1}{4}F^2 + [(\partial_\mu - ieA_\mu)\phi]^*(\partial_\mu - ieA_\mu)\phi - m^2\phi^*\phi - \frac{g}{2}(\phi^*\phi)^2 \\ &= -\frac{1}{4}F^2 + \partial_\mu\phi^*\partial_\mu\phi + \underbrace{ie\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi}_{J_\mu} A_\mu + (e^2A^2 - m^2)\phi^*\phi - \frac{g}{2}(\phi^*\phi)^2 \end{aligned} \tag{266}$$

with $m^2 < 0$ which leads to the spontaneous breakdown of the global phase symmetry and the appearance of the condensate

$$\phi(x) = \rho(x)e^{i\Theta(x)} = \rho(x). \tag{267}$$

Notice that the Goldstone-modes correspond to the field $\Theta(x)$ and gauge invariance allows us to set $\Theta(x) = 0$. This is the gauge choice which removes the Goldstone-modes from the physical sector in a manifest manner. The Lagrangian becomes

$$L = -\frac{1}{4}F^2 + (\partial\rho)^2 + (e^2A^2 - m^2)\rho^2 - \frac{g}{2}\rho^4 \tag{268}$$

in this gauge. We now separate off the vacuum expectation value from the scalar field by writing $\rho = \rho_0 + \chi$ where

$$\rho_0 = \sqrt{\frac{-m^2}{g}}. \tag{269}$$

The dynamics of the fluctuations around the condensate is given by the new Lagrangian

$$L = -\frac{1}{4}F^2 + (\partial\chi)^2 + 2m^2\chi^2 - \frac{g}{2}(4\rho_0\chi^3 + \chi^4) + e^2A^2(\rho_0 + \chi)^2 - m^2\rho_0^2 - \frac{g}{2}\rho_0^4 \tag{270}$$

which contains a massive gauge boson,

$$m_A^2 = e^2\rho_0^2 = -\frac{e^2m^2}{g^2} \tag{271}$$

and a massive scalar particle,

$$m_H^2 = -4m^2 \tag{272}$$

which is real, ie. its particle number is not conserved and is coupled to the gauge field in a non-minimal manner. Both masses introduced in this manner are relevant at low energies and the high energy processes of the model involves massless gauge bosons and a massive, charged scalar particle.

The case of non-Abelian gauge theories is a bit more complicated. Let us consider the Lagrangian

$$\begin{aligned} L &= -\frac{1}{4}F^2 + [(\partial_\mu - igA_\mu)\phi]^\dagger(\partial_\mu - igA_\mu)\phi - U(\phi) \\ &= -\frac{1}{4}F^2 + \partial_\mu\phi^\dagger\partial_\mu\phi + \underbrace{ig\phi^\dagger \overset{\leftrightarrow}{\partial}_\mu \tau^\alpha\phi}_{J_\mu^\alpha} A_\mu^\alpha + g^2A_\mu^\alpha A^{\beta\mu}\phi^\dagger\tau^\alpha\tau^\beta\phi - U(\phi) \end{aligned} \tag{273}$$

for the field $\phi^a(x)$ where the minimum of the potential $U(\phi)$ is degenerate. The mass matrix for the gauge bosons in the vacuum with spontaneously broken symmetry,

$$M_{\alpha\beta}^2 = \phi_0^\dagger\tau^\alpha\tau^\beta\phi_0 = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{M}^2 \end{pmatrix} \tag{274}$$

has rank $\mathrm{Rank}\mathcal{M}^2 = n_b$ where n_b is the number of broken directions in the symmetry group, $||\tau_b\phi_0|| > 0$. Therefore we have a massive gauge boson corresponding to each broken continuous symmetry.

E. Dual superconductor model for quark confinement

The Higgs-phase of a gauge theory has massive gauge bosons and their field strength becomes short ranged as in the superconducting phase of solids. We shall first outline a simple effective theory, a model containing the relevant degrees of freedom and applicable in a window of scales only, for the Bardeen-Cooper-Schrieffer superconducting phase.

BCS superconductivity: The phonon-mediated attractive interaction between electrons is weak in solids but it is not screened. As a result, it dominates the screened Coulomb-repulsion of electrons at long distances. The low energy physics of an electron gas with finite density is two-dimensional because the typical particle-hole excitations are close the the Fermi surface, a two-dimensional manifold. An arbitrarily weak interaction is sufficient in two-dimensions to create bound states. Thus one expects the appearance of e^-e^- bound states in solids at sufficient low temperature, with k_BT being smaller than the binding energy of the pair. The formation of these bound states, Cooper-pairs, is one ingredient of the BCS ground state of conventional superconductors. The other important point is that the binding energy is strong enough to overcome the kinetic energy arising from the localization of the electrons in the loosely-bound Cooper-pairs. As a result the Cooper-pairs form a condensate. This is the BCS vacuum.

The Cooper-pairs are spinless bosons with charge $2e$ and they will be described by a non-hermitian scalar field $\phi(x)$. The Lagrangian for the system of interacting Cooper-pairs, (266), is valid up to binding energies of the Cooper-pairs. This model is considered in the Higgs-phase where the photons are massive and the electromagnetic interaction is strongly suppressed beyond the distance scale m_A^{-1}.

Such a screening is the dynamical origin of the Meissner effects, the concentration of magnetic field in the superconductor into narrow flux tubes. Imagine a superconductor of size $L \gg m_A^{-1}$ placed into homogeneous magnetic field. The magnetic force lines try to dilute themselves within the BCS vacuum where the magnetic field decays rapidly. But there are no magnetic charges to remove the magnetic flux. What can the magnetic field do in order to decrease its magnitude preserving the relation $\boldsymbol{\partial}\cdot\boldsymbol{B}=0$? It piles up the magnetic flux which entered into the superconductor into a narrow flux tube and drives it through the system. Within this flux tube the magnetic field is long-ranged because the flux must leave the large superconductor. Therefore the magnetic field pays the price in energy for the restoration of the normal, non-superconducting vacuum of QED within the flux tube. The energy density in the normal vacuum with $\langle 0|\phi^\dagger(x)\phi(x)|0\rangle = 0$ is higher than that in the symmetry broken vacuum by $|m^2|^2/2g$ thus the increase of the energy density of the false vacuum within the flux tube of radius R is $\Delta E = r^2\pi|m^2|^2/2g$ per unit length. But there is another price, as well, the magnetic flux should be independent of r due to the absence of magnetic charges and this requires $B \approx \Phi_M/r^2\pi$ within the superconductor to forward magnetic flux Φ_M, yielding the additional magnetic energy $E_M = r^2\pi\Phi_M^2/4\pi^3r^4$ per unit length. The minimization of the energy

$$E(r) = \frac{r^2\pi|m^2|^2}{2g} + \frac{\Phi_M^2}{4r^2\pi^2} \tag{275}$$

in r gives $r_{\rm tube} = (g\Phi_M^2/2\pi^3|m^2|^2)^{1/4}$ and

$$E_{\rm tube} = \frac{\Phi_M|m^2|}{\sqrt{2g\pi}}. \tag{276}$$

This is naturally a rough estimate what happens, the interactions among magnetic flux tubes, induced by the charges of the solid might make energetically favorable to split the flux tube into smaller units. We ignore this possibility for simplicity.

Magnetic monopoles: It was Dirac's observation that classical electrodynamics can be extended in a natural manner to include magnetic charges. Let us consider the Maxwell-equations in our units,

$$\begin{aligned} \rho_e &= \boldsymbol{\nabla}\cdot\boldsymbol{E}, & \boldsymbol{j}_e &= \boldsymbol{\nabla}\times\boldsymbol{B} - \partial_0\boldsymbol{E} \\ 0 &= \boldsymbol{\nabla}\cdot\boldsymbol{B}, & 0 &= \boldsymbol{\nabla}\times\boldsymbol{E} - \partial_0\boldsymbol{B} \end{aligned} \tag{277}$$

which can be written in relativistic notation as

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{278}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu = (A_0, -\boldsymbol{A})$, $\boldsymbol{E} = -\partial_0\boldsymbol{A} - \boldsymbol{\nabla}A_0$, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and the second equation is the Bianchi identity for the dual field strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{279}$$

The generalization of the Maxwell-equations for magnetic current j^μ_m is

$$\begin{aligned} \rho_e &= \boldsymbol{\nabla}\cdot\boldsymbol{E}, \qquad & \boldsymbol{j}_e &= \boldsymbol{\nabla}\times\boldsymbol{B} - \partial_0\boldsymbol{E} \\ \rho_m &= \boldsymbol{\nabla}\cdot\boldsymbol{B}, \qquad & \boldsymbol{j}_m &= \boldsymbol{\nabla}\times\boldsymbol{E} - \partial_0\boldsymbol{B} \end{aligned} \tag{280}$$

or

$$\begin{pmatrix} \partial\cdot F \\ \partial\cdot\tilde{F} \end{pmatrix} = \begin{pmatrix} j_e \\ j_m \end{pmatrix}. \tag{281}$$

The price of having magnetic charge (pole) in the system is the singular space-time dependence of the vector potential A_μ but this singularity does no show up in the electric and magnetic field for Dirac's magnetic monopole, ie. magnetic charge $q = 2\pi n/e$ with integer n.

Let us consider two static magnetic monopoles, one with magnetic charge q and another with magnetic charge $-q$ at distance R. The magnetic flux emanating from one charge goes into the other one and one recover Coulomb force law in the normal vacuum. But the interaction of magnetic charges placed into a superconductor changes in a fundamental manner due to the Meissner-effect which concentrates the magnetic field into a straight flux tube between the charges. The result is the static interaction potential between these two magnetic charges is linear in the separation,

$$U(R) = \sigma R, \tag{282}$$

for large enough R. The coefficient σ, the energy of the flux tube per unit length is called string tension. The lesson is that the Meissner-effect can be summed up by saying that magnetic charges experience confining interaction if the vacuum contains a condensate of the electric charges.

Dual symmetry: The generalized Maxwell-equations (281) display an $O(2)$ dual symmetry,

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \to \omega \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}, \quad \begin{pmatrix} j_e \\ j_m \end{pmatrix} \to \omega \begin{pmatrix} j_e \\ j_m \end{pmatrix}, \quad \omega \in O(2). \tag{283}$$

The usual field strength tensor identifies the electric and magnetic fields,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix} \tag{284}$$

and its dual is given by

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix}. \tag{285}$$

Thus the duality transformation (283) implies

$$\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \to \omega \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}. \tag{286}$$

Dual superconductor: Let us apply the duality transformation

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{287}$$

on a normal superconductor. In the resulting system we shall have a condensate of magnetic charges and the dual Meissner-effect consists of the formation of flux-tubes between static electric charges. The electric charges are confined in a dual superconductor which serves as a simple, relativistic model for confinement of charges.

There has been a long series of unsuccessful attempts to find Dirac's magnetic monopoles in Nature despite the simple possibility of including it into the framework of the time honored Maxwell-equations. The possible reason, suggested by the numerical simulation of QED on a space-time lattice is that the magnetic charge, even if they exist, are confined in extremely small, magnetically neutral bound states.

The dual superconductor model for quark confinement is based on the Abelian gauge theory defined by the diagonal subgroup $U(1)\otimes U(1) \subset SU(3)$ in QCD.

VII. RENORMALIZATION GROUP AND ASYMPTOTIC FREEDOM

The renormalization group was first conceived in High Energy Physics to remove the UV divergences of Quantum Field theory.

A. Regularization

The first appearance of UV divergence, Eq. (65), is not serious because the divergence can be eliminated by the normal ordering prescription. The perturbation expansion usually generates divergent contributions in each order and their elimination is far from being obvious. The divergences arise in perturbation expansion due to the summation over intermediate states, like in the last line of Eq. (148). For along while it was an open question if these divergences reflect the true properties of Quantum Field Theories or are the artifacts of the perturbation expansion. This latter alternative is not so surprising in the light of the asymptotic nature of the perturbation expansion. In fact, the point $g = 0$ is usually not a regular one on the plane of complex coupling constant because the infinitesimally small but negative coupling strength already produces Hamiltonians which are unbounded from below and no ground state exists. But the numerical simulations of field theoretical models suggests that the divergences are genuine.

The divergences appear because we insist on using the same equations of motion down to zero distances in space or up to arbitrary high energies. They indicate that the continuous structure of space-time and the rules of Quantum Mechanics, extended for multi-particle systems are incompatible. Which one to preserve? There is no experimental evidence for the continuous structure of space-time and it is difficult to imagine to have any observation in the future which could support a limiting procedure of mathematics. Thus what is left is to be pragmatic and use Lagrangian, equations of motion etc. only in for scales where they can be tested experimentally. We have to introduce a minimal distance r_0 or a maximal energy $\Lambda = 2\pi/r_0$ and ignore or cut off any state beyond these limits in our Fock-space. These limits reflect our ignorance of physics and keep the mathematics corresponding to verifiable and observed phenomena well defined. In the case of a scalar particle all momentum integral as in Eq. (148) must be restrained to the region $|p| < \Lambda$.

B. Renormalization

The price of eliminating the divergences by the introduction of a cutoff is that our predictions depend on an additional parameter, the cutoff. To make things worst this new parameter reminds us our ignorance only. What to do with it, how to set its value? Whenever an observation is made and it is compared with the results of the corresponding calculations we should put the cutoff safely away form the scales of observations where we know what we are doing. To find a general purpose theory, applicable for any scales we seek the limit $\Lambda \to \infty$. This limit will be made by relying on another complication, namely that the parameters of the Lagrangian with interaction have no clear physical meaning. The Lagrangian for free fields contain the physical mass of the particles but interactions 'dress up' the physical particles and change their mass. In a similar manner, the coupling constants parametrize the strength of interactions but have no unambiguous, observable importance in the dynamics of the real, 'dressed' particles.

Let us consider QED where two experimental results O_a with $a = 1, 2$, say the slow electron-electron elastic Thomson-cross section and an atomic energy level, are used to fix the free parameters, e and m of the QED Lagrangian. As a result we need the solution of the set of non-linear equations

$$\begin{aligned} O_1 &= F_1(e, m, \Lambda) \\ O_2 &= F_2(e, m, \Lambda) \end{aligned} \tag{288}$$

where $F_a(e, m, \Lambda)$ denotes expressions obtained in some calculation, for e and m. The solution, $m = m(O_1, O_2, \Lambda)$ and $e = e(O_1, O_2, \Lambda)$ is called the renormalized trajectory of QED in the space of free parameters, on the (m, e) plane. Different trajectories are characterized by different experimental inputs, O_a, and a given trajectory whose points are characterized by the parameter Λ should represent the same theory. Is this really the case? let us consider a third observable $O_3(p)$, corresponding to the momentum scale p, and its theoretical expression $F_3(p, e, m, \Lambda)$. Now the question is whether the quantity

$$O_3^{\text{theor.}}(p) = F_3(p, e(O_1, O_2, \Lambda), m(O_1, O_2, \Lambda), \Lambda) \tag{289}$$

independent of Λ? What happens in perturbative QED is that the cutoff-dependence of the right hand side appears through factors $\mathcal{O}(m/\Lambda)$ and $\mathcal{O}(p/\Lambda)$ and the expression converges for $\Lambda \to \infty$. In this case this equation represents a

theoretical prediction of the theory for observation far enough form the cutoff scale, $p \ll \Lambda$. For $p \approx \Lambda$ our ignorance is too close to make any predictions. This results holds in general and any observable, calculated along the renormalized trajectory converges when the cutoff is removed.

The evaluation of observables along the renormalized trajectory, by means of parameters which are functions of the experimental input and the cutoff, is called renormalization. the theory where every observable converges in this limit is called renormalizable. Non-renormalizable theories contain interactions which can not be reproduced at arbitrarily short distance within the framework of Quantum Field Theory. These theories are supposed to be effective, applicable in a certain scale interval only.

The perturbative condition of renormalizability is given by power counting. Let us imagine a theory with a single coupling constant g for simplicity and an observable O calculated in the framework of the perturbation expansion,

$$O = \sum_{n=0}^{\infty} g^n I_n. \tag{290}$$

We denote the mass dimension of quantities, such as this observable by $[O]$. No other dimension occurs in our units where $\hbar = c = 1$. The coefficients, I_n, are given in terms of momentum integrals of the form

$$I_n = \int_{m_0<|p_1|,\ldots,|p_k|<\Lambda} d^4p_1 \cdots d^4p_k \frac{N(p_1,\ldots,p_k)}{D(p_1,\ldots,p_k)}, \tag{291}$$

where $N(p_1,\ldots,p_k)$ and $D(p_1,\ldots,p_k)$ are polynomials and both IR and UV cutoffs, m_0 and Λ, respectively, are imposed. We introduce the primitive degree of divergence,

$$\omega(I_n) = 4k + [N(p_1,\ldots,p_k)] - [D(p_1,\ldots,p_k)], \tag{292}$$

as the mass dimension of the integral. The contribution to the integral of the regime where all components of the momentum variables $p_1,\ldots,p_k$ diverge in such a manner that the ratio of the components is constant is

$$\int_{m_0}^{\Lambda} dp p^{4k-1+[N]-[D]} = \begin{cases} \left(\frac{\Lambda}{m_0}\right)^{4k+[N]-[D]} & 4k-1+[N]-[D] \neq 0 \\ \ln\frac{\Lambda}{m_0} & 4k-1+[N]-[D] = 0 \end{cases}. \tag{293}$$

The integrals I_n with negative primitive degrees of divergence are UV finite. The power of the UV divergence in Λ is $\omega(I_n)$ for $\omega(I_n) > 0$. Finally, integrals with $\omega(I_n) = 0$ are logarithmically divergent.

Let us forget for a moment that divergences may arise from other regions of the integration domain as well and see how different orders of the perturbation expansion diverge when the UV cutoff is removed. The relation

$$[O] = n[g] + [I_n], \tag{294}$$

giving

$$\omega(I_n) = [O] - n[g] \tag{295}$$

shows that there are stronger and stronger power divergences as we increase the order of the perturbation expansion for $[g] < 0$. This theory is called perturbatively non-renormalizable. The theory with $[g] = 0$ is called perturbatively renormalizable. It has UV divergent contributions at each order of the perturbation expansion but they belong to the same divergence structure. Finally, there are finite number of UV divergent orders when $[g] > 0$. This is a perturbatively super-renormalizable theory.

When the divergences coming from other regions of the integration domains are taken into account one can show by induction in the order of the perturbation expansion that the appropriate adjustment of the free parameter of the Lagrangian, $g \to g(\Lambda)$, $m \to m(\Lambda)$, etc. is enough to remove the divergent contributions to observables for perturbatively renormalizable or super-renormalizable theories. The UV divergences can not be balanced order-by-order in perturbatively non-renormalizable theories by means of the adjustment of the free parameters of the theory.

For theories with massless particles IR divergences might appear due to the form $G(p) = -1/(p^2 + i\epsilon)$ of the propagator. Notice that power of the IR and UV divergences is the same only their sign is the contrary according to Eq. (293). Thus perturbatively non-renormalizable massless theories are IR stable and perturbatively super-renormalizable theories have IR unstable, non-perturbative vacuum. The renormalizable theories represent a compromise between the divergence structure of the IR and UV domain.

Thus theories whose Lagrangian contains parameters with non-negative mass dimensions only are perturbatively renormalizable. For theories with bosons with non-vanishing spin, such as gauge theories, additional care is needed.

For example, the massive spin one propagator, shown in Eq. (128), is $\mathcal{O}\left(p^0\right)$ in momentum space for large p because the dimensionless combination $p^\mu p^\nu/m^2$ which is not seen by power counting but leads to divergences in the longitudinal, ie. gauge dependent sector of the theory and renders massive gauge theories non-renormalizable. The Lagrangian of massless gauge fields contains dimensionless coupling constants and massless gauge bosons. One can show that the formal gauge invariance of the Lagrangian is sufficient to guarantee the absence of radiative corrections to the gauge boson mass in each order of the perturbation expansion. Thus gauge theories are perturbatively renormalizable.

C. Renormalization group

The renormalization group was first constructed in Quantum Field Theory where the adjustment

Observations always involve scales, the time of measurement, the size or mass of the object measured, etc. The change of these scales results in the change of the observed numbers. In other words, the observed quantities depends on the scale of observations. The consequence of this rather general remark is the absence of constants in physics. What was believed to be constant is actually a functions which may vary rather slowly in a certain window of scales.

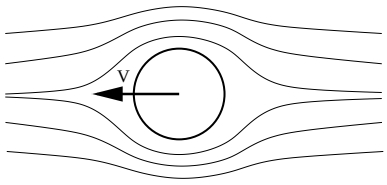


FIG. 8: A rigid ball moving with velocity $\boldsymbol{V}$, immersed into a fluid.

Let us consider a rigid ball immersed into a viscous fluid. What is its mass? There is now problem in the absence of fluid. Mass is a parameter relating force and acceleration, both unambiguously defined and measured for an isolated ball. But part of the viscous fluid moves along the ball and contribute to its mass. How could one find its value? Let us suppose that we can measure the total energy, $E(\boldsymbol{V})$, of the fluid and ball when the latter moves with a velocity $\boldsymbol{V}$. A possible definition of the mass M, based on the usual kinetic energy expression is

$$M(\boldsymbol{V}) = \frac{2[E(\boldsymbol{V}) - E(0)]}{\boldsymbol{V}^2}, \tag{296}$$

is obviously velocity dependent. Any other attempt to define mass for this interacting system brings in functions instead of a single constant.

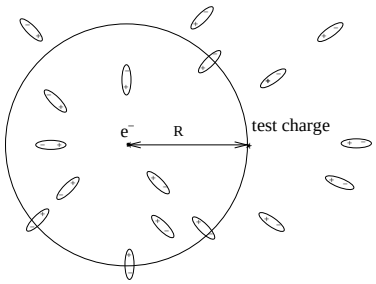


FIG. 9: The polarization cloud around a charge in a classical, polarizable medium.

As another example let us consider an electron, e^-, first inserted into a polarizable medium. The resulting static polarization, shown in Fig. 9, is spherically symmetrical. The Coulomb-force F, experienced by an infinitesimal test charge q, placed at distance R from the electron can be obtained by concentrating all charges around the electron within a sphere of radius R into the position of the electron and ignoring the rest. The small dipoles, induced by the

electrons Coulomb field are neutral therefore the only contribution to the charge within the sphere, beyond that of the electron, come from the dipoles crossing the surface of the sphere. The electron attracts the positive charge of the dipoles therefore the total charge within the sphere, $e(R)$, is always less than the charge of the electron, $e(R) < e$. The charge seen at distance R, $e(R)$, can be defined by the help of the Coulomb law,

$$e(R) = \frac{F(R)R^2}{q}, \tag{297}$$

where $F(R)$ is the force acting on the test charge q. The electron charge, usually assumed to be constant, is actually is the value of the function $e(R)$ for large, macroscopic distances. Similar screening takes place in the vacuum of QED, too. The electron polarizes the Dirac-see, for example the charges in the vacuum at a given instant consist of the original charge plus the e^+e^- virtual vacuum-polarization dipoles created by virtual photons as shown in Fig. 10. A screening phenomenon, similar to the classical polarizable medium, occurs resulting in a distance dependent elementary charge.

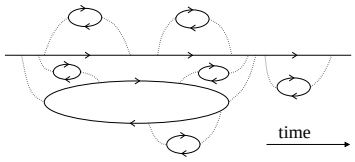


FIG. 10: Virtual charges in the Dirac-see.

The scale-dependence of the charge appears in the simplest manner when the corrections to Coulomb-law is sought in the vacuum. The perturbative series of the Wilson-loop of Fig. 7 is

$$\begin{aligned}\langle 0|w[\gamma]|0\rangle &= \langle 0|T[e^{-ie\int dz\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z)}e^{-ie\int_x^y d\gamma^\mu A_\mu(\gamma)}]|0\rangle \\ &= \sum_{m,n=0}^{\infty}\frac{(-ie)^{n+m}}{m!n!}\int dz_1\cdots dz_m\int d\gamma_1^{\mu_1}\cdots d\gamma_n^{\mu_n} \\ &\quad\times\langle 0|T[\bar{\psi}(z_1)\gamma^{\nu_1}\psi(z_1)A_{\nu_1}(z_1)\cdots\bar{\psi}(z_m)\gamma^{\nu_m}\psi(z_m)A_{\nu_m}(z_m)A_{\nu_1}(\gamma_1)\cdots A_{\nu_n}(\gamma_n)]|0\rangle,\end{aligned} \tag{298}$$

and Fig. 11 shows some of its graphs. It is clear that the effective charge,

$$e^2(R) = -U(R)R, \tag{299}$$

defined in terms of the static potential,

$$U(R) = \lim_{T\to\infty}\frac{1}{T}\ln\langle 0|w[\gamma]|0\rangle, \tag{300}$$

is distance dependent.

The renormalization group method is a systematic way of finding the scale dependence in physical quantities by taking into account the "dressing" phenomena generated by interactions around the "bare", naive degrees of freedom. One defines "running" coupling constants, $g(\mu)$, parameters of the system which characterize the dynamics at a given momentum scale μ. For example, the mass of a scalar particle $m(\mu)$ could be defined at the momentum scale μ by the expression

$$G(p^2 = \mu^2) = \frac{1}{\mu^2 - m^2(\mu)} \tag{301}$$

for the propagator in the momentum space. This mass is not a constant and the interaction induces a non-trivial scale dependence. One can construct such a physically motivated, so called renormalized versions for all parameters occurring in the Lagrangian. They have physical meaning in contrary to the original parameters of the Lagrangian, called bare parameters. Since the scale dependence is generated by interactions an infinitesimal change of the observational scale, $\mu \to \mu + \delta\mu$, implies infinitesimal amount of interactions and could be computed in leading order in the perturbation expansion, using $\delta\mu/\mu$ as small parameters, if we could allow all possible interactions occurring

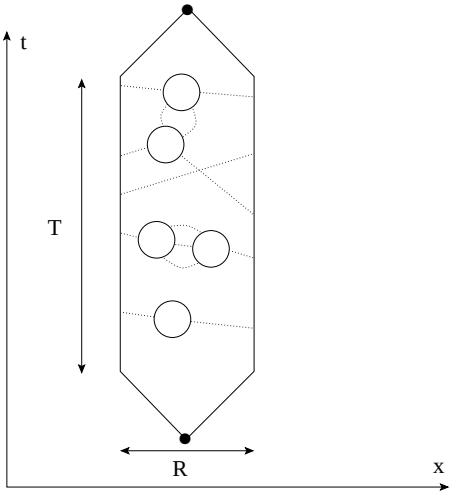


FIG. 11: Few graphs contributing to the radiative corrections to the Wilson-loop.

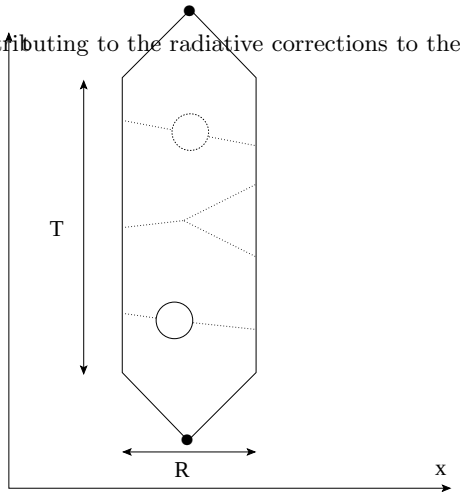


FIG. 12: Leading order graphs contributing to the running coupling constants in QCD.

explicitly in our Lagrangian. But instead we usually have few coupling constants only and they have to regenerate all. This is the reason that in this case the beta-function, defined by

$$\beta = \mu \frac{dg(\mu)}{d\mu}, \tag{302}$$

receive contributions to every order in the perturbation expansion. The multiplicative factor μ on the right hand side is inserted to make the beta-function dimensionless. The beta-functions depends on the running coupling constants (running masses treated as running coupling constants), the cutoff drops out during renormalization if the theory is renormalizable.

The leading order contribution to the beta function of QED comes from the lowest, $\mathcal{O}\left(e^4\right)$ graph in Fig. 11,

$$\beta(e) = \frac{e^3}{12\pi^2} + \mathcal{O}\left(e^5\right). \tag{303}$$

The leading order graph, contributing to the running of the coupling constants in QCD, are shown in Fig. 12. Their contributions give

$$\beta(g) = -\frac{11 - \frac{2}{3}n_f}{(4\pi)^2} g^3 + \mathcal{O}\left(g^5\right) \tag{304}$$

for n_f quark flavor.

The integration of the renormalization group equation

$$\mu \frac{dg(\mu)}{d\mu} = \beta(g) = \beta_0 g^3 + \mathcal{O}\left(g^5\right) \tag{305}$$

gives

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 - 2g^2(\mu_0)\beta_0 \ln \frac{\mu^2}{\mu_0^2}}. \tag{306}$$

This result represents a resummation of infinite higher order contributions of the perturbation series. What is implied in the integration of the renormalization group equation is that the running coupling constant rather then the initial value characterizes the strength of interactions. What is complicated is the "dressing up" of the physical particles by the elementary excitations, vacuum-polarizations, using the bare parameters of the Lagrangian defined with the help of the cutoff. But once we have obtained a running coupling constant at a given scale μ, we can use make low order perturbative calculation at that scale by using the running coupling constant as small parameter because this latter characterizes the strength of interaction just at that scale. In other words, the running coupling constant comprises lot of the perturbative corrections of the original perturbation expansion of the bare theory.

The resummation carried out by solving the renormalization group equation can easily be understood by the following analogy. Let us consider the differential equation

$$\frac{dx(t)}{dt} = f(x(t), t). \tag{307}$$

The straightforward perturbation expansion of Quantum Field Theory is analogous to the strategy when the solution of this equation is sought by expanding the right hand side around $x = 0$,

$$\frac{dx(t)}{dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(0,t)}{\partial x^n} x^n(t). \tag{308}$$

This is the good strategy for $x(t) \approx$ but not otherwise. What we can do for large values of the $x(t)$ is to expand around a closed enough point, $x_0(t)$,

$$\frac{dx(t)}{dt} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(x_0(t),t)}{\partial x^n} [x(t) - x_0(t)]^n. \tag{309}$$

What is the best choice? It is clearly $x_0(t) = x(t)$. But a simple numerical quadrature for the integration, such as

$$\begin{aligned} x(t + \Delta t) &= x(t) + \Delta t f(x(t), t) \\ &= x(t) + \Delta t \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(x_0(t),t)}{\partial x^n} [x(t) - x_0(t)]^n \end{aligned} \tag{310}$$

shows that the base of the expansion should be somewhere in between $x(t - \Delta t)$ and $x(t)$, say at

$$x_0(t) = x(t + c\Delta t) \approx x(t) + c\Delta t f(x(t), t), \tag{311}$$

c being a fixed number. What we have achieved is not really shocking but is still a new small parameter for the expansion of the right hand side of the differential equation,

$$x(t + \Delta t) = x(t) + \Delta t f(x(t), t) + \mathcal{O}\left(\Delta t^2\right). \tag{312}$$

This expansion now has more chance to converge than in Eq. (308) and the simple algorithm, based on the leading order contribution, becomes exact in the differential equation limit, $\Delta \to 0$ because we carry our base point for the expansion along the solution. This explains the remark made above Eq. (302) because the taking into account all coupling constants is the analogy of keeping all term in the series (308), an procedure which is equivalent with the leading order expansion results with well chosen, close enough base point, ie. $x_0(t)$ is the analogy of the running coupling constant.

What kind of radiative corrections are resummed by the integration of the renormalization group equation? Once the bare, dimensionless coupling constant is mall enough the only problem with perturbation expansion may arise from the not fast enough decrease of the coefficient I_n in Eq. (290). This may happen only if the integration over intermediate states receive unexpectedly large contribution which can happen either from too many high energy modes or too many low energy modes. Thus the UV or IR divergences are the only danger for the perturbation expansion. These divergences can be power-like or logarithmic in the cutoffs or the observational scale μ. Let us suppose that the theory considered has no mass parameter in the Lagrangian and all coupling constants are dimensionless. Then logarithmic

divergences are left only. One can see that the integration of the renormalization group equation corresponds to the resummation of the logarithmically diverging, dangerous terms of the perturbation expansion.

Theories, such a QCD with $n_f \leq 16$, having negative beta-functions in the perturbative domain become weakly interacting at short distances and are called asymptotically free. But notice that unless the beta function has a zero and changes sign towards larger values of the coupling constant asymptotic freedom implies strong coupling at large distances. In particular, the coupling constant (306) diverges at

$$\mu_L^2 = \mu_0^2 e^{\frac{1}{2\beta_0 g^2(\mu_0^2)}}, \tag{313}$$

called Landau scale. Such an anti-screening, the increase of the charge with the distance, opens the way for quark confinement in QCD, a genuine non-perturbative effect.

The beta-function of QED is positive, meaning that the charge increases at short distances and becomes infinite at the Landau-scale, obtained in Eq. (313) by the replacement $e \to g$. The electromagnetic interactions can not be maintained beyond this energy scale. Due to the smallness of the electric charge on the mass shell of a free-electron, $e^2(m_{el}^2) \approx 1/137$, the Landau-scale is well beyond the unification scale of the electromagnetic and weak interactions and this problem, the non-renormalizability of QED, is an interesting but purely mathematical issue.

The numerical simulation of quantum field theories suggest that only asymptotically free theories are renormalizable, all non-asymptotically free theory develop Landau-scale and their cutoff can not be removed. Since non-Abelian gauge theories are the only theory which can be asymptotically free in four dimensions an important lesson for the Standard Model is that it is non-renormalizable due to its $U(1)$ and Higgs-sectors.

VIII. STANDARD MODEL

The Standard model covers all three interactions which are observed with their quantum effects. The strong interaction is not really unified with the electromagnetic and weak interactions, QCD and the unified electro-weak theories are simply placed beside in the Standard Model.

A. Fermi theory of the weak interaction

The unified electro-weak theory was constructed historically from the Fermi contact interaction model for weak interaction, developed at the beginning by the help of low energy processes.

1. β-decay

The first theory of the weak interaction, constructed by Fermi, is based on the weak current. The angle dependence of the cross section of the β-decay

$$n \to p + e^- + \bar{\nu}_e \tag{314}$$

suggests that the the interaction Lagrangian, responsible for this process is

$$L_\beta = -\sqrt{2} G_\beta \bar{e}(x)\gamma^\mu \nu_e(x) \bar{\psi}_p(x)\gamma_\mu \psi_n(x) + c.c., \tag{315}$$

where $e(x)$ and $\nu_e(x)$ are the electron and neutrino fields and $G_\beta = (1.1473 \pm 0.0006) \cdot 10^{-5}\text{GeV}^{-2}$. The neutrino mass, extracted by using the energy-momentum conservation turned out to be zero. The interaction Lagrangian in terms of the quark field which generates the same transition is

$$L_\beta = -\sqrt{2} G_\beta \bar{e}(x)\gamma^\mu \nu_e(x) \bar{u}(x)\gamma_\mu d(x) + c.c. \tag{316}$$

Neutrinos exist in left handed version only therefore we can make the replacement

$$\nu_e \to \frac{1}{2}(1-\gamma^5)\nu_e. \tag{317}$$

in the Lagrangian. The breakdown of the space inversion symmetry, generated by the γ^5 matrix is observed in the hadronic processes, too, and the extended interaction Lagrangian turns out to be of the form

$$L_\beta = -\frac{G_\beta}{\sqrt{2}} \bar{e}(x)\gamma^\mu(1-\gamma^5)\nu_e(x) \bar{u}(x)\gamma_\mu(1-1.225\gamma^5)d(x) + c.c. \tag{318}$$

The coefficient of the γ^5 matrix is not unity in the hadronic current due to renormalizations, arising from the strong interaction.

Similar Lagrangian,

$$L_\mu = -\frac{G_\mu}{\sqrt{2}}\bar{e}(x)\gamma^\mu(1-\gamma^5)\nu_e(x)\bar{\mu}(x)\gamma_\mu(1-\gamma^5)\nu_\mu(x) + c.c., \tag{319}$$

with $G_\mu \approx 1.02 G_\beta$ can be extracted from leptonic processes, such as from the μ-decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{320}$$

2. Cabibbo angle

There is no reason to expect that quark flavors, the eigenstates of the interaction Hamiltonian, agree for the weak and strong interactions. It was found experimentally that when the quark flavors, defined by the strong interaction are used then the interaction Lagrangian for weak processes still displays the same u quark field but requires the replacement

$$d \to d_C = \cos\theta_C d + \sin\theta_C s. \tag{321}$$

Therefore, the weak interaction Lagrange function is

$$L_\beta = -\frac{G_\beta}{\sqrt{2}}\bar{e}(x)\gamma^\mu(1-\gamma^5)\nu_e(x)\bar{u}(x)\gamma_\mu(1-1.225\gamma^5)[\cos\theta_C d(x) + \sin\theta_C s(x)] + c.c. \tag{322}$$

where the Cabibbo angle θ_C is defined by

$$\cos\theta_C = \frac{G_\beta}{G_\mu}. \tag{323}$$

The current numerical value is

$$\sin\theta_C = 0.21. \tag{324}$$

3. Universality

The similar occurrences of the hadronic and leptonic currents in different, elementary weak interaction processes suggests the generalization

$$L = -\frac{G_\beta}{\sqrt{2}} J_\mu J^\mu + c.c. \tag{325}$$

where the weak current

$$J_\mu = J_\mu^{lept} + J_\mu^{hadr}, \tag{326}$$

is the sum of the leptonic and hadronic contributions,

$$\begin{aligned} J_\mu^{lept} &= \bar{e}\gamma^\mu(1-\gamma^5)\nu_e + \bar{\mu}\gamma_\mu(1-\gamma^5)\nu_\mu, \\ J_\mu^{hadr} &= \bar{u}\gamma_\mu(1-1.225\gamma^5)(\cos\theta_C d + \sin\theta_C s), \end{aligned} \tag{327}$$

respectively. This universality of the weak interaction has been confirmed in careful experimental studies.

4. Neutral current

The weak currents, introduced above doe not cover all observed weak processes. The flavor and lepton type conserving weak processes, such as

$$\begin{aligned} \bar{\nu}_\mu + e^- &\to \bar{\nu}_\mu + e^-, \\ u + \nu_e &\to u + \nu_e, \end{aligned} \tag{328}$$

do not involve change of electric charge and can be generated by means of introducing a neutral current in the interaction Lagrangian

$$L = -\frac{G_n}{\sqrt{2}} J^n_\mu J^{n\mu} + c.c. \tag{329}$$

where the neutral current is build up by means of terms like $\bar{u}\gamma^\mu(1-\gamma^5)u$.

5. Generalization of the Fermi-model

The contact interaction, introduced above is not satisfactory because it is non-renormalizable. This means that phenomena generated by it are dominated by high energy processes. One would like to find a theory of weak interaction which can be localized at a certain energy range and this is possible for normalizable theories only.

The solution of the problem was sought in the direction suggested by QED where the

$$e^- + e^- \to e^- + e^- \tag{330}$$

elastic scattering amplitude is given in the leading order of the perturbation expansion by the effective interaction action

$$S^{eff}_{em} = -\frac{G_e}{\sqrt{2}} \int dxdy J^{em}_\mu(x) D^{\mu\nu}(x-y) J^{em\ \nu}(y) \tag{331}$$

expressed in terms of the (neutral) electric current $J^{em}_\mu = \bar{e}\gamma^\mu e$ and the photon propagator $D^{\mu\nu}(x-y)$. Though such an interaction is non-renormalizable QED is based on a more elementary, local interaction Lagrangian which is renormalizable. Can we have a similar rescue operation for the weak interaction? The current-current interaction is short ranged as opposed to the electromagnetic interactions, therefore the interaction of the weak current should be associated with the exchange of a sufficiently massive vector boson. In the case of a single current one tries the model

$$L = -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 + \frac{1}{2} m^2_W W^2 - \frac{1}{2\alpha}(\partial_\mu W^\mu)^2 + g J^\mu A_\mu. \tag{332}$$

The static Yukawa-potential, generated by the exchange of a massive W-particle,

$$U(r) = \frac{g^2}{4\pi r} e^{-r m_W}, \tag{333}$$

suggests the replacement

$$U(r) \to \frac{g^2}{m^2_W} \delta^{(3)}(\boldsymbol{x}) \tag{334}$$

at low energies, at distances longer than m^{-1}_W. Thus the massive vector boson exchange appears as a contact interaction at low energies with $G \approx g^2/m^2_W$. By choosing $g^2 = 1/137$ we find $m_W \approx 80 GeV$.

The massive vector boson propagator,

$$D^{\mu\nu}(x) = \int \frac{dp}{(2\pi)^4} e^{-ipx} \left[\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2_W} \right) \frac{1}{p^2 - m^2_W + i\epsilon} + \frac{p^\mu p^\nu}{m^2_W} \frac{1}{p^2 - \alpha m^2_W + i\epsilon} \right]. \tag{335}$$

is a symmetric tensor and as such its Fourier transform contains the multiplicative factor $p^\mu p^\nu / m^2_W$. This makes the longitudinal part of the propagator $\mathcal{O}\left(p^0\right)$ which is in contrast with the normalizable, $\mathcal{O}\left(p^{-2}\right)$ transverse part. The mass couples the transverse and the longitudinal helicity states of the W-boson and the unusually slow decay of the propagator for large momenta renders the theory non-renormalizable. This problem, the construction of a renormalizable interaction which reduces to a current-current contact interaction at low energy, blocked the development of particle physics for several decades.

B. Unified electro-weak theory

The solution of the aforementioned problem, the mass generation for vector bosons in a renormalizable manner, is provided by spontaneous symmetry breaking. In fact, it has been mentioned that the spontaneous symmetry breaking is a low energy phenomenon. It may produce mass for certain particles at high energy but the particle mass, generated in this manner disappears at high energy where the issue of renormalizability is settled.

It is important that the original Lagrangian, before spontaneous symmetry breaking be gauge invariant. This property assures the decoupling of the longitudinal modes of the gauge boson which is the trouble maker at high energy.

We outline now the construction of the Lagrangian for the unified electro-weak theory.

1. Vector bosons

There are four currents, participating in the electro-weak interactions, two charged and a neutral weak current and a neutral electromagnetic one. The vector bosons, coupled to these currents will be denoted by W^-, W^+, Z^0 and A. The electromagnetic interactions suggests the presence of a $U(1)$ gauge group and it is natural to extend this gauge symmetry to $SU(2)\otimes U(1)$ because the group $SU(2)$ has just three generators. We shall see that the weak currents couple to the left handed version of the fermions therefore the gauge group is $SU(2)_L\otimes U(1)$. These two independent gauge groups imply two independent coupling constants, g and g', appearing in the Lagrangian

$$L_{vb} = -\frac{1}{2}\mathrm{tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{336}$$

where

$$\begin{aligned} W_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + ig[W_\mu, W_\nu], \qquad W_\mu = W_\mu^a T^a, \qquad T^a = \frac{\sigma^a}{2}, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \tag{337}$$

2. Higgs sector

We want to break the global subgroup of $SU(2)\otimes U(1)$ in such a manner that one of the four gauge fields, A_μ, remain massless. This can be achieved by using $SU(2)_L$ doublet scalar Higgs field

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1^1 + i\phi_1^2\\ \phi_2^1 + i\phi_2^2\end{pmatrix}. \tag{338}$$

The eigenvalue y_H of the charge Y for the Higgs field, appearing in the covariant derivative

$$D_\mu = \partial_\mu + igW_\mu^a T^a + ig'B_\mu Y \tag{339}$$

will be set later.

The Lagrangian for the Higgs field is assumed to be

$$L_H = \frac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{2}\mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{340}$$

with $\mu^2 < 0$ and the vacuum expectation value of the Higgs field is taken to be

$$\langle 0|\phi|0\rangle = \begin{pmatrix}0\\ \phi_0\end{pmatrix}, \tag{341}$$

where

$$\phi_0 = \sqrt{\frac{-\mu^2}{\lambda}} \tag{342}$$

in the mean-field approximation.

The gauge field corresponding to the unbroken symmetry group directions satisfies the equation

$$(gW^aT^a + g'BY)\langle 0|\phi|0\rangle = \frac{1}{2}\phi_0 \begin{pmatrix} g(W^1 - iW^2) \\ -gW^3 + 2g'By_H \end{pmatrix} = 0. \tag{343}$$

The linear combination of the gauge fields which is orthogonal to the three real vector field appearing in this equation,

$$g'W^3 + \frac{g}{2y_H}B, \tag{344}$$

is identified with the photon field.

The mass term for the gauge bosons,

$$\langle 0|\phi^\dagger|0\rangle(-igW^a_\mu T^a - ig'B_\mu y_H)(igW^a_\mu T^a + ig'B_\mu y_H)\langle 0|\phi|0\rangle, \tag{345}$$

can be written as

$$\frac{g^2\phi_0^2}{4}W^+_\mu W^{-\mu} + \frac{\phi_0^2}{4}(g^2 + 4g'^2y_H^2)Z_\mu Z^\mu \tag{346}$$

in terms of the normal modes

$$\begin{aligned} W^\pm_\mu &= W^1_\mu \mp iW^2_\mu, \\ W^3_\mu &= \frac{1}{\sqrt{g^2 + 4g'^2y_H^2}}(gZ_\mu + 2g'y_HA_\mu), \\ B_\mu &= \frac{1}{\sqrt{g^2 + 4g'^2y_H^2}}(-2g'y_HZ_\mu + gA_\mu). \end{aligned} \tag{347}$$

The value $y_H = 1/2$ is usually set at this stage.

3. Quarks and leptons

The fermions are rearranged in three families,

$$\begin{aligned} 1: &\quad \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L, \quad \begin{pmatrix}u \\ d_s\end{pmatrix}_L, \quad e_R, \quad u_R, \quad d_{sR} \\ 2: &\quad \begin{pmatrix}\nu_\mu \\ \mu\end{pmatrix}_L, \quad \begin{pmatrix}c \\ s_s\end{pmatrix}_L, \quad \mu_R, \quad c_R, \quad s_{sR} \\ 3: &\quad \begin{pmatrix}\nu_\tau \\ \tau\end{pmatrix}_L, \quad \begin{pmatrix}t \\ b_s\end{pmatrix}_L, \quad \tau_R, \quad t_R, \quad b_{sR}, \end{aligned} \tag{348}$$

where the quarks flavors are defined by the strong interactions.

It will be shown that one quark flavor direction can be chosen to be identical for the weak and strong interactions. We use this freedom to introduce unique flavor for the u, c and t quarks and allow Cabbibo-mixing for the d, s and b quarks whose strong interaction directions are denoted by d_s, s_s and b_s.

The charged $SU(2)_L$ gauge fields, $W^\pm_\mu$, mediate interaction among the components of the $SU(2)_L$ doublets. The neutral weak and electric currents couple to the gauge fields Z^0_μ and A_μ corresponds to two orthogonal linear superpositions of the W^3_μ and B_μ. The kinetic energy with minimal couplings is of the form

$$\begin{aligned} L_{quark-lept} &= (\bar\nu_e, \bar e)_L i\not{D} \begin{pmatrix}\nu_e \\ e\end{pmatrix}_L + (\bar u, \bar d_s)_L i\not{D} \begin{pmatrix}u \\ d_s\end{pmatrix}_L + \bar e_R i\not{\tilde D} e_R + \bar u_R i\not{\tilde D} u_R + \bar d_{sR} i\not{\tilde D} d_{sR} \\ &+(\bar\nu_\mu, \bar\mu)_L i\not{D} \begin{pmatrix}\mu \\ \nu_\mu\end{pmatrix}_L + (\bar c, \bar s_s)_L i\not{D} \begin{pmatrix}u \\ s_s\end{pmatrix}_L + \bar\mu_R i\not{\tilde D}\mu_R + \bar c_R i\not{\tilde D} c_R + \bar s_{sR} i\not{\tilde D} s_{sR} \\ &+(\bar\nu_\tau, \bar\tau)_L i\not{D} \begin{pmatrix}\tau \\ \nu_\tau\end{pmatrix}_L + (\bar t, \bar b_s)_L i\not{D} \begin{pmatrix}t \\ b_s\end{pmatrix}_L + \bar\tau_R i\not{\tilde D}\tau_R + \bar t_R i\not{\tilde D} t_R + \bar b_{sR} i\not{\tilde D} b_{sR}, \end{aligned} \tag{349}$$

where the $U(1)$ covariant derivative $\tilde D = \partial_\mu + ig'B_\mu Y$ has been introduced.

The mixing of the W^3_μ and B_μ fields in forming the physical normal modes,

$$\begin{aligned} W^3_\mu &= \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \\ B_\mu &= -\sin\theta_W Z_\mu + \cos\theta_W A_\mu, \end{aligned} \tag{350}$$

is parametrized by the weak mixing angle θ_W,

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \qquad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}. \tag{351}$$

The detailed form of the lepton interaction Lagrangian is

$$\begin{aligned} L_{i,lept} &= -\frac{g}{2}(W^+\bar\nu_{eL}\gamma^\mu e_L + W^-\bar e_L\gamma^\mu\nu_{eL}) - \sqrt{g^2+g'^2}\Big[(\bar\nu_e,\bar e)_L\gamma^\mu\Big(T^3(\cos^2\theta_W Z_\mu + \cos\theta_W\sin\theta_W A_\mu) \\ &\quad +Y(\sin\theta_W\cos\theta_W A_\mu - \sin^2\theta_W Z_\mu)\Big)\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L + \bar e_R\gamma^\mu Y(\sin\theta_W\cos\theta_W A_\mu - \sin^2\theta_W Z_\mu)e_R\Big] \\ &= -\frac{g}{2}(W^+\bar\nu_{eL}\gamma^\mu e_L + W^-\bar e_L\gamma^\mu\nu_{eL}) \\ &\quad -\frac{1}{2}\sqrt{g^2+g'^2}\Big[Z_\mu\bar\nu_{eL}\gamma^\mu\nu_{eL} + A_\mu(2\sin\theta_W\cos\theta_W\bar e_L\gamma^\mu e_L - 2y_R\sin\theta_W\cos\theta_W\bar e_R\gamma^\mu e_R) \\ &\quad +Z_\mu\Big((\cos^2\theta_W - \sin^2\theta_W)\bar e_L\gamma^\mu e_L + 2y_R\sin^2\theta_W\bar e_R\gamma^\mu e_R\Big)\Big], \end{aligned} \tag{352}$$

where the absence of photon-neutrino coupling in the observed processes has been taken into account by the choice

$$y_{L,lept} = -\frac{1}{2}. \tag{353}$$

The usual electric current,

$$J^{em}_\mu = \bar e_R\gamma^\mu e_R + \bar e_L\gamma^\mu e_L \tag{354}$$

has been achieved by setting

$$y_{R,lept} = -1. \tag{355}$$

When the whole interaction Lagrangian, $L_{quark-lept}$, is expressed in terms of the gauge field normal modes one finds

$$L_{quark-lept} = -g(W^+_\mu J^{ch\mu} + W^-_\mu J^{ch\mu\dagger}) - \sqrt{g^2+g'^2}Z_\mu J^{n\mu} - eA_\mu J^{em\mu} \tag{356}$$

where

$$e = -\frac{gg'}{\sqrt{g^2+g'^2}}, \tag{357}$$

and

$$\begin{aligned} J^{ch}_\mu &= (\bar\nu_e,\bar e)_L\gamma^\mu(T^1+iT^2)\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L = \bar\nu_{eL}\gamma^\mu e_L, \\ J^{em}_\mu &= -(\bar\nu_e,\bar e)_L\gamma^\mu(T^3+Y)\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L - \bar e_R\gamma_\mu(T_3+Y)e_R = \bar e_R\gamma^\mu e_R + \bar e_L\gamma^\mu e_L, \\ J^n_\mu &= (\bar\nu_e,\bar e)_L\gamma^\mu\left[T^3 - \sin^2\theta_W(T^3+Y)\right]\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L - \bar e_R\gamma_\mu\sin^2\Theta_W Y e_R \\ &= \frac{1}{2}\bar\nu_{eL}\gamma^\mu\nu_{eL} - \frac{1}{2}\bar e_L\gamma^\mu e_L - \sin^2\theta_W J^{em}_\mu. \end{aligned} \tag{358}$$

The second equation yields the electric charge

$$Q = T^3 + Y \tag{359}$$

which in turn requires $y_{L,quark} = \frac{1}{6}$. Since $T^3 = 0$ the $U(1)$ charge for right handed quarks,

$$y_{R,u\ quark} = \frac{2}{3}, \qquad y_{R,d_s\ quark} = -\frac{1}{3}, \tag{360}$$

is flavor dependent.

4. Mixing

The mass term, the gradient free quadratic part of the Lagrangian in the quantum fluctuations, is responsible of two different kinds of mixing. On the one hand, it mixes the left and right hand states of massive fermions. On the other hand, it defines the quark flavor states of the weak interaction.

The $SU(2)_L$ gauge invariance allows the mass term to couple the left and right handed singlets only therefore it may contain the combinations

$$\phi^\dagger \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_L = \phi_1^\dagger \psi_1 + \phi_2^\dagger \psi_2, \qquad \phi^{tr} \epsilon \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_L = \phi_1 \psi_2 - \phi_2 \psi_1, \tag{361}$$

where

$$\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{362}$$

is the metric tensor for $SU(2)$ spinors. The $U(1)$ gauge invariance requires that the Y charge of the mas term is vanishing. Since

$$y_H = y_{L,lept} - y_{R,lept} \tag{363}$$

the only allowed mass term for leptons is

$$L_{m,lept} = -(\bar{e}, \bar{\mu}, \bar{\tau})_R \mathcal{C}_{lept} \begin{pmatrix} \phi^\dagger \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0}, \tag{364}$$

where the family mixing is parametrized by the 3×3 complex matrix $\mathcal{C}_{lept}$. The flavor dependent Y-charge of the right handed quarks allows both kinds of mass term for quarks,

$$L_{m,q} = -(\bar{d}_s, \bar{s}_s, \bar{b}_s)_R \mathcal{C}_{quark1} \begin{pmatrix} \phi^\dagger \begin{pmatrix} u \\ d_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} c \\ s_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} t \\ b_s \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0} - (\bar{u}, \bar{c}, \bar{t})_R \mathcal{C}_{quark2} \begin{pmatrix} \phi^{tr} \epsilon \begin{pmatrix} u \\ d_s \end{pmatrix}_L \\ \phi^{tr} \epsilon \begin{pmatrix} c \\ s_s \end{pmatrix}_L \\ \phi^{tr} \epsilon \begin{pmatrix} t \\ b_s \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0}. \tag{365}$$

After having identified the mixing of the left and right handed fermions we parametrize the allowed family mixing by bringing the most general mass term, $L_{m,lept} + L_{m,q}$, into a canonical form by means of basis transformations. The key observation is that the family independence of the Y-charge allows the arbitrary $U(3)$ transformations

$$\begin{aligned} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R &\to \mathcal{U}_{lept} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_R, & \begin{pmatrix} d_s \\ s_s \\ b_s \end{pmatrix}_R &\to \mathcal{U}_{quark1} \begin{pmatrix} d_s \\ s_s \\ b_s \end{pmatrix}_R, & \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R &\to \mathcal{U}_{quark2} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_R, \\ \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L &\to \mathcal{V}_{lept}^\dagger \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L, & \begin{pmatrix} d_s \\ s_s \\ b_s \end{pmatrix}_L &\to \mathcal{V}_{quark}^\dagger \begin{pmatrix} d_s \\ s_s \\ b_s \end{pmatrix}_L, & \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L &\to \mathcal{V}_{quark}^\dagger \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L, \end{aligned} \tag{366}$$

on the families. Furthermore, other arbitrary $U(3)$ transformations

$$\begin{aligned} \mathcal{C}_{lept} &\to \mathcal{U}_{lept}^\dagger \mathcal{C}_{lept} \mathcal{V}_{lept}, \\ \mathcal{C}_{quark1} &\to \mathcal{U}_{quark1}^\dagger \mathcal{C}_{quark1} \mathcal{V}_{quark}, \\ \mathcal{C}_{quark2} &\to \mathcal{U}_{quark2}^\dagger \mathcal{C}_{quark2} \mathcal{V}_{quark} \end{aligned} \tag{367}$$

are allowed on the family mixing matrices introduced so far.

The matrix $\mathcal{C}_{lept}\mathcal{C}_{lept}^{\dagger}$ s hermitian and its eigenvalues, m_e^2, m_μ^2 and m_τ^2 are non-negative. In the basis for leptons where $\mathcal{C}_{lept}\mathcal{C}_{lept}^{\dagger}$ is diagonal one has

$$\mathcal{C}_{lept} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} W \tag{368}$$

with W being an $U(3)$ matrix. We carry out a further basis transformations $\mathcal{V}_{lept} = W^{\dagger}$ and arrive at the lepton mass matrix

$$\mathcal{C}_{lept} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{369}$$

We proceed in a similar manner in the case of the quark families in determining the canonical basis. The only difference is that we have the same left handed doublets in both mass terms. The usual choice is to diagonalize the mass matrix involving the right handed states of the upper components of the doublets. The result is the mass matrix

$$\mathcal{C}_{quark2} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \tag{370}$$

for the u, c and t flavor states. But the matrix $\mathcal{C}_{quark1}$, corresponding to the right handed states of the lower components of the doublets can now only be multiplied from the left by an arbitrary $U(3)$ matrix and the parametrization

$$\mathcal{C}_{quark1} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \mathcal{V}^{\dagger} \tag{371}$$

is the most general. The transformation $\mathcal{U}_{quark1} = \mathcal{V}^{\dagger}$ on the right handed states $(d_s, s_s, b_s)_R$ brings us to the form

$$\mathcal{C}_{quark1} = \mathcal{V} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \mathcal{V}^{\dagger} \tag{372}$$

where $\mathcal{V}$ is the Kobayashi-Maskawa-Cabbibo (KMC) matrix, the generalization of the Cabbibo-mixing for three families.

The form of the KMC matrix can be simplified by performing $U(1)^3$ phase transformations on the quark families,

$$\mathcal{V} \to \begin{pmatrix} e^{-i\phi_1} & 0 & 0 \\ 0 & e^{-i\phi_2} & 0 \\ 0 & 0 & e^{-i\phi_3} \end{pmatrix} \mathcal{V} \begin{pmatrix} e^{i\chi_1} & 0 & 0 \\ 0 & e^{i\chi_2} & 0 \\ 0 & 0 & e^{i\chi_3} \end{pmatrix}. \tag{373}$$

Such a transformation with $\phi_i = \chi_j$ leave the KMC matrix invariant thus we can eliminate five parameters in this manner. The remaining four parameter (the group $U(3)$ is nine dimensional) of the KCM matrix are denoted by $0 \le \theta_i \le \pi/2$, $i = 1, 2, 3$ and $0 \le \delta \le 2\pi$, yielding

$$\mathcal{V} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} \\ -s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 e^{i\delta} & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} \end{pmatrix} \tag{374}$$

with $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$.

C. Lagrange function of the Standard Model

The strong interaction is included in the Standard Model by extending the gauge symmetry to $SU(3)_c \otimes SU(2)_L \otimes U(1)$. The corresponding Lagrangian is written in the form

$$L = -\frac{1}{2}\mathrm{tr}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\mathrm{tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{1}{2}(D_\mu\phi)^{\dagger}D^\mu\phi - V(\phi) + \bar{\psi}i\not{D}\psi + L_m \tag{375}$$

where the field strength tensors

$$\begin{aligned} G_{\mu\nu} &= (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu)\frac{\lambda^a}{2}, \\ W_{\mu\nu} &= (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu)\frac{\sigma^a}{2}, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \end{aligned} \tag{376}$$

correspond to the gauge fields,

$$V(\phi) = \frac{1}{2}\mu^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2, \tag{377}$$

is the Higgs-potential and the covariant derivative

$$D_\mu = \partial_\mu + ig_s G^a_\mu \frac{\lambda^a}{2} + igW^a_\mu \frac{\sigma^a}{2} + ig' B_\mu Y \tag{378}$$

contains the generators λ^a, σ^a and Y which are taken in the appropriate representations. The mass term for the fermions is

$$\begin{aligned} L_m &= -(\bar{e},\bar{\mu},\bar{\tau})_R \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \begin{pmatrix} \phi^\dagger \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0} \\ &\quad -(\bar{d}_s,\bar{s}_s,\bar{b}_s)_R \mathcal{V} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \mathcal{V}^\dagger \begin{pmatrix} \phi^\dagger \begin{pmatrix} u \\ d_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} c \\ s_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} t \\ b_s \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0} - (\bar{u},\bar{c},\bar{t})_R \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} \begin{pmatrix} \phi^\dagger \begin{pmatrix} u \\ d_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} c \\ s_s \end{pmatrix}_L \\ \phi^\dagger \begin{pmatrix} t \\ b_s \end{pmatrix}_L \end{pmatrix} \frac{1}{\phi_0}. \end{aligned} \tag{379}$$

After the spontaneous breaking of three out of the four dimensions of the symmetry $SU(2)_L \otimes U(1)$ the Higgs-field expectation value is written as

$$\phi(x) = \begin{pmatrix} 0 \\ \phi_0 + \phi_1 \end{pmatrix}, \tag{380}$$

where

$$\phi_0 = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{381}$$

This leads finally to the Lagrangian

$$\begin{aligned} L &= -\frac{1}{2}\mathrm{tr}G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\mathrm{tr}W_{\mu\nu}W^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + W^+_\mu W^{-\mu} m^2_W(1+\rho)^2 + \frac{1}{2}Z^\mu Z_\mu m^2_Z(1+\rho)^2 \\ &\quad + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2_H}{2}\phi^2_1 - \frac{m^2_H}{2\phi_0}\phi^3_1 - \frac{m^2_H}{4\phi^2_0}\phi^4_1 + \sum_\ell \{\bar{\nu}_{\ell L} i\partial\!\!\!/ \nu_{\ell L} + \bar{\ell}_L[i\partial\!\!\!/ - m_\ell(1+\rho)]\ell_L\} \\ &\quad + \sum_q \{\bar{q}[i\partial\!\!\!/ - m_q(1+\rho)]q\} - g_s J^{sa\mu} G^a_\mu - g(W^+_\mu J^{ch\mu} + W^-_\mu J^{ch\mu\dagger}) - \sqrt{g^2+g'^2}Z_\mu J^{n\mu} - eA_\mu J^{em\mu} \end{aligned} \tag{382}$$

with

$$\rho(x) = \frac{\phi_1(x)}{\phi_0}, \tag{383}$$

and

$$m^2_H = -2\mu^2. \tag{384}$$

The lower component of the weak $SU(2)_L$ hadronic doublets are

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = \mathcal{V}^\dagger \begin{pmatrix} d_s \\ s_s \\ b_s \end{pmatrix} \tag{385}$$

which give the currents

$$\begin{aligned}
J^{sa}_\mu &= (\bar{u},\bar{c},\bar{t})\frac{\lambda^a}{2}\gamma^\mu \begin{pmatrix} u \\ c \\ t \end{pmatrix} + (\bar{d},\bar{s},\bar{b})\frac{\lambda^a}{2}\gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \\
J^{ch\mu} &= (\bar{\nu}_e,\bar{\nu}_\mu,\bar{\nu}_\tau)_L\gamma_\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_L + (\bar{u},\bar{c},\bar{t})_L\gamma^\mu\mathcal{V}\begin{pmatrix} d \\ s \\ b \end{pmatrix}_L, \\
J^{em}_\mu &= -(\bar{\nu}_e,\bar{e})_L\gamma^\mu(T^3+Y)\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L - \bar{e}_R\gamma_\mu(T_3+Y)e_R + \text{quarks} \\
&= \bar{e}\gamma^\mu e + \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma_\mu d, \\
J^n_\mu &= (\bar{\nu}_e,\bar{\nu}_\mu,\bar{\nu}_\tau)\gamma^\mu\frac{1}{2}L\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} + (\bar{e},\bar{\mu},\bar{\tau})\gamma^\mu\left(-\frac{1}{2}L+\sin^2\theta_W\right)\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \\
&\quad +(\bar{u},\bar{c},\bar{t})\gamma^\mu\left(\frac{1}{2}L-\frac{2}{3}\sin^2\theta_W\right)\begin{pmatrix} u \\ c \\ t \end{pmatrix} + (\bar{d},\bar{s},\bar{b})\gamma^\mu\left(-\frac{1}{2}L+\frac{1}{3}\sin^2\theta_W\right)\begin{pmatrix} d \\ s \\ b \end{pmatrix}
\end{aligned} \tag{386}$$

with

$$L = \frac{1-\gamma_5}{2}. \tag{387}$$

Note that the KMC matrix drops from the neutral current, there is no weak-flavor changing neutral current.

There are 18 free parameters in the model, the three coupling constants, g_s, g, g', corresponding to the three semisimple subgroup of the gauge group, the boson masses, m_W^2, m_H^2, the fermion masses, m_e, m_μ, m_τ, m_u, m_d, m_c, m_s, m_t, m_b, and the four parameters of the KMC matrix, θ_1, θ_2, θ_3, δ.

D. Epilogue

What part of physics is covered by the Standard Model? If the model is renormalizable then it covers interactions up to some finite energy. The upper energy scale is about hundred GeV and this is where the model becomes "empty". But alias, the Standared Model is not perturbative at high energy and the perturbative proof of renormalizability, honored by a Nobel price, is not much help at high energy. The problem comes from the non asymptotically free sectors of the model, the $U(1)$ gauge field and the self interactions of the Higgs-particles. Above 500 GeV perturbation expansion breaks down and numerical simulations are left to guide us. According to them the Standard Model can not function beyond this order of magnitude of energy and necessarily new physics is to be found.

Appendix A: Classical Field theory

Our goal in Section is to obtain equations of motion which are local in space-time and are compatible with certain symmetries in a systematic manner. The basic principle is to construct equations which remain invariant under nonlinear transformations of the coordinates and the time. We shall see that this property renders the resulting equations more useful.

1. Variational principle

Field theory is a dynamical system containing degrees of freedom, denoted by $\phi(\boldsymbol{x})$, at each space point $\boldsymbol{x}$. The coordinate $\phi(\boldsymbol{x})$ can be a single real number (real scalar field) or consist n-components (n-component field). Our

goal is to provide an equation satisfied by the trajectory $\phi_{cl}(t, \boldsymbol{x})$. The index cl is supposed to remind us that this trajectory is the solution of a classical (as opposed to a quantum) equation of motion.

This problem will be simplified in two steps. First we restrict $\boldsymbol{x}$ to a single value, $\boldsymbol{x} = \boldsymbol{x}_0$. The n-component field $\phi(\boldsymbol{x}_0)$ can be thought as the coordinate of a single point particle moving in n-dimensions. We need the equation satisfied by the trajectory of this particle. The second step of simplification is to reduce the n-dimensional function $\phi(\boldsymbol{x}_0)$ to a single point on the real axis.

a. Single point on the real axis

Problem: identification of a point on the real axis, $x_{cl} \in \mathbb{R}$, in a manner which is independent of the reparametrization of the real axis.

Solution: Find a function with vanishing derivative at x_{cl} only:

$$\frac{df(x)}{dx}_{|x=x_{cl}} = 0 \tag{A1}$$

To check the reparametrization invariance of this equation we introduce new coordinate y by the function $x = x(y)$ and find

$$\frac{df(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{df(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0 \tag{A2}$$

Variational principle: There is simple way of rewriting Eq. (A1). Let us perform an infinitesimal variation of the coordinate $x \to x + \delta x$, and write

$$\begin{aligned} f(x_{cl} + \delta x) &= f(x_{cl}) + \delta f(x_{cl}) \\ &= f(x_{cl}) + \delta x \underbrace{f'(x_{cl})}_{0} + \frac{\delta x^2}{2} f''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \end{aligned} \tag{A3}$$

The variation principle, equivalent of Eq. (A1) is

$$\delta f(x_{cl}) = \mathcal{O}\left(\delta x^2\right), \tag{A4}$$

stating that x_{cl} is characterized by the property that an infinitesimal variation around it, $x_{cl} \to x_{cl} + \delta x$, induces an $\mathcal{O}\left(\delta x^2\right)$ change in the value of $f(x_{cl})$.

b. Non-relativistic point particle

Problem: identification of a trajectory in a coordinate choice independent manner.

Variational principle: Let us identify a trajectory $x_{cl}(t)$ by specifying the coordinate at the initial and final time, $x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f$ (by assuming that the equation of motion is of second order in time derivatives) and consider a variation of the trajectory $x(t)$: $x(t) \to x(t) + \delta x(t)$ which leaves the initial and final conditions invariant (ie. does not modify the solution). Our function $f(x)$ of the previous section becomes a functional, called action

$$S[x(\cdot)] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \tag{A5}$$

involving the Lagrangian $L(x(t), \dot{x}(t))$. (The symbol $x(\cdot)$ in the argument of the action functional is supposed to remind us that the variable of the functional is a function. It is better to put a dot in the place of the independent variable of the function $x(t)$ otherwise the notation $S[x(t)]$ can be mistaken with an embedded function $S(x(t))$.) The

variation of the action is

$$\begin{aligned}\delta S[x(\cdot)] &= \int_{t_i}^{t_f} dtL\left(x(t)+\delta x(t),\dot{x}(t)+\frac{d}{dt}\delta x(t)\right)-\int_{t_i}^{t_f} dtL(x(t),\dot{x}(t)) \\ &= \int_{t_i}^{t_f} dt\Big[L(x(t),\dot{x}(t))+\delta x(t)\frac{\partial L(x(t),\dot{x}(t))}{\partial x}+\frac{d}{dt}\delta x(t)\frac{\partial L(x(t),\dot{x}(t))}{\partial \dot{x}}+\mathcal{O}\left(\delta x(t)^2\right) \\ &\quad -\int_{t_i}^{t_f} dtL(x(t),\dot{x}(t))\Big] \\ &= \int_{t_i}^{t_f} dt\delta x(t)\left[\frac{\partial L(x(t),\dot{x}(t))}{\partial x}-\frac{d}{dt}\frac{\partial L(x(t),\dot{x}(t))}{\partial \dot{x}}\right]+\underbrace{\delta x(t)}_{0}\frac{\partial L(x(t),\dot{x}(t))}{\partial \dot{x}}\Bigg|_{t_f}^{t_i}+\mathcal{O}\left(\delta x(t)^2\right) \end{aligned} \tag{A6}$$

The variational principle amounts to the suppression of the integral in the last line for an arbitrary variation, yielding the Euler-Lagrange equation:

$$\frac{\partial L(x,\dot{x})}{\partial x}-\frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}}=0 \tag{A7}$$

The generalization of the previous steps for a n-dimensional particle gives

$$\frac{\partial L(\boldsymbol{x},\dot{\boldsymbol{x}})}{\partial \boldsymbol{x}}-\frac{d}{dt}\frac{\partial L(\boldsymbol{x},\dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}}=0. \tag{A8}$$

It is easy to check that the Lagrangian

$$L=T-U=\frac{m}{2}\dot{\boldsymbol{x}}^2-U(\boldsymbol{x}) \tag{A9}$$

leads to the usual Newton equation

$$m\ddot{\boldsymbol{x}}=-\boldsymbol{\nabla}U(\boldsymbol{x}). \tag{A10}$$

It is advantageous to introduce the generalized momentum:

$$p=\frac{\partial L(x,\dot{x})}{\partial \dot{x}} \tag{A11}$$

which allows to write the Euler-Lagrange equation as

$$\dot{p}=\frac{\partial L(x,\dot{x})}{\partial x} \tag{A12}$$

The coordinate not appearing in the Lagrangian in an explicit manner is called cyclic coordinate,

$$\frac{\partial L(x,\dot{x})}{\partial x_{cycl}}=0. \tag{A13}$$

For each cyclic coordinate there is a conserved quantity because the generalized momentum of a cyclic coordinate, p_{cycl} is conserved according to Eqs. (A11) and (A13).

c. Scalar field

Problem: identification of the equation of motion for an n-component field, $\phi_a(x)$, $a=1,\ldots,n$. (Notation: $x=(t,\boldsymbol{x})$.)

Variational principle: let us consider a variation of the trajectory $\phi(x)$:

$$\phi(x)\to\phi(x)+\delta\phi(x),\qquad \delta\phi(t_i,\boldsymbol{x})=\delta\phi(t_f,\boldsymbol{x})=0. \tag{A14}$$

The variation of the action

$$S[\phi(\cdot)]=\int_V \underbrace{dtd^3x}_{dx}L(\phi,\partial\phi) \tag{A15}$$

is

$$\begin{aligned}\delta S &= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\delta\partial_\mu\phi_a\right) + \mathcal{O}\left(\delta^2\phi\right) \\ &= \int_V dx \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a}\delta\phi_a + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\partial_\mu\delta\phi_a\right) + \mathcal{O}\left(\delta^2\phi\right) \\ &= \int_{\partial V} ds^\mu \delta\phi_a \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a} + \int_V dx\delta\phi_a \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a}\right) + \mathcal{O}\left(\delta^2\phi\right) \end{aligned} \tag{A16}$$

The first term for $\mu = 0$,

$$\int_{\partial V} ds^0 \delta\phi_a \frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} = \int_{t=t_f} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} - \int_{t=t_i} d^3x \underbrace{\delta\phi_a}_{0} \frac{\partial L(\phi,\partial\phi)}{\partial\partial_0\phi_a} = 0 \tag{A17}$$

is vanishing because there is no variation at the initial and final time. When $\mu = j$ then

$$\int_{\partial V} ds^j \delta\phi_a \frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a} = \int_{x_j=\infty} ds^j \delta\phi_a \underbrace{\frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a}}_{0} - \int_{x_j=-\infty} ds^j \delta\phi_a \underbrace{\frac{\partial L(\phi,\partial\phi)}{\partial\partial_j\phi_a}}_{0} = 0 \tag{A18}$$

and it is still vanishing because we are interested in the dynamics of localized systems and the interactions are supposed to be short ranged. Therefore, $\phi = 0$ at the spatial infinities and the Lagrangian is vanishing. The suppression of the second term gives the Euler-Lagrange equation

$$\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a} = 0. \tag{A19}$$

Examples:

1. Free scalar particle:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \quad \Longrightarrow \quad (\Box + m^2)\phi(x) = 0 \tag{A20}$$

2. Self interacting scalar particle:

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - U(\phi) \quad \Longrightarrow \quad (\partial_\mu\partial^\mu + m^2) = -U'(\phi) \tag{A21}$$

3. Free fermions:

$$\begin{aligned} L &= \bar\psi[i\partial_\mu\gamma^\mu - m]\psi \to \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar\psi)\gamma^\mu\psi] - m\bar\psi\psi \quad \Longrightarrow \quad (i\partial_\mu\gamma^\mu - m)\psi(x) = 0 \\ \gamma^0 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1}\end{pmatrix}, \quad \gamma^\ell = \begin{pmatrix} 0 & \sigma^\ell \\ -\sigma^\ell & 0\end{pmatrix} \end{aligned} \tag{A22}$$

4. Yukawa model (proton + σ meson):

$$L = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{M^2}{2}\sigma^2 + \bar\psi[i\partial_\mu\gamma^\mu - m - g\sigma]\psi \tag{A23}$$

2. Noether theorem

The reparametrization invariance of the Euler-Lagrange equation shows that there is a conserved current for each continuous symmetry.

Symmetry: A transformation of the space-time coordinates $x^\mu \to x'^\mu$, and the field $\phi_a(x) \to \phi'_a(x)$ preserves the equation of motion. Since the equation of motion is obtained by varying the action, the action should be preserved

by the symmetry transformations. A slight generalization is that the action can in fact be changed by a surface term which does not influence its variation, the equation of motion at finite space-time points. Therefore, the symmetry transformations satisfy the condition

$$L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu \Lambda^\mu \tag{A24}$$

with a certain vector function $\Lambda^\mu(x')$.

Continuous symmetry: There are infinitesimal symmetry transformations, in an arbitrary small neighborhood of the identity, $x^\mu \to x^\mu + \delta x^\mu$, $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$. Examples: Rotations, translations in the space-time, and $\phi(x) \to e^{i\alpha}\phi(x)$ for a complex field.

Conserved current: $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \boldsymbol{\partial} v j = -\int_{\partial V} d\boldsymbol{s} \cdot \boldsymbol{j} \tag{A25}$$

It is useful to distinguish external and internal spaces, corresponding to the space-time and the values of the field variable. Eg.

$$\phi_a(x): \quad \underbrace{\mathbb{R}^4}_{\text{external space}} \quad \to \quad \underbrace{\mathbb{R}^m}_{\text{internal space}} \quad . \tag{A26}$$

Internal and external symmetry transformations act on the internal or external space, respectively.

a. Point particle

The main points of the construction of the Noether current for internal symmetries can be best understood in the framework of a particle.

To find the analogy of the internal symmetries let us consider a point particle with the continuous symmetry $\boldsymbol{x} \to \boldsymbol{x} + \epsilon \boldsymbol{f}(\boldsymbol{x})$ for infinitesimal ϵ,

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x} + \epsilon \boldsymbol{f}(\boldsymbol{x}), \dot{\boldsymbol{x}} + \epsilon(\dot{\boldsymbol{x}} \cdot \boldsymbol{\partial})\boldsymbol{f}(\boldsymbol{x})) + \mathcal{O}\left(\epsilon^2\right). \tag{A27}$$

Let us introduce a new, time dependent coordinates, $\boldsymbol{y}(t) = \boldsymbol{y}(\boldsymbol{x}(t))$, based on the solution of the equation of motion, $\boldsymbol{x}_{cl}(t)$, in such a manner that one of them will be $y^1(t) = \epsilon(t)$, where $\boldsymbol{x}(t) = \boldsymbol{x}_{cl}(t) + \epsilon(t)\boldsymbol{f}(\boldsymbol{x}_{cl}(t))$. There will be $n-1$ other new coordinates, y^ℓ, $\ell = 2, \ldots, n$ whose actual form is not interesting for us. The Lagrangian in terms of the new coordinates is defined by $L(\boldsymbol{y}, \dot{\boldsymbol{y}}) = L(\boldsymbol{y}(\boldsymbol{x}), \dot{\boldsymbol{y}}(\boldsymbol{x}))$. The ϵ-dependent part assumes the form

$$L(\epsilon, \dot{\epsilon}) = L(\boldsymbol{x}_{cl} + \epsilon \boldsymbol{f}(\boldsymbol{x}_{cl}), \dot{\boldsymbol{x}}_{cl} + \epsilon(\dot{\boldsymbol{x}}_{cl} \cdot \boldsymbol{\partial})\boldsymbol{f}(\boldsymbol{x}_{cl}) + \dot{\epsilon}\boldsymbol{f}(\boldsymbol{x}_{cl})) + \mathcal{O}\left(\epsilon^2\right). \tag{A28}$$

What is the equation of motion of this Lagrangian? Since the solution is $\epsilon(t) = 0$ it is sufficient to retain the $\mathcal{O}(\epsilon)$ contributions in the Lagrangian only,

$$L(\epsilon, \dot{\epsilon}) \to L^{(1)}(\epsilon, \dot{\epsilon}) = \epsilon \frac{\partial L(\boldsymbol{x}_{cl}, \dot{\boldsymbol{x}}_{cl})}{\partial \boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x}_{cl}) + \frac{\partial L(\boldsymbol{x}_{cl}, \dot{\boldsymbol{x}}_{cl})}{\partial \dot{\boldsymbol{x}}} [\epsilon(\dot{\boldsymbol{x}}_{cl} \cdot \boldsymbol{\partial})\boldsymbol{f}(\boldsymbol{x}_{cl}) + \dot{\epsilon}\boldsymbol{f}(\boldsymbol{x}_{cl})] \tag{A29}$$

up to an ϵ-independent constant. The corresponding Euler-Lagrange equation is

$$\frac{\partial L^{(1)}(\epsilon, \dot{\epsilon})}{\partial \epsilon} - \frac{d}{dt}\frac{\partial L^{(1)}(\epsilon, \dot{\epsilon})}{\partial \dot{\epsilon}} = 0. \tag{A30}$$

(this is the point where the formal invariance of the equation of motion under nonlinear, time dependent transformations of the coordinates is used). According to Eq. (A27) ϵ is a cyclic coordinate,

$$\frac{\partial L(\epsilon, \dot{\epsilon})}{\partial \epsilon} = 0 \tag{A31}$$

and its generalized momentum,

$$p_\epsilon = \frac{\partial L(\epsilon, \dot{\epsilon})}{\partial \dot{\epsilon}} \tag{A32}$$

is conserved.

The external space transformation corresponds to the shift of the time, $t \to t+\epsilon$ which induces $x(t) \to x(t-\epsilon) = x(t) - \epsilon\dot{x}(t)$ for infinitesimal ϵ. This is a symmetry as long as the Hamiltonian (and the Lagrangian) does not contain explicitly the time. In fact, the action changes by a boundary contribution only which can be seen by expanding the Lagrangian in time around $t-\epsilon$,

$$\int_{t_i}^{t_f} dt L(x(t),\dot{x}(t)) = \int_{t_i}^{t_f} dt \left[L(x(t-\epsilon),\dot{x}(t-\epsilon)) + \epsilon \frac{dL(x(t),\dot{x}(t))}{dt} \right] + \mathcal{O}\left(\epsilon^2\right) \tag{A33}$$

and as a result the variational equation of motion remains unchanged. But the continuation of the argument is slightly different from the case of internal symmetry. We consider ϵ as a time dependent function which generates a transformation of the coordinate, $x(t) \to x(t-\epsilon(t)) = x(t) - \epsilon(t)\dot{x}(t) + \mathcal{O}\left(\epsilon^2\right)$. The Lagrangian of $\epsilon(t)$ as new coordinate for $x(t) = x_{cl}(t)$ is

$$\begin{aligned}
L^{(1)}(\epsilon,\dot{\epsilon}) &= L(x_{cl}(t-\epsilon),\dot{x}_{cl}(t-\epsilon)) - L(x_{cl}(t),\dot{x}_{cl}(t)) \\
&= -\epsilon\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial x} - \frac{d\epsilon\dot{x}_{cl}}{dt}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}} + \mathcal{O}\left(\epsilon^2\right) \\
&= \underbrace{-\epsilon\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial x} - \epsilon\ddot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}}}_{-\epsilon\frac{dL(x_{cl},\dot{x}_{cl})}{dt}} - \dot{\epsilon}\dot{x}_{cl}\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}} + \mathcal{O}\left(\epsilon^2\right) \\
&= -\epsilon\left[\frac{dL(x_{cl},\dot{x}_{cl})}{dt} - \frac{d}{dt}\left(\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}}\dot{x}_{cl}\right)\right] - \frac{d}{dt}\left(\frac{\partial L(x_{cl},\dot{x}_{cl})}{\partial \dot{x}_{cl}}\epsilon\dot{x}_{cl}\right) + \mathcal{O}\left(\epsilon^2\right)
\end{aligned} \tag{A34}$$

up to an ϵ-independent constant. Its Euler-Lagrange equation (A30) assures the conservation of the energy,

$$H = \frac{\partial L(x,\dot{x})}{\partial \dot{x}}\dot{x} - L(x,\dot{x}). \tag{A35}$$

b. Internal symmetries

An internal symmetry transformation of field theory acts on the internal space only. We shall consider linearly realized internal symmetries for simplicity where

$$\delta x^\mu = 0, \quad \delta\phi_a(x) = \epsilon \underbrace{\tau_{ab}}_{generator} \phi_b(x). \tag{A36}$$

This transformation is a symmetry,

$$L(\phi,\partial\phi) = L(\phi + \epsilon\tau\phi, \partial\phi + \epsilon\tau\partial\phi) + \mathcal{O}\left(\epsilon^2\right). \tag{A37}$$

Let us introduce new "coordinates", ie. new field variable, $\Phi(\phi)$, in such a manner that $\Phi^1(x) = \epsilon(x)$ where $\phi(x) = \phi_{cl}(x) + \epsilon(x)\tau\phi_{cl}(x)$, $\phi_{cl}(x)$ being the solution of the equations of movement. The linearized Lagrangian for $\epsilon(x)$ is

$$\begin{aligned}
\tilde{L}(\epsilon,\partial\epsilon) &= L(\phi_{cl} + \epsilon\tau\phi(x), \partial\phi_{cl} + \partial\epsilon\tau\phi(x) + \epsilon\tau\partial\phi(x)) \\
&\to \epsilon\tau\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\phi} + [\partial\epsilon\tau\phi(x) + \epsilon\tau\partial\phi(x)]\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\partial\phi}.
\end{aligned} \tag{A38}$$

The symmetry, Eq. (A37), indicates that ϵ is a cyclic coordinate and the equation of motion

$$\frac{\partial\tilde{L}(\epsilon,\partial\epsilon)}{\partial\epsilon} - \partial_\mu\frac{\partial\tilde{L}(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon} = 0. \tag{A39}$$

shows that the current,

$$J^\mu = -\frac{\partial\tilde{L}(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon} = -\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\tau\phi \tag{A40}$$

defined up to a multiplicative constant as the generalized momentum of ϵ, is conserved. Notice that (i) we have an independent conserved current corresponding to each independent direction in the internal symmetry group and (ii) the conserved current is well defined up to a multiplicative constant only.

Examples:

1. n-component real scalar field: ϕ_a, $a = 1, \cdots, n$, the symmetry group is $G = O(n)$,

$$\begin{aligned} L &= \frac{1}{2}(\partial\phi)^2 - V(\phi^2) \\ \delta\phi &= \epsilon^a \tau^a \phi, \quad \tau^a \in o(n) \\ J^a_\mu &= -\partial_\mu \phi \tau^a \phi \end{aligned} \tag{A41}$$

2. Single complex scalar field: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, symmetry: $G = U(1)$, $\phi(x) \to e^{i\alpha}\phi(x)$

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - V\left(\frac{1}{2}(\phi_1^2 + \phi_2^2)\right) \\ &= \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^\dagger\phi - V(\phi^\dagger\phi) \end{aligned} \tag{A42}$$

Two ways of treating a complex variable in complex calculus: $z \to \binom{z}{z^*}$ or $z = z_1 + iz_2 \to \binom{z_1}{z_2}$

(a) $\binom{\phi}{\phi^*}$:

$$\begin{aligned} \begin{pmatrix}\phi \\ \phi^*\end{pmatrix} \quad &: \quad \begin{pmatrix}\phi \\ \phi^*\end{pmatrix} \to \begin{pmatrix}e^{i\alpha}\phi \\ e^{-i\alpha}\phi^*\end{pmatrix}, \quad \delta\begin{pmatrix}\phi \\ \phi^*\end{pmatrix} = i\alpha\begin{pmatrix}\phi \\ -\phi^*\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi \\ \phi^*\end{pmatrix}, \quad \tau = i\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \\ J &= -\frac{\partial L}{\partial\partial_\mu\phi}\tau\phi = -i\left(\frac{\partial L}{\partial\partial_\mu\phi}\phi - \frac{\partial L}{\partial\partial_\mu\phi^*}\phi^*\right) = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) = i\phi^* \overleftrightarrow{\partial}_\mu \phi \end{aligned} \tag{A43}$$

(b) $\binom{\phi_1}{\phi_2}$:

$$\begin{aligned} \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \quad &: \quad \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}[\cos\alpha\phi_1 - \sin\alpha\phi_2 + i(\cos\alpha\phi_2 + \sin\alpha\phi_1)] \\ &\delta\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \alpha\begin{pmatrix}-\phi_2 \\ \phi_1\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}, \quad \tau = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} \\ J &= -\frac{\partial L(\phi, \partial\phi_1)}{\partial\partial_\mu\phi}\tau\phi = -\left(-\frac{\partial L}{\partial\partial_\mu\phi_1}\phi_2 + \frac{\partial L}{\partial\partial_\mu\phi_2}\phi_1\right) = \partial_\mu\phi_1\phi_2 - \partial_\mu\phi_2\phi_1 \\ &= \frac{i}{2}[\partial_\mu(\phi_1 + i\phi_2)^*(\phi_1 + i\phi_2) - (\phi_1 + i\phi_2)^*\partial_\mu(\phi_1 + i\phi_2)] = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) \end{aligned} \tag{A44}$$

3. n-component complex scalar field: ϕ_a, $a = 1, \cdots, n$, $G = U(n)$

$$\begin{aligned} L &= \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi) \\ \delta\phi &= \epsilon^a\tau^a\phi, \quad \delta\phi^\dagger = \epsilon^a(\phi\tau^a)^\dagger = -\epsilon^a\phi^\dagger\tau^a \\ J^a_\mu &= -\partial_\mu\phi^\dagger\tau^a\phi + \partial_\mu\phi(\tau^a)^{\mathrm{tr}}\phi^\dagger = -\partial_\mu\phi^\dagger\tau^a\phi + \phi^\dagger\tau^a\partial_\mu\phi = \phi^\dagger\tau^a\overleftrightarrow{\partial}_\mu\phi \end{aligned} \tag{A45}$$

4. Electron: ψ, $G = U(1)$, $\psi \to e^{i\alpha}\psi$, $\bar\psi \to e^{-i\alpha}\bar\psi$, $\tau = i\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$

$$\begin{aligned} L &= \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar\psi)\gamma^\mu\psi] - m\bar\psi\psi \\ J_\mu &= \frac{1}{2}\bar\psi\gamma^\mu\psi + \frac{1}{2}\bar\psi\gamma^\mu\psi = \bar\psi\gamma^\mu\psi \end{aligned} \tag{A46}$$

c. External symmetries

When external and internal space transformation are performed simultaneously we have to follow the variation of the domain of integration in the space-time. The resulting equations are useful because they cover all previous cases considered, too.

The most general transformations leaving the action invariant may act in the external space, too. Therefore, let us consider the translation $x^\mu \to x'^\mu = x^\mu + \delta x^\mu$ and $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$ where $\delta\phi(x) = -\delta x^\mu \partial_\mu \phi(x)$. The variation of the action is

$$\begin{aligned} \delta S &= \int_V dx \delta L + \int_{V'-V} dx L \\ &= \int_V dx \delta L + \int_{\partial V} dS_\mu \delta x^\mu L \end{aligned} \tag{A47}$$

according to Fig. 13 and it can be written as

$$\delta S = \int_V dx \left(\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \phi} \right) \delta\phi + \int_{\partial V} dS_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi} \delta\phi + \delta x^\mu L \right) \tag{A48}$$

due to the relation

$$\partial_\mu \delta\phi(x) = \partial_\mu [\phi(x - \delta x^\mu(x)) - \phi(x)] = \partial_\mu \phi(x - \delta x^\mu(x)) - \partial_\mu \phi(x) = \delta \partial_\mu \phi(x) \tag{A49}$$

which holds as before, in case of a field independent variation. The variation principle, imposed in the inner points of the region V cancels the first integral on the right hand side, leading to

$$\delta S = \int_{\partial V} dS_\mu \delta x^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi \right). \tag{A50}$$

This integral is vanishing for arbitrary volume V and shift δx^μ if the current

$$J^\mu = \epsilon^\nu \left(L g^\mu_\nu - \frac{\partial L}{\partial \partial_\mu \phi} \partial_\nu \phi \right) \tag{A51}$$

is conserved. Therefore, the canonical energy-momentum tensor,

$$T_c^{\mu\nu} = \frac{\partial L}{\partial \partial_\mu \phi} \partial^\nu \phi - L g^{\mu\nu} \tag{A52}$$

containing the four Noether currents obeys the conservation law

$$\partial_\mu T_c^{\mu\nu} = 0. \tag{A53}$$

Accordingly,

$$P^\nu = \int d^3x T_c^{0\nu} \tag{A54}$$

can be identified by the energy-momentum vector and we have the parametrization

$$T_c^{\mu\nu} = \begin{pmatrix} \epsilon & c\boldsymbol{p} \\ \frac{1}{c}\boldsymbol{S} & \sigma \end{pmatrix} \tag{A55}$$

where ϵ is the energy density, $\boldsymbol{p}$ is the momentum density, $\boldsymbol{S}$ is the density of the energy flux and σ^{jk} is the flux of p^k in the direction j.

When Lorentz transformations and translations are performed simultaneously then we have $\delta x^\mu = a^\mu + \omega^\mu_\nu x^\nu$ and $\delta\phi = \Lambda^{\nu\mu}\omega_{\mu\nu}\phi \neq 0$ for field with non-vanishing spin and the conserved current is

$$J^\mu = \frac{\partial L}{\partial \partial_\mu \phi} (\Lambda^{\nu\kappa} \omega_{\kappa\nu} \phi - \delta x^\nu \partial_\nu \phi) + \delta x^\mu L. \tag{A56}$$

Let us simplify the expressions by introducing the tensor

$$f^{\mu\nu\kappa} = \frac{\partial L}{\partial \partial_\mu \phi} \Lambda^{\nu\kappa} \phi \tag{A57}$$

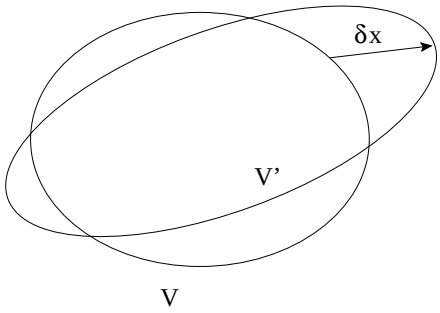


FIG. 13: Deformation of the volume in the external space.

and write

$$J^\mu = f^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L. \tag{A58}$$

By the cyclic permutation of the indices $\mu\nu\kappa$ we can define another tensor

$$\tilde{f}^{\mu\nu\kappa} = \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi \tag{A59}$$

which is antisymmetric in the first two indices,

$$\begin{aligned}\tilde{f}^{\nu\mu\kappa} &= \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\kappa\nu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\nu\mu}\right)\phi \\ &= \left(-\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi \\ &= -\tilde{f}^{\mu\nu\kappa}\end{aligned} \tag{A60}$$

and verifies the equation

$$\begin{aligned}\tilde{f}^{\mu\nu\kappa}\omega_{\nu\kappa} &= \left(\frac{\partial L}{\partial\partial_\mu\phi}\Lambda^{\nu\kappa} + \frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\kappa\mu} - \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa} - \left(\frac{\partial L}{\partial\partial_\nu\phi}\Lambda^{\mu\kappa} + \frac{\partial L}{\partial\partial_\kappa\phi}\Lambda^{\mu\nu}\right)\phi\omega_{\nu\kappa} \\ &= f^{\mu\nu\kappa}\omega_{\nu\kappa}.\end{aligned} \tag{A61}$$

As a result we can replace $f^{\mu\nu\kappa}$ by it in Eq. (A58),

$$\begin{aligned}J^\mu &= \tilde{f}^{\mu\nu\kappa}\omega_{\kappa\nu} - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \tilde{f}^{\mu\nu\kappa}\partial_\nu(\delta x_\kappa) - \frac{\partial L}{\partial\partial_\mu\phi}\delta x^\nu\partial_\nu\phi + \delta x^\mu L \\ &= \delta x_\kappa\left(g^{\mu\kappa}L - \frac{\partial L}{\partial\partial_\mu\phi}\partial^\kappa\phi - \partial_\nu\tilde{f}^{\mu\nu\kappa}\right) + \partial_\nu(\tilde{f}^{\mu\nu\kappa}\delta x_\kappa\phi).\end{aligned} \tag{A62}$$

The last term $J'^\mu = \partial_\nu(\tilde{f}^{\mu\nu\kappa}\delta x_\kappa\phi)$ gives a conserved current thus can be dropped and the conserved Noether current simplifies as

$$J^\mu = T^{\mu\nu}(a_\nu + \omega_{\nu\kappa}x^\kappa) = T^{\mu\nu}a_\nu + \frac{1}{2}(T^{\mu\nu}x^\kappa - T^{\mu\kappa}x^\nu)\omega_{\nu\kappa} \tag{A63}$$

where we can introduced the symmetric energy momentum tensor

$$T^{\mu\nu} = T_c^{\mu\nu} + \partial_\kappa\tilde{f}^{\mu\kappa\nu} \tag{A64}$$

Due to

$$\int_{\partial V} dS_\mu\partial_\kappa\tilde{f}^{\mu\kappa\nu} = \int_V \partial_\mu\partial_\kappa\tilde{f}^{\mu\kappa\nu} = 0 \tag{A65}$$

the energy momentum extracted from $T^{\mu\nu}$ and $T_c^{\mu\nu}$ agree and and the tensor

$$M^{\mu\nu\sigma} = T^{\mu\nu}x^\sigma - T^{\mu\sigma}x^\nu. \tag{A66}$$

is conserved,

$$\partial_\mu M^{\mu\nu\sigma} = 0, \tag{A67}$$

yielding the relativistic angular momentum

$$J^{\nu\sigma} = \int d^3x(T^{0\nu}x^\sigma - T^{0\sigma}x^\nu). \tag{A68}$$

with the usual non-relativistic spatial structure. The energy-momentum tensor $T^{\mu\nu}$ is symmetric because the conservation of the relativistic angular momentum, Eq. (A67) gives

$$0 = \partial_\rho M^{\rho\mu\nu} = \partial_\rho(T^{\rho\mu}x^\nu - T^{\rho\nu}x^\mu) = T^{\nu\mu} - T^{\mu\nu}. \tag{A69}$$

3. Construction of the Lagrangian

The construction of a Lagrangian for a given dynamics is usually governed by three principles.

1. The fundamental laws of physics are expected to be local in space-time.
2. Most of the observed conservation laws arise from the presence of a continuous symmetry.
3. If the theory is assumed to be valid until arbitrary short distance scales, ie. the physics observed is supposed to be extrapolated down to zero distance, then the theory must be renormalizable.

The importance of locality is often overlooked. There are no doubt correlations among observations performed at large distances in space-time, their explanation is just the goal of physics. But we believe that the structure of differential equations and Quantum Mechanics allows us to develop a formalism (Lagrangian, variational principle, quantization laws, ...) where all equations are local in space-time.

On the level of the action this is usually expressed as the applicability of the gradient expansion where the field variables are supposed to have non-singular, slow enough dependence on the space-time coordinates. In other words, any singularity arising form a theory of a given resolution in the space-time should originate from microscopic mechanism which is captured by equations and non-singular quantities accessible by observations with better space-time resolution. According the the gradient expansion the action functional for a relativistic scalar field can be written as

$$S[\phi] = \int dx[Z_0(\phi(x)) + Z_2(\phi(x))\Box\phi(x) + Z_4^{(1)}(\phi(x))\Box^2\phi(x) + Z_4^{(2)}(\phi(x))(\Box\phi(x))^2 + \mathcal{O}\left(\partial^4\right)]. \tag{A70}$$

Note that Lorentz invariance suppresses the odd powers of the derivatives. The choice

$$Z_0(\phi) = -\frac{m^2}{2}\phi^2 - U(\phi), \qquad Z_2(\phi) = -\frac{\phi}{2}, \qquad V_n = 0 \quad (n \geq 4) \tag{A71}$$

corresponds to the Lagrangian (A21).

The Landau-Ginzburg double expansion in the amplitude of the field and the gradient is the assumption of the polynomial ansatz like

$$Z_0(\phi) = \sum_{n=1}^{\infty} \frac{g_n}{n!}\phi^n \tag{A72}$$

for the coefficient functions. The renormalization group is used to assess the effects of higher order terms in this double expansion. The general rule with notable exceptions is that higher dimension of a coupling constant is in expressed in energy units ($\hbar = c = 1$) means less important effects at large distances.

Appendix B: Internal symmetry groups and their representations

1. Representation of groups

Definition: The representation of a group G in a linear space $\mathcal{H}$ is the set of operators $U(g) : \mathcal{H} \to \mathcal{H}$, $g \in G$ which preserves the group structure, $U(g_1 g_2) = U(g_1)U(g_1)$.

Example: Translations and rotations in Quantum Mechanics: $\boldsymbol{x} \to \boldsymbol{x}+\boldsymbol{a}$, $U(\boldsymbol{a})\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}-\boldsymbol{a})$, $\boldsymbol{x} \to R\boldsymbol{x}$, $R^{tr}R = \mathbb{1}$, $U(R)\psi(\boldsymbol{x}) = \psi(R^{-1}\boldsymbol{x})$.

Definition: Two representations, $U(g)$ and $U'(g)$ are equivalent if they are related by a change of basis, $U(g) = A^\dagger U'(g)A$.

Definition: The subspace $\mathcal{H}' \subset \mathcal{H}$ is called invariant if $U(g)\mathcal{H}' \subset \mathcal{H}'$ for each $g \in G$.

Definition: A representation is irreducible if there is no non-trivial invariant subspace ($\mathcal{H}' = \emptyset$ or $\mathcal{H}$).

Example: Translations: The set of functions $\psi_{\boldsymbol{y}}(\boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{y})$ can formally be considered as a base for the wave functions due to the identity

$$\psi(\boldsymbol{x}) = \int d^3y \psi_{\boldsymbol{y}}(\boldsymbol{x})\psi(\boldsymbol{y}). \tag{B1}$$

Starting with an arbitrary basis function we can arrive at another arbitrary basis element by applying well chosen translations. Therefore there is no non-trivial invariant subspace. The representation is unitary,

$$\langle\phi|\mathbb{1}|\psi\rangle = \langle\phi|\psi\rangle = \int d^3x \phi^*(\boldsymbol{x})\psi(\boldsymbol{x}) = \int d^3x \phi^*(\boldsymbol{x}-\boldsymbol{a})\psi(\boldsymbol{x}-\boldsymbol{a}) = \langle\phi|U^\dagger(\boldsymbol{a})U(\boldsymbol{a})|\psi\rangle \tag{B2}$$

Rotations: The basis functions concentrated on a sphere are mixed only under rotations thus the representation has non-trivial invariant subspaces. These are given by the multiplets $H_\ell = \{\sum_m c_m|\ell, m\rangle\}$.

Definition: A representation is reducible if it is not irreducible, ie. there is non-trivial invariant subspace. If its orthogonal complement is is also invariant then the representation is called decomposable.

Definition: A representation is unitary if $\mathcal{H}$ is equipped with a scalar product and $U^\dagger(g)U(g) = \mathbb{1}$ for all $g \in G$.

Theorem: If a unitary representation is reducible then it is decomposable.

Definition: A representation is a direct sum if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_a$ are two orthogonal invariant spaces for $a = 1, 2$ and $U(g) = U_1(g) \oplus U_2(g)$ with $U_a(g)$ being a representation on $\mathcal{H}_a$.

Theorem: A decomposable representation is the direct sum of irreducible representations.

The importance of irreducible representations of the symmetry is that their vectors can be called elementary particles.

2. Continuous groups

The discrete symmetry groups are rather simple in High Energy Physics, they are inversions in different spaces. The time and space inversions are discrete external symmetries, the charge inversion (particle-anti particle exchange) is a discrete subgroup of the internal symmetries. Their representations can easily be constructed.

The question of continuous or Lie groups is far more involved. These groups are equipped with continuous topology and are characterized by their multiplication law,

$$\omega(x) \cdot \omega(y) = \omega(F(x, y)) \tag{B3}$$

where x and y are n-dimensional vectors for an n-dimensional group and the function $F(x, y)$ describes the multiplication law.

The set of functions $F^\alpha(x, y)$ gives rise to a rather complicated structure and offers not too much help in identifying the possible continuous groups. The first step of progress in this direction comes from the construction of the Lie-algebra, consisting of the generators of the group. The infinitesimal transformation in the vicinity of the identity of an n-dimensional continuous group can formally be written as

$$\omega = \mathbb{1} + \epsilon^\alpha \tau^\alpha + \mathcal{O}\left(\epsilon^2\right) \tag{B4}$$

TABLE I: Real classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N)$	general linear group	$\det A \neq 0^a$	N^2	$\{\tau : \text{real } N \times N \text{ matrices}\}$
$SL(N)$	special linear group	$\det A = 1$	$N^2 - 1$	$\mathrm{tr}\tau = 0^b$
$O(N)$	orthogonal group	$A^{tr}A = \mathbb{1}^c$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$
$SO(N)$	special orthogonal group	$A^{tr}A = \mathbb{1}$, $\det A = 1$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$, $\mathrm{tr}\tau = 0$

[a]The matrix A is supposed to be an element of the group in question.
[b]$\det(\mathbb{1} + \epsilon\tau) = 1 + \epsilon\mathrm{tr}\tau + \mathcal{O}\left(\epsilon^2\right)$
[c]$\det A^{tr}A = (\det A)^2 = 1$ and $\det A = \pm 1$.

with ϵ^a as free, continuous, real parameters. (Einstein convention is always assumed except explicitly stated otherwise.) The identity

$$\begin{aligned} \psi(\boldsymbol{x} - \boldsymbol{a}) &= e^{-\boldsymbol{a}\cdot\boldsymbol{\partial}}\psi(\boldsymbol{x}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{p}}\psi(\boldsymbol{x}) \\ \psi(R_z^{-1}(\alpha)\boldsymbol{x}) &= e^{-\alpha\frac{\partial}{\partial\phi}}\psi(\boldsymbol{x}) = e^{-\frac{i}{\hbar}\alpha L_z}\psi(\boldsymbol{x}) \end{aligned} \tag{B5}$$

where $R_z(\alpha)$ is the rotation matrix by angle α around the axis z indicate that the generators (defined in Quantum Mechanics as the coefficient of $-i/\hbar$ times the continuous parameter in the infinitesimal transformation) of translations and rotations are the momentum and the angular momentum operators. The appearance of the generators in the exponent is not limited to these examples. The equation

$$\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n(\frac{a}{n}+\mathcal{O}\left(n^{-2}\right))} = e^a \tag{B6}$$

is valid for the the generators,

$$\lim_{n\to\infty}\left(1 + \frac{a\tau}{n}\right)^n = e^{a\tau}, \tag{B7}$$

showing that the repeating of an appropriately chosen infinitesimal transformation many times generates any finite group element.

The linear superpositions of the objects τ^α by using real coefficients are the generators of the continuous group. The group multiplication assures that the commutator of the generators is always a generator,

$$[\tau^\alpha, \tau^\beta] = f^{\alpha\beta\gamma}\tau^\gamma. \tag{B8}$$

In other words, the set of generators are closed for multiplication given by the commutator. The algebraic structure obtained in this manner is called Lie-algebra of the group. The real numbers f^{abc} are called structure constants of the Lie-algebra. The group structure has another important consequence, namely that the Lie-algebra is unique in the sense that the same set of generators and structure constants can be found by considering the vicinity of any other group elements. The importance of the Lie-algebra is that it fixes the Lie-group locally, ie. the structure of the group in the vicinity of any element.

The classification of the possible continuous symmetry groups is made simple by Ado's theorem asserting that any finite dimensional Lie-algebra is identical with a subspace of the generators of the matrix group $GL(N)$ (GL=General Linear group), consisting of non-singular $N \times N$ real matrices, for sufficiently large N. Thus any continuous group is locally identical with a subgroup of $GL(N)$ for certain N. In order to cover all continuous groups it is sufficient to study the matrix groups. The important matrix groups are called classical matrix groups and are shown in Tables I and II. The rest, the exceptional groups have not yet found application in physics.

The typical symmetry group in quantum physics is $U(N)$. Let us suppose that we have N equivalent states for a system. The symmetry group is then $U(N)$, being the largest set of matrices which mix these states with complex coefficients and preserve the scalar product, a natural requirement for a symmetry transformation. The gauge theory structure for the fundamental interactions shows that the overall $U(1)$ phase symmetry of Quantum Mechanics gives rise to electrodynamics after gauging. Thus what is left to discuss as the non-trivial consequence of the degeneracy is the symmetry group $SU(N)$ because $U(N) = U(1) \otimes SU(N)$.

3. Irreducible representations of classical matrix groups

The finite dimensional irreducible representations of classical matrix groups will be constructed in terms of tensors.

TABLE II: Complex classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N,C)$	complex general linear group	$\det A \neq 0$	$2N^2$	$\{\tau : \text{complex } N \times N \text{ matrices}\}$
$SL(N,C)$	complex special linear group	$\det A = 1$	$2N^2-2$	$\mathrm{tr}\tau = 0$
$U(N)$	unitary group	$A^\dagger A = \mathbb{1}$[a]	N^2	$\tau^\dagger = -\tau$
$SU(N)$	special unitary group	$A^\dagger A = \mathbb{1}$, $\det A = 1$	N^2-1	$\tau^\dagger = -\tau$, $\mathrm{tr}\tau = 0$

[a] $\det A^\dagger A = (\det A)^* \det A = |\det A|^2 = 1$

Fundamental representations: Let us start with the fundamental, N-dimensional representations of $GL(N,C)$. The fundamental representation for contravariant vectors in $\mathbb{C}^N$ is

$$x^j \to \omega^j{}_{j'} x^{j'} \qquad j, j' = 1, \ldots, N. \tag{B9}$$

The fundamental representation for covariant vectors is

$$x_j \to x_{j'} (\omega^{-1})^{j'}{}_j. \tag{B10}$$

These representations are defined in such a manner that the scalar product $x_i y^i$ is an invariant. One can introduce the complex conjugate fundamental representations,

$$x^{\dot{j}} \to (\omega^{\dot{j}}{}_{\dot{j}'})^* x^{\dot{j}'} \qquad x_{\dot{j}} \to x_{\dot{j}'} ((\omega^{-1})^{\dot{j}'}{}_{\dot{j}})^* \tag{B11}$$

and we shall use dotted index for the conjugate representation vectors. The fundamental representations are obviously irreducible.

Tensor representations: Higher representations are obtained by generalizing these formulae for tensors. A tensor of type $(m,n,\dot{m},\dot{n})$ is represented by the set of $N^{m+n+\dot{m}+\dot{n}}$ numbers $T^{j_1,\ldots,j_m,\dot{j}_1,\ldots,\dot{j}_{\dot{m}}}_{k_1,\ldots,k_n,\dot{k}_1,\ldots,\dot{k}_{\dot{n}}}$ following the transformation rules

$$T^{j_1,\ldots,j_m,\dot{j}_1,\ldots,\dot{j}_{\dot{m}}}_{k_1,\ldots,k_n\dot{k}_1,\ldots,\dot{k}_{\dot{n}}} \to \omega^{j_1}_{j'_1} \cdots \omega^{j_m}_{j'_m} (\omega^{\dot{j}_1}_{\dot{j}'_1})^* \cdots (\omega^{\dot{j}_{\dot{m}}}_{\dot{j}'_{\dot{m}}})^* T^{j'_1,\ldots,j'_m,\dot{j}'_1,\ldots,\dot{j}'_{\dot{m}}}_{k'_1,\ldots,k'_n\dot{k}'_1,\ldots,\dot{k}'_{\dot{n}}} (\omega^{-1})^{k'_1}{}_{k_1} \ldots (\omega^{-1})^{k'_n}{}_{k_n} ((\omega^{-1})^{\dot{k}'_1}{}_{\dot{k}_1})^* \cdots ((\omega^{-1})^{\dot{k}'_{\dot{n}}}{}_{\dot{k}_{\dot{n}}})^* \tag{B12}$$

The representations $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ and $(0,0,0,1)$ are clearly equivalent with the fundamental representations mentioned above.

The tensor representation representations are reducible. In fact, the contraction of a contravariant and a covariant index is an invariant operation, the remaining lower order tensor corresponds to an invariant subspace. To eliminate these subspaces we introduce traceless tensors, defined by requiring vanishing contraction for an arbitrary pair of covariant and contravariant indices. Note that no contraction allowed between normal and dotted indices because it creates no $GL(N,C)$ invariant.

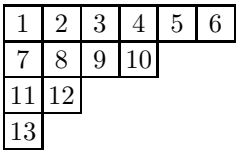
The traceless tensors still give reducible representations. The way to see it is to notice that the symmetry-type of a tensor with respect to the permutation of the indices of the same type does not change during the transformation (B12) where the same rule applies to every index, independently of each others. One realizes here that the tensors of the representation $(m,n,\dot{m},\dot{n})$ actually provide a representation of the group of permutations of m, n, $\dot{m}$ and $\dot{n}$ objects, $S_m \otimes S_n \otimes S_{\dot{m}} \otimes S_{\dot{n}}$,

$$T^{j_1,\ldots,j_m,\dot{j}_1,\ldots,\dot{j}_{\dot{m}}}_{k_1,\ldots,k_n\dot{k}_1,\ldots,\dot{k}_{\dot{n}}} \to T^{j_{\pi_t(1)},\ldots,j_{\pi_t(m)},\dot{j}_{\dot{\pi}_t(1)},\ldots,\dot{j}_{\dot{\pi}_t(\dot{m})}}_{k_{\pi_c(1)},\ldots,k_{\pi_c(1)},\dot{k}_{\dot{\pi}_c(1)},\ldots,\dot{k}_{\dot{\pi}_c(\dot{n})}}, \qquad \pi_t \otimes \pi_c \otimes \dot{\pi}_t \otimes \dot{\pi}_c \otimes \in S_m \otimes S_n \otimes S_{\dot{m}} \otimes S_{\dot{n}}. \tag{B13}$$

The $GL(N,C)$ transformation rule, (B12), does not mix different representations of the symmetric group thus traceless tensors of the type $(m,n,\dot{m},\dot{n})$ corresponding to a given irreducible representations of the group $S_m \otimes S_n \otimes S_{\dot{m}} \otimes S_{\dot{n}}$ form an invariant subspace. Weyl's theorem asserts that these invariant subspaces can not be reduced anymore, they are irreducible with respect to $GL(N,C)$, as well. What is left to work out is the identification of the irreducible representations of the symmetric group S_n.

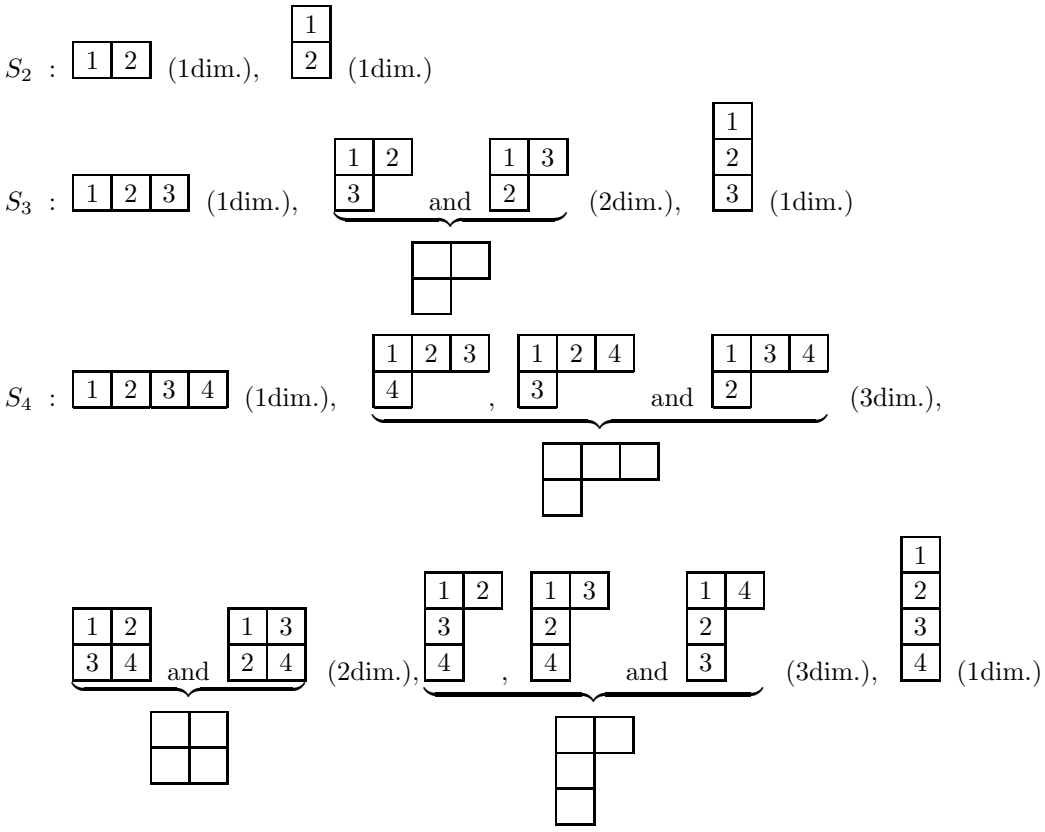
Irreducible representations of the symmetric group: The operators projecting tensors of the type $(m,0,0,0)$ into the irreducible representations of S_m are called Young-tableau projectors. A Young-tableau, σ, of order m is the distribution of m objects, usually the numbers $1,\ldots,m$, into different vertical columns, such as for example with

$m = 13$,



(B14)

The sum of the length of the columns is m. We put the columns beside each other in such a manner that highest lying object of each column appear in the same row and their length is changing in a monotonous manner. The resulting rearrangement of the indices of the tensor is a Young-tableau. The projection operator π_σ of a Young-tableau σ consists of first the symmetrization over the permutations of objects in the same rows, followed by the antisymmetrization of objects in columns. One can prove that there is a unique correspondence between irreducible representations of S_m and Young-tableaux with m objects. The dimension of an irreducible representation labeled by a given Young-tableau σ is the number of different fillings, the different distributions of the numbers $1, \ldots, m$ into the positions in the tableau by taking into account the given symmetry patterns with respect to permutation. The list of irreducible representations for $m = 2, 3$ and 4 is



(B15)

Irreducible representations of $GL(N, C)$ and $SL(N, C)$: The tensor of the type $(m, n, \dot{m}, \dot{n})$ is said to belong to Young-tableaux $(\sigma, \tau, \dot{\sigma}, \dot{\tau})$ if

$$T^{j_1,\ldots,j_m,\dot{j}_1,\ldots,\dot{j}_{\dot{m}}}_{k_1,\ldots,k_n\dot{k}_1,\ldots,\dot{k}_{\dot{n}}} = T^{j_{\pi_\sigma(1)},\ldots,j_{\pi_\sigma(m)},\dot{j}_{\pi_{\dot{\sigma}}(1)},\ldots,\dot{j}_{\pi_\sigma(\dot{m})}}_{k_{\pi_\tau(1)},\ldots,k_{\pi_\tau(n)}\dot{k}_{\pi_{\dot{\tau}}(1)},\ldots,\dot{k}_{\pi_{\dot{\tau}}(\dot{n})}} . \tag{B16}$$

The relation between the irreducible representations of the group $GL(N, C)$ and the traceless tensors with certain Young-tableaux $(\sigma, \tau, \dot{\sigma}, \dot{\tau})$ is one-to-one according to Weyl's theorem. The irreducible representations of the group $GL(N, C)$ remain irreducible when the group is restricted to $GL(N, C) \to SL(N, C)$.

Irreducible representations of $U(N)$ and $SU(N)$: The unitarity condition

$$(\omega\omega^\dagger)^j{}_{\dot{k}} = \omega^j{}_\ell \delta^\ell_{\dot{\ell}} (\omega^\dagger)^{\dot{\ell}}{}_{\dot{k}} = \delta^j_{\dot{k}} \tag{B17}$$

shows the emergence of a new mixed invariant obtained by contracting a normal index with a dotted one as far as unitary transformations are concerned. In other words, two types of indices show independent transformation laws out of four. One usually retains tensors with a normal and a dotted indices, both are covariant or contravariant. The irreducible representations of the unitary group have a one-to-one correspondence with the tensors of the type $(m, 0, \dot{m}, 0)$ with Young-tableaux $(\sigma, \dot{\sigma})$. The irreducible representations of $U(N)$ remain irreducible when the group is restricted to $U(N) \to SU(N)$.

Irreducible representations of $GL(N)$ and $SL(N)$: The are only two fundamental representations for real matrix groups and the irreducible representations of the group $GL(N)$ belong to traceless tensors of the type (m, n) with m contravariant and n covariant indices with given Young-tableau permutation symmetries, (σ, τ). The irreducible representations of $GL(N)$ remain irreducible when the group is restricted to $GL(N) \to SL(N)$.

Irreducible representations of $O(N)$ and $SO(N)$: The orthogonality condition

$$(\omega\omega^{tr})^{j}{}_{k} = \omega^{j}{}_{\ell}\delta^{\ell}_{\ell'}(\omega^{tr})^{\ell'}_{k} = \delta^{j}_{k} \tag{B18}$$

shows that only one fundamental representation is independent. Thus the irreducible representations of the orthogonal group have a one-to-one correspondence with the tensors of the type $(m, 0)$ with Young-tableaux σ. The irreducible representations of $O(N)$ remain irreducible when the group is restricted to $O(N) \to SO(N)$.

4. Definition of elementary particles

Let us consider a relativistic many-particle quantum system with N equivalent particle states. The underlying internal symmetry group is $SU(N)$ cf. the remark at the end of Section B 2 above. The multi-particle state with n particles and $\dot{n}$ anti-particles transform as a tensor with exclusively contravariant (or only covariant) indices of the type $(n, \dot{n})$ under the internal symmetry transformations. The simplest family of states displaying all consequences of the symmetry is an irreducible representation space. Its vectors can be called elementary particles of the symmetry $SU(N)$.

As a simple example let us see the fundamental representations $(1, 0, 0, 0)$ and $(0, 0, 1, 0)$ of the group $SU(n)$, usually denoted by the dimension of the multiplet n and $\bar{n}$, respectively. The two-particle states give the representations

$$n \otimes n = \underbrace{\frac{n(n+1)}{2}}_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}} \oplus \underbrace{\frac{n(n-1)}{2}}_{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}}, \tag{B19}$$

realized by the symmetrical and antisymmetrical two-index $SU(n)$ tensors. The addition of further particles requires the use of the generalization of the Clebsh-Gordon coefficients of the rotation group. The particle-anti particle states belong to the representations $n \otimes \bar{n} = 1 \oplus n^2 - 1$, where 1 stands for the singlet, the trace of the tensor $(1, 0, 1, 0)$ and $n^2 - 1$ corresponds to linear space of traceless tensors.

The physical particles show external and internal symmetries. The true elementary particles are defined as the vectors in the irreducible representations of the direct product of the Poincare group (space-time translations and Lorentz transformations) and the internal symmetry group. The former factor is well known since long time but a breakthrough is achieved in High Energy Physics at each enlargement of the second factor. For example:

Isospin symmetry: Protons and neutrons, or in general up and down quarks, participate in an equivalent manner in strong interactions. The resulting approximate symmetry (by ignoring electromagnetic and weak forces) gives rise of the $SU(2)$ isospin symmetry group. The fundamental representation 2 with isospin half is the (u, d) doublet. The quark-anti quark pair gives rise to the representations

$$2 \otimes \bar{2} = 1 \oplus 3, \tag{B20}$$

where 1 is an isospin 0 singlet meson and 3 stands for isospin 1 triplet mesons. The comparison of the electric charge Q and the projection of the isospin, defined by the eigenvalues I_3 of σ_3 (the Pauli-matrices $\boldsymbol{\sigma}$ being the generators of the $SU(2)$ group) yields the relation

$$Q = \frac{B}{2} + I_3 \tag{B21}$$

where B is the baryon charge (number of quarks minus anti-quarks divided by three). The exact isospin symmetry implies the vanishing commutator of the Hamiltonian with the generators of the isospin transformations. The symmetry breaking part of the Hamiltonian can be written as a sum over tensor-operators of the isospin group. It is assumed that this sum involves the simplest tensor operators, the generators G_α only, $H = H_0 + g_\alpha G_\alpha$. The coefficients g_α, $\alpha = 1, 2$ and 3 can be determined by perturbation expansion.

Eightfold way: The isospin symmetry can be extended by including states with non-vanishing strangeness. The resulting $SU(3)$ flavor symmetry group is broken in a stronger manner than the isospin symmetry but is still a useful phenomenological tool. The fundamental representations are the $3 = (u, d, s)$ quark and $\bar{3} = (\bar{u}, \bar{d}, \bar{s})$ anti-quark triplets. Among the irreducible representations those are found among the observed, ie. isolated hadronic states only which have vanishing triality, the number of quarks minus anti-quarks, taken modulo 3. The mechanism, responsible for the removal of multi-quark states from the asymptotic states of scattering experiments with non-vanishing triality is called quark confinement. The allowed, non-confined irreducible representations are given in terms of tensors of type $(m, 0, \dot{m}, 0)$ with $T = m - \dot{m}(\mathrm{mod}3) = 0$. Baryons, made of three valence quarks belong to the representation provided by flavor tensors of the type $(3, 0, 0, 0)$. This 27 dimensional representation is reducible,

$$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}\ (3^3\mathrm{dim.}) = \boxed{1}\boxed{2}\boxed{3}\ (10\mathrm{dim.})\ \oplus\ \underbrace{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\quad\text{and}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}}_{\begin{array}{|c|c|}\hline &\\\hline &\\\cline{1-1}\end{array}}\ (8+8\mathrm{dim.})\ \oplus\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ (1\mathrm{dim.}). \tag{B22}$$

The mesons are found in the representation $(1, 0, 1, 0)$. Since three particle antisymmetric state is an $SU(3)$ singlet, the anti-particle is similar to an antisymmetrical two-particle state and the meson multiplets are

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\otimes\boxed{3}\ (3^2\mathrm{dim.}) = \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\ (8\mathrm{dim.})\ \oplus\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ (1\mathrm{dim.}). \tag{B23}$$

Appendix C: Lorentz group

The Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{C1}$$

are defined by requiring the invariance of the scalar product

$$x \cdot y = x^\mu g_{\mu\nu} y^\nu = x^\mu y_\nu. \tag{C2}$$

Therefore, we have the condition

$$x \cdot y = x^{\mu'} \Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'} y^{\nu'} \tag{C3}$$

or

$$g = \Lambda^{\mathrm{tr}} \cdot g \cdot \Lambda. \tag{C4}$$

for the transformation matrix Λ.

The Lorentz group is 6 dimensional, 3 dimensions correspond to three-dimensional rotations and three other directions belong to Lorentz-boosts, parametrized by the three-velocity $\boldsymbol{v}$ relating the inertial systems. Let us denote the the parallel and perpendicular projection of the three-coordinate on the velocity $\boldsymbol{v}$ by $\boldsymbol{x}_\parallel$ and $\boldsymbol{x}_\perp$, respectively,

$$\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp, \quad \boldsymbol{x}_\parallel \cdot \boldsymbol{x}_\perp = \boldsymbol{v} \cdot \boldsymbol{x}_\perp = 0. \tag{C5}$$

We can then write a general Lorentz transformation in a three-dimensional notation as

$$\boldsymbol{x}' = \alpha(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \gamma \boldsymbol{x}_\perp, \quad t' = \beta\left(t - \frac{\boldsymbol{x} \cdot \boldsymbol{v}}{\tilde{c}^2}\right) \tag{C6}$$

The invariance of the length,

$$c^2t^2 - \boldsymbol{x}^2 = c^2\beta^2\left(t - \frac{\boldsymbol{x}\cdot\boldsymbol{v}}{\tilde{c}^2}\right)^2 - \alpha^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)^2 - \gamma\boldsymbol{x}_\perp^2, \tag{C7}$$

yields the relations

$$\begin{aligned} \gamma &= \pm 1, \quad v = 0 \implies \gamma = 1 \\ \tilde{c} &= c \\ \alpha &= \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \end{aligned} \tag{C8}$$

and leads to the expression

$$x'_\parallel = \frac{x_\parallel - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vx_\parallel}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{C9}$$

for boosts.

Notice that inversions in space-time are special Lorentz transformations. In fact, the space-inversion $P : (t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$, and the time-inversion $T : (t, \boldsymbol{x}) \to (-t, \boldsymbol{x})$, are realized by the Lorentz transformation matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{C10}$$

1. Proper Lorentz group

By taking the determinant of both side of Eq. (C4), $\det g = \det g(\det \Lambda)^2$, we find that $\det \Lambda = \pm 1$. The 00 component of the same equation, $1 = g_{00} = (\Lambda^0{}_0)^2 - \sum_j (\Lambda^j{}_0)^2$ shows that $|\Lambda^0{}_0| > 1$. We have, therefore, four disconnected components in the Lorentz group:

1. $\det \Lambda = 1$, $\Lambda^0{}_0 > 1$. This component is a proper subgroup because it contains the identity and is called proper Lorentz group and is usually denoted by $L_+^\uparrow$.
2. $\det \Lambda = 1$, $\Lambda^0{}_0 < 1$, obtained from $L_+^\uparrow$ by time inversion, $T : (x^0, \boldsymbol{x}) \to (-x^0, \boldsymbol{x})$.
3. $\det \Lambda = -1$, $\Lambda^0{}_0 > 1$, obtained from $L_+^\uparrow$ by spatial inversion, $P : (x^0, \boldsymbol{x}) \to (x^0, -\boldsymbol{x})$.
4. $\det \Lambda = -1$, $\Lambda^0{}_0 < 1$, obtained from $L_+^\uparrow$ by time and space inversion, $TP : (x^0, \boldsymbol{x}) \to (-x^0, -\boldsymbol{x})$.

These are disconnected because a Lorentz transformation matrix can not be brought into another component by the infinitesimal change of its matrix elements.

2. Spinors

Let us combine the four components of the space-time vector $(t, \boldsymbol{x})$ into a 2×2 hermitian matrix,

$$x \to \hat{x} = t\mathbb{1} + \boldsymbol{x}\boldsymbol{\sigma}, \tag{C11}$$

$\boldsymbol{\sigma}$ being the Pauli-matrices. This establishes an invertible mapping between the set of four vectors and the 2×2 hermitian matrices with the property

$$\det \hat{x} = t^2 - \boldsymbol{x}^2. \tag{C12}$$

The linear transformations

$$\hat{x} \to g\hat{x}g^\dagger \tag{C13}$$

where $g \in SL(2,C)$ preserve $\det \hat{x}$ because $\det(g\hat{x}g^\dagger) = |\det g|^2 \det \hat{x} = \det \hat{x}$ and must correspond to Lorentz transformations of the corresponding four-vectors. Moreover, it is easy to see that any proper Lorentz transformation can be written as (C13). In fact, write $g = g^0 \mathbb{1} + i\boldsymbol{g}\boldsymbol{\sigma}$ where g^μ are complex numbers and note that the $SL(2C)$ group is 6 dimensional because this parametrization satisfies the constraint $\det g = g^{02} + \boldsymbol{g}^2 = 1$. The transformations with real g^μ we have $g \in SU(2)$, (C13) leaves the time unchanged and these transformations realize 3 dimensional rotations, $SO(3)$. It is easy to see that the 3 dimensional family of imaginary g^μ correspond to Lorentz-boosts. We have thus established the isomorphism $L_+^\uparrow \approx SL(2,C)$.

We can introduce two kinds of Lorentz spinors with the transformation rules,

$$\chi^a \to g^a{}_b \chi^b, \qquad \phi^{\dot{a}} \to g^{\dot{a}}{}_{\dot{b}}{}^* \phi^{\dot{b}}, \tag{C14}$$

where the dot is a recall that the index corresponds to the complex conjugate representation but both types of indices take the values 1 or 2. The metric tensor G_{ab} defines the scalar product $\chi^a G_{ab} \xi^b$ which is invariant for

$$G_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 = -G^{ab} = G^{\dot{a}\dot{b}} = -G^{\dot{a}\dot{b}} \tag{C15}$$

because

$$\chi^a G_{ab} \xi^b \to \chi^a \xi^b \underbrace{(g^1{}_a g^2{}_b - g^2{}_a g^1{}_b)}_{G_{ab} \det g = G_{ab}}. \tag{C16}$$

The Lorentz tensors can have four types of indices, $T^{a_1,\ldots,\dot{a}_1,\ldots}_{b_1,\ldots,\dot{b}_1,\ldots}$. The contraction of a pair of upper and a lower indices of the same type, $T^{a_1,\ldots,\dot{a}_1,\ldots}_{a_1,\ldots,\dot{b}_1,\ldots}$ or $T^{a_1,\ldots,\dot{a}_1,\ldots}_{b_1,\ldots,\dot{a}_1,\ldots}$ produces tensors with less indices, thus all covariant tensors can be obtained by considering the families $T^{a_1,\ldots,\dot{a}_1,\ldots}$ only. These representations are irreducible when these tensors are symmetric with respect to the permutation of the indices of the same type. The transformation rule of the tensor $T^{a_1,\ldots,a_m,\dot{a}_1,\ldots,\dot{a}_n}$ is denoted by (m,n), ie. $\chi \sim (1,0)$, $\phi \sim (0,1)$ and $\hat{x} \sim (1,1)$.

Note that the tensors (m,n) and (n,m) are equivalent from the point of view of three dimensional rotations, $SU(2)$. This is because the relation

$$G\boldsymbol{\sigma}G = \boldsymbol{\sigma}^* \tag{C17}$$

establishes an isomorphism between the two representations in Eqs. (C14).

Appendix D: Dirac equation

1. Square root of the Klein-Gordon equation

The Schrödinger equation for a free particle,

$$i\partial_0 \psi = -\frac{1}{2m}\Delta\psi \tag{D1}$$

is not covariant, ie. does not transform in a simple manner under Lorentz transformations. In fact, the Lorentz boosts mix space and time and all space-time derivative should appear in the same order in a covariant equation. We should either have second order derivative with respect to the time or first order with respect to the spatial coordinates. The former idea gives the Klein-Gordon equation,

$$(\partial_0^2 - \Delta + m^2)\psi = 0, \tag{D2}$$

by performing the replacement $p^\mu = (p^0, \boldsymbol{p}) \to (i\partial_0, -i\boldsymbol{\partial})$ in the dispersion relation $E^2 - \boldsymbol{p}^2 = m^2$. Dirac equation results from the latter strategy, by taking the "square root" of the Klein-Gordon equation.

Let us suppose that we can write the first order equation of motion in the form

$$i\partial_0 \psi = (-i\boldsymbol{\alpha}\cdot\boldsymbol{\partial} + \beta m)\psi, \tag{D3}$$

where $\boldsymbol{\alpha}$ and β are formal symbols whose properties can be read off by considering the square of this equation,

$$\begin{aligned} -\partial_0^2\psi &= (-i\boldsymbol{\alpha}\cdot\boldsymbol{\partial} + \beta m)^2\psi \\ &= [-\alpha_j\alpha_k\partial_j\partial_k + \beta^2 m^2 - mi(\alpha_j\beta + \beta\alpha_j)\partial_j]\psi \\ &= [-\{\alpha_j,\alpha_k\}\partial_j\partial_k + \beta^2 m^2 - mi[\alpha_j,\beta] + \partial_j]\psi, \end{aligned} \tag{D4}$$

where the commutator $[A,B]=AB-BA$ and anti-commutator $\{A,B\}=AB+BA$ has been introduced and the identity

$$\begin{aligned}\alpha_j\alpha_k\partial_j\partial_k &= \frac{1}{2}(\{\alpha_j,\alpha_k\}+[\alpha_j,\alpha_k])\frac{1}{2}(\{\partial_j,\partial_k\}+[\partial_j,\partial_k])\\ &= \frac{1}{2}(\{\alpha_j,\alpha_k\}+[\alpha_j,\alpha_k])\{\partial_j,\partial_k\}\\ &= \frac{1}{4}(\{\alpha_j,\alpha_k\}+[\alpha_j,\alpha_k])\{\partial_j,\partial_k\}\\ &= \frac{1}{2}\{\alpha_j,\alpha_k\}\partial_j\partial_k \end{aligned} \tag{D5}$$

was used in the last equation. We arrive at the Klein-Gordon equation by requiring

$$\{\alpha_j,\alpha_k\}=2\delta_{j,k},\quad \beta^2=\mathbb{1},\quad \{v\alpha,\beta\}=0. \tag{D6}$$

A covariant notation is established by introducing $\gamma^\mu=(\beta,\beta\boldsymbol{\alpha})$ and writing the Dirac equation as

$$(i\gamma^\mu\partial_\mu-m)\psi(x)=(i\partial\!\!\!/-m)\psi(x)=0 \tag{D7}$$

where the constraints (D6) read as

$$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu,\nu}=2\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}. \tag{D8}$$

The simplest realization of the algebraic properties (D6) requires 4×4 matrices,

$$\beta=\gamma^0=\begin{pmatrix}\mathbb{1}&0\\0&-\mathbb{1}\end{pmatrix},\qquad \boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix},\qquad \boldsymbol{\gamma}=\begin{pmatrix}0&\boldsymbol{\sigma}\\-\boldsymbol{\sigma}&0\end{pmatrix}, \tag{D9}$$

in the standard representation where

$$\boldsymbol{\sigma}=\left(\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\right) \tag{D10}$$

are the Pauli matrices.

The solution of the Dirac equation is a four component complex field $\psi_a(x)$, $a=1,\ldots,4$ and according the constraint (D8) is automatically solves the Klein-Gordon equation, too,

$$(\partial\!\!\!/+m)(\partial\!\!\!/-m)\psi(x)=(\Box-m^2)\psi=0. \tag{D11}$$

Observe that the hermitian conjugate of the Dirac-matrices satisfy the equation

$$\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0 \tag{D12}$$

in either representation which justifies the introduction of the Dirac-conjugate $\bar\psi=\psi^\dagger\gamma^0$. In fact, the Dirac-conjugate of Eq. (D7),

$$(i\gamma^{\mu*}\partial_\mu+m)\psi^*=0\qquad\Longrightarrow\qquad \bar\psi(i\gamma^\mu\partial_\mu+m)=0 \tag{D13}$$

where the derivative act to the left remains covariant. The matrix $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3$ with the property

$$\{\gamma^5,\gamma^\mu\}=0 \tag{D14}$$

will play an important role. It is hermitian, $\gamma^{5\dagger}=\gamma^5$ and the relation $(\gamma^5)^2=\mathbb{1}$ assures that its eigenvalues are ±1.

2. Relativistic covariance

The transformation properties of the field $\psi(x)$ will be obtained first for proper Lorentz transformations which have infinitesimal forms. After that space-time inversions will be included.

a. Proper Lorentz group

The Lorentz transformation acts on the field $\psi_a(x)$ as

$$\psi(x) \to \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x) \tag{D15}$$

or

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x). \tag{D16}$$

For spinless particle (Klein-Gordon equation) $\psi(x)$ is a single component scalar and $S(\Lambda) = 1$. The Dirac equation (D7) transforms into

$$\left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = \left(i\gamma^\mu \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{(\Lambda^{-1})^\nu{}_\mu} \frac{\partial}{\partial x^\nu} - m\right) S(\Lambda)\psi(x) = 0 \tag{D17}$$

To compare it with the original Dirac equation we multiply Eq. (D7) by $S(\Lambda)$,

$$S(\Lambda)(i\gamma^\mu\partial_\mu - m)\psi(x) = S(\Lambda)(i\gamma^\mu\partial_\mu - m)S^{-1}(\Lambda)S(\Lambda)\psi(x) = 0. \tag{D18}$$

This is equivalent with (D17) by imposing the condition

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu \gamma^\nu, \tag{D19}$$

expressing that the Dirac matrices γ^μ represent tensor operators, ie. the effect of a proper Lorentz transformation can be obtained either by acting on the vector index μ or by a basis transformation in the Dirac-spinor space.

The actual form of $S(\Lambda)$ can easiest be found for an infinitesimal Lorentz transformation

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu, \qquad (\Lambda^{-1})^\mu{}_\nu = g^\mu{}_\nu - \omega^\mu{}_\nu, \tag{D20}$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (because $x \cdot y = (\Lambda x)\cdot(\Lambda y) \approx (x + \omega x)\cdot(y + \omega y) \approx x \cdot y + x \cdot (\omega^{tr} + \omega) \cdot y$). We write

$$S(\Lambda) = \mathbb{1} - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \mathcal{O}\left(\omega^2\right), \qquad S^{-1}(\Lambda) = \mathbb{1} + \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu} + \mathcal{O}\left(\omega^2\right) \tag{D21}$$

and the substitution into Eq. (D19) yields

$$\left(\mathbb{1} - \frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right)\gamma^\kappa\left(\mathbb{1} + \frac{i}{4}\sigma^{\mu'\nu'}\omega_{\mu'\nu'}\right) = (g^\kappa{}_\nu - g^{\kappa\mu}\omega_{\mu\nu})\gamma^\nu + \mathcal{O}\left(\omega^2\right) \tag{D22}$$

and

$$[\gamma^\kappa, \sigma^{\mu\nu}] = 2i(g^{\kappa\mu}\gamma^\nu - g^{\kappa\nu}\gamma^\mu). \tag{D23}$$

This equation can be satisfied by the matrix

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{D24}$$

The finite Lorentz transformation

$$S(\Lambda) = e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}} \tag{D25}$$

can easily be obtained in the chiral representation where $\sigma_{\mu\nu}$ is block-diagonal,

$$\sigma_{0i} = -i\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}, \qquad \sigma_{ij} = \epsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}. \tag{D26}$$

We shall need below $S^\dagger(\Lambda)$ which can be calculated by means of the property (D12),

$$\sigma^{\dagger\mu\nu} = \frac{i}{2}\gamma^0[\gamma^\mu, \gamma^\nu]\gamma^0 = \gamma^0\sigma^{\mu\nu}\gamma^0 \tag{D27}$$

which yields

$$S^\dagger(\Lambda) = \gamma^0 e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\gamma^0. \tag{D28}$$

The spatial components of the antisymmetric tensor $\omega_{\mu\nu}$ parameterize spatial rotations therefore the spin operator is defined by

$$\Sigma^j = \frac{1}{2}\epsilon^{jk\ell}\sigma^{k\ell} = \gamma^5\gamma^0\gamma^j = \begin{pmatrix}\sigma^j & 0\\ 0 & \sigma^j\end{pmatrix} \tag{D29}$$

and the other three components, ω_{0j} correspond to Lorentz boosts.

Chiral fermions The block-diagonality (D26) shows that the $\psi(x)$ appears as the direct sum of two two-component Pauli-spinors under proper Lorentz transformations (without space-time inversions). This explains the name bi-spinor for the solution of the Dirac-equation. The bi-spinors form a reducible representation of the proper Lorentz group because the two Pauli-spinors are irreducible with respect to this group. Space-time inversions, cf. section D 2 b map the two Pauli-spinors into each other and render the bi-spinor an irreducible representation of the full Lorentz group. The explicit form

$$\gamma^5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{D30}$$

in the chiral representation shows that the eigenvalue of γ^5, called chirality, distinguishes the two irreducible spinor representations in the bi-spinor space.

The two irreducible spinors are coupled to each other during the time evolution by the mass term. To understand this better let us start consider a massless fermion field satisfying the Dirac-equation

$$i\partial_0\psi = H_{m=1}\psi = -i\boldsymbol{\alpha}\cdot\boldsymbol{\partial}\psi. \tag{D31}$$

Due to the anti-commutator relation (D14) γ^5 commutes with the massless Dirac-Hamiltonian, $[H_{m=0}, \gamma^5] = 0$ and chirality is conserved by the time evolution. The projection operators

$$\underset{L}{P_R} = \frac{1}{2}(\mathbb{1} \pm \gamma^5) \tag{D32}$$

correspond to the $\pm$ chirality subspaces and the bi-spinors $P_R\psi$ and $P_L\psi$ are called right or left handed spinors. This names originate from the opposite helicity (projection of the spin on the direction of momentum) of the right and left handed spinors. In fact, the chirality is flipped by complex conjugation which preserves the spin but inverts the momentum.

The mass term does not commute with γ^5 therefore it mixes the chirality.

Let us consider N free fermions with degenerate masses described by the Lagrangian

$$L = \psi(i\partial\!\!\!/ - m)\psi \tag{D33}$$

where

$$\psi(x) = \begin{pmatrix}\psi^1(x)\\ \vdots \\ \psi^N(x)\end{pmatrix}. \tag{D34}$$

The symmetry under the transformations

$$\psi(x) \to e^{-i\alpha_V^a\tau^a}\psi(x), \qquad \bar\psi(x) \to \bar\psi(x)e^{i\alpha_V^a\tau^a} \tag{D35}$$

where τ^a are the generators of the group $U(N)$ assures the degeneracy of the mass spectrum. The corresponding Noether current,

$$J_\mu^a(x) = \bar\psi(x)\gamma^\mu\tau^a\psi(x) \tag{D36}$$

is a four-vector. The chiral transformations,

$$\psi(x) \to e^{-i\alpha_A^a\tau\gamma^5}\psi(x), \qquad \bar\psi(x) \to \bar\psi(x)e^{-i\alpha_A^a\tau\gamma^5} \tag{D37}$$

which represent a symmetry for massless fermions only and the chiral invariant dynamics does not mix fermions states with different chirality. The Noether-current of chiral symmetry,

$$J_\mu^{5a}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\tau^a\psi(x) \tag{D38}$$

is axial(pseudo)-vector. One talks therefore the symmetry $U_V(N) \otimes U_A(N)$ for massless fermions and the symmetry $U_V(N)$ for massive fermions. Another way to express the symmetry $U_V(N) \otimes U_A(N)$ is to use chiral fermions when we have the symmetry group $U_L(N) \otimes U_R(N)$ with the left and right transformations given by

$$\psi(x) \to e^{-i\alpha_L^a\tau}P_L\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)P_L e^{i\alpha_L^a\tau} \tag{D39}$$

and

$$\psi(x) \to e^{-i\alpha_R^a\tau}P_R\psi(x), \qquad \bar{\psi}(x) \to \bar{\psi}(x)P_R e^{i\alpha_R^a\tau}, \tag{D40}$$

respectively.

b. Inversions

Space Inversion: The space-inversion, $P : (t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$, transforms the Dirac-matrices as

$$S(P)\gamma^\mu S(P)^{-1} = g^{\mu\mu}\gamma^\nu \tag{D41}$$

(no summation over μ) according to Eq. (D19). The solution of this equation is

$$S(P) = e^{i\phi_P}\gamma^0 \tag{D42}$$

where ϕ_P is an arbitrary phase and we find the unitary representation

$$U(P) : \psi(t, \boldsymbol{x}) \to \psi'(t, -\boldsymbol{x}) = e^{i\phi_P}\gamma^0\psi(t, \boldsymbol{x}), \tag{D43}$$

$U^\dagger(P) = U(P)$ for the wave functions.

Time Inversion: The time inversion is defined in classical physics as the transformation $T : (t, \boldsymbol{x}) \to (-t, \boldsymbol{x})$ of the space-time coordinates. In Quantum Mechanics the situation becomes more involved because there is no operator representing the time and the inversion is actually a "motion reversal", it produces a time evolution as a movie played from the back to the beginning. Let us assume that $U(T)$ the operator representing time inversion in Quantum Mechanics is chosen in such a manner that $U(T)\boldsymbol{x}U^{-1}(T) = \boldsymbol{x}$ and $U(T)\boldsymbol{p}U^{-1}(T) = -\boldsymbol{p}$ and consider the commutation relation $[x_j, p_k] = i\delta_{jk}$ for time reversed operators, $[U(T)x_jU^{-1}(T), U(T)p_kU^{-1}(T)] = U(T)i\delta_{jk}U^{-1}(T)$. To recover the usual commutation relation we have to upgrade $U(T)$ to an anti-linear operator which involves a complex conjugation, such as $U(T) : \phi(t, \boldsymbol{x}) \to \phi^*(-t, \boldsymbol{x})$ for the wave function of a spinless particle.

We construct here the representation $U(T)\psi(t, \boldsymbol{x}) = S(T)\bar{\psi}(-t, \boldsymbol{x})$ on the wave functions of spin half particles. The time-inverted wave function $U(T)\psi(t, \boldsymbol{x})$ satisfies the free Dirac-equation,

$$(i\gamma^\mu\partial_\mu - m)S(T)\bar{\psi}(-t, \boldsymbol{x}) = 0 \tag{D44}$$

which is compatible with the Dirac-equation

$$(i\gamma^{\mu tr}\partial_\mu + m)\bar{\psi}(t, \boldsymbol{x}) = 0 \tag{D45}$$

for $\bar{\psi}(x)$ if

$$S(T)\gamma^{0tr} = \gamma^0 S(T), \qquad S(T)\boldsymbol{\gamma}^{tr} = -\boldsymbol{\gamma}S(T) \tag{D46}$$

which is satisfied by

$$S(P) = ie^{i\phi_T}\gamma^3\gamma^1\gamma^0 \tag{D47}$$

where ϕ_T is an arbitrary phase.

Charge conjugation: Charge conjugation turns the wave function of a particle with charge q into those of a charge $-q$. The charge q is defined in Quantum Mechanics as the parameter characterizing the phase transformation of the

wave function under gauge transformation, $\psi(x) \to e^{ie\Phi(x)}\psi(x)$ therefore, the charge conjugation must be anti-linear and involve a complex conjugation.

To find the representation of the charge conjugation, $U(C) : \psi(x) \to S(C)\bar{\psi}(x)$, we have to introduce an external vector potential by generalizing the minimal coupling procedure $\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}$ to the relativistic case, $\partial_\mu \to \partial_\mu - ieA_\mu$, leading to the Dirac-equation

$$[i\gamma^\mu \partial_\mu + e\gamma^\mu A_\mu(x) - m]\psi(x) = [i\not{\partial} + \not{A}(x) - m]\psi(x) = 0. \tag{D48}$$

The wave function $S(C)\bar{\psi}(x)$ satisfies similar equation with $e \to -e$ if

$$S(C)\gamma^{\mu tr} = -\gamma^\mu S(T) \tag{D49}$$

which is satisfied by

$$S(C) = ie^{i\phi_C}\gamma^2\gamma^0 \tag{D50}$$

where ϕ_C is an arbitrary phase.

The CPT theorem assures that $\phi_C + \phi_P + \phi_T$ is integer multiple of 2π for any relativistic, local Quantum Field Theory, ie.

$$U(CPT)\psi(x) = i\gamma^5\psi(-x). \tag{D51}$$

c. Bilinears

The transformation property, (D19), allows us to construct useful bilinears of the field $\psi(x)$. By means of the transformation rule

$$\bar{\psi}(x) \to \bar{\psi}'(x') = \bar{\psi}(x)\gamma^0 S^\dagger(\Lambda)\gamma^0 = \bar{\psi}(x)S^{-1}(\Lambda) \tag{D52}$$

one finds

$$\bar{\psi}'(x')\Gamma\psi'(x') = \bar{\psi}(x)S^{-1}(\Lambda)\Gamma S(\Lambda)\psi(x) \tag{D53}$$

for any $4\times$ four matrix Γ. A complete set of matrices,

$$\begin{aligned} \Gamma_S &= \mathbb{1} \\ \Gamma_V^\mu &= \gamma^\mu \\ \Gamma_T^{\mu\nu} &= \sigma^{\mu\nu} \\ \Gamma_A^\mu &= \gamma^5\gamma^\mu \\ \Gamma_P &= \gamma^5, \end{aligned} \tag{D54}$$

leads to scalar ($\bar{\psi}\Gamma_S\psi$), vector ($\bar{\psi}\Gamma_V^\mu\psi$), tensor ($\bar{\psi}\Gamma_T^{\mu\nu}\psi$), axial-vector ($\bar{\psi}\Gamma_A^\mu\psi$), pseudo-scalar ($\bar{\psi}\Gamma_P\psi$).

3. Bispinors

There is a more involved but more illuminating way of deriving the Dirac-equation. We start by recalling that the elementary particles are defined by the irreducible representations of the corresponding symmetries. In case of the proper Lorentz group the simplest irreducible representations are given in Eqs. (C14). We shall use the spinors ξ^a and $\eta_{\dot{b}}$, they correspond to spin half fermions which obeys well defined transformation rules with respect to the proper Lorentz group but space-time inversions are not recognized, as neutrinos.

When space-time inversions are supposed to be symmetry transformations then we need the representation of the full Lorentz group. Let us consider the case of space-inversions. The angular momentum, $\boldsymbol{L} = \boldsymbol{x} \otimes \boldsymbol{p}$ is a pseudo vector because is remains invariant under space inversion and we have $[P, \boldsymbol{L}] = 0$. Therefore the representation of P on the spinors must be a multiplication by a complex number, $\xi^a \to (\pi\xi)^a = \pi\xi^a$, according to the Schur-lemma (stating that a linear transformation which commutes with all symmetry transformation in an irreducible representation must be c-number times the identity operator). Two consecutive inversions restore the three vectors. According to Eqs. (C13) and (C14) the most general spatial rotation which preserves the three vectors is $g = \pm\mathbb{1}$. Therefore, $\pi^2 = \pm 1$

and $\pi = \pm 1$ or $\pi = \pm i$. It is usually the convention $\pi = i$ is assumed. (The choice matters for charged, massless particles only.)

But there are problems with the Lorentz-boosts. Let us denote the Lorentz-boost transformation which carries out a change $\boldsymbol{v}$ in the velocity by $\Lambda(\boldsymbol{v})$ then we have $P\Lambda(\boldsymbol{v}) = \Lambda(-\boldsymbol{v})P$ indicating that $[P, \Lambda(\boldsymbol{v})] \neq 0$, the space-inversion can not be represented by a simple multiplication in the presence of Lorenz-boosts. The most economic extension of the irreducible representation of the proper Lorentz group to the full one is obtained by putting together two spinors into a bispinors,

$$\psi = \begin{pmatrix} \xi^a \\ \eta_{\dot{a}} \end{pmatrix}, \tag{D55}$$

and realizing space inversion by the exchange,

$$P : \begin{cases} \xi^a \to \pi \eta_{\dot{a}=a} \\ \eta_{\dot{a}} \to \pi \xi^{a=\dot{a}} \end{cases} . \tag{D56}$$

The two spinors describe particles with different chirality because the space inversion flips the sing of the three-momentum but not the spin and they define the chiral representation for the Dirac-equation.

In order to obtain covariant equation of motion for a free particle which is linear in the momentum $\hat{p} = p^{a\dot{b}}$ we can contract upper and lower indices of the same type,

$$\begin{aligned} p^{a\dot{b}}\eta_{\dot{b}} &= m\xi^a \\ p_{\dot{b}a}\xi^a &= m\eta_{\dot{b}} \end{aligned} \tag{D57}$$

where m is a constant of mass dimension. (It is easy to see that a rescaling of the spinors allows us to use the same constant in both lines.) Let us use the parametrization $p^{a\dot{b}} = (p^0 + \boldsymbol{p\sigma})^{a\dot{b}}$. The relations

$$p_{\dot{a}b} = (GpG^{\mathrm{tr}})^*_{\dot{a}b} = (p^0 + G(\boldsymbol{p\sigma})^* G^{\mathrm{tr}})_{\dot{a}b} = (p^0 - \boldsymbol{p\sigma})_{\dot{a}b} \tag{D58}$$

allow us to write the equations of motion as

$$\begin{aligned} (p^0 + \boldsymbol{p\sigma})\eta &= m\xi \\ (p^0 - \boldsymbol{p\sigma})\xi &= m\eta, \end{aligned} \tag{D59}$$

or as (D7) with

$$\beta = \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{D60}$$

in the chiral representation. Note then now

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D61}$$

These considerations show that the role of the mass in the dynamics of spin half fermions is to couple the left and two simplest irreducible representations of the proper Lorentz group, the right and left handed neutrino fields.

4. Plane waves

The physical content of plane wave solution of the free Dirac-equation is discussed now for non-vanishing mass.

The relativistic dispersion relation $E^2 = m^2 + \boldsymbol{p}^2$ allows both positive and negative energy values. Let us introduce plane waves with positive and negative energies,

$$\psi_k^{(+)}(x) = e^{-ik\cdot x}u(k), \qquad \psi_k^{(-)}(x) = e^{ik\cdot x}v(k) \tag{D62}$$

where $k^0 = \omega_{\boldsymbol{k}} \geq 0$ and the bi-spinors $u(k)$ and $v(k)$ satisfy

$$(\not{k} - m)u(k) = (\not{k} + m)v(k) = 0 \tag{D63}$$

with $k^2 = m^2$ due to Eq. (D11). The construction of the spinors $u(k)$ and $v(k)$ starts in their rest frame ($m^2 > 0$),

$$(\gamma^0 - 1)u(k_{rf}) = (\gamma^0 + 1)v(k_{rf}) = 0, \tag{D64}$$

with $k_{rf} = (m, \mathbf{0})$. The solution in the standard representation is

$$u^{(1)}(k_{rf}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)}(k_{rf}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v^{(1)}(k_{rf}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v^{(1)}(k_{rf}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{D65}$$

The bi-spinors corresponding to an arbitrary energy-momentum k on the mass shell, $k^2 = m^2$ is given by

$$\begin{aligned} u^{(\alpha)}(k) &= \frac{\not{k} + m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} u^{(\alpha)}(k_{rf}) = \begin{pmatrix} \sqrt{\frac{m+\omega_{\boldsymbol{k}}}{2m}}\phi^{(\alpha)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}}\phi^{(\alpha)} \end{pmatrix} \\ v^{(\alpha)}(k) &= \frac{-\not{k} + m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} v^{(\alpha)}(k_{rf}) = \begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}}\chi^{(\alpha)} \\ \sqrt{\frac{m+\omega_{\boldsymbol{k}}}{2m}}\chi^{(\alpha)} \end{pmatrix} \end{aligned} \tag{D66}$$

according to Eq. (D11) where $k^0 = \omega_{\boldsymbol{k}} = \sqrt{m^2 + \boldsymbol{k}^2}$, $\phi^{(\alpha)}$ and $\chi^{(\alpha)}$ are k-independent, two-component spinors. The normalization of the bi-spinor is chosen in such a manner that the relations

$$\bar{u}^{(\alpha)}(k)u^{(\beta)}(k) = -\bar{v}^{(\alpha)}(k)v^{(\beta)}(k) = \delta_{\alpha,\beta}, \qquad \bar{u}^{(\alpha)}(k)v^{(\beta)}(k) = \bar{v}^{(\alpha)}(k)u^{(\beta)}(k) = 0 \tag{D67}$$

hold.

It is now easy to construct the projection operator corresponding to the positive and negative energy spinors,

$$P_+(k) = \sum_{\alpha=1}^{2} u^{(\alpha)}(k) \otimes \bar{u}^{(\alpha)}(k) = \frac{\not{k} + m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} \frac{1+\gamma^0}{2} \frac{\not{k} + m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} = \frac{\not{k} + m}{2m} \tag{D68}$$

and

$$P_-(k) = -\sum_{\alpha=1}^{2} v^{(\alpha)}(k) \otimes \bar{v}^{(\alpha)}(k) = \frac{\not{k} - m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} \frac{1-\gamma^0}{2} \frac{\not{k} - m}{\sqrt{2m(m+\omega_{\boldsymbol{k}})}} = \frac{-\not{k} + m}{2m}, \tag{D69}$$

respectively by means of the identity $(\not{k} + m)\gamma^0(\not{k} + m) = 2\omega_{\boldsymbol{k}}(\not{k} + m)$.

The four linearly independent plane wave solutions of the free Dirac equation corresponding to a given energy-momentum k on the mass shell, $k^2 = m^2$ can be classified by the projection of the spin on the quantization axis and the sign of the energy.

Appendix E: Time dependence in Quantum Mechanics

The time has no operator counterpart in usual, so called Schrödinger representation of Quantum Mechanics in contrary to the coordinates. As a result the field operator, introduced as the analogy of the coordinate operator, should be time independent in contrary to the spirit of classical field theory. This problem is settled by the introduction of the Heisenberg representation of the time evolution. The perturbation expansion is simplified by defining a further, interaction representation.

1. Representations of time dependence

Schrödinger representation: The time evolution of the state of the system is generated by the Schrödinger equation

$$i\partial_t |\psi(t)\rangle_S = H|\psi(t)\rangle_S \tag{E1}$$

and the operators are time independent, $A = A_S$, in this representation. The solution of the Schrödinger equation is

$$|\psi(t)\rangle_S = e^{-i(t-t_i)H}|\psi(t_i)\rangle_S. \tag{E2}$$

Heisenberg representation: The time evolution is placed in the operators and the state are time independent in this representation. The former is achieved by the similarity transformation

$$|\Psi(t)\rangle_H = e^{i(t-t_i)H}|\Psi(t)\rangle_H \tag{E3}$$

which induces the transformation

$$A_H(t) = e^{i(t-t_i)H}A_S e^{-i(t-t_i)H} \tag{E4}$$

for the operators in order to recover the same matrix elements in both representations. Therefore, the operators satisfy the Heisenberg equation of motion,

$$i\partial_t A_H(t) = [A_H, H], \tag{E5}$$

with the initial conditions $A_H(t_i) = A_S$. Notice that the two representations are identical at the initial time $t = t_i$.

Interaction representation: Let us suppose that $H = H_0 + H_1$ where H_0 and H_1 represent the free, easily diagonalizable dominant part and the small, complicated interaction part of the Hamiltonian, respectively. It is rather complicated to use the perturbation expansion of the Heisenberg equation. It seems as a reasonable compromise between these two preceding representations to place the time dependence, generated by the simple part of the Hamiltonian into the operators and leave the complicated but supposedly small part of the time dependence for the states where the usual Rayleigh-Schrödinger perturbation expansion is relatively simple.

For this end we define

$$|\Psi(t)\rangle_i = e^{i(t-t_i)H_0}|\Psi(t)\rangle_S \tag{E6}$$

which induces the transformation

$$A_i(t) = e^{i(t-t_i)H_0}A_S e^{-i(t-t_i)H_0} \tag{E7}$$

for the operators. The state vector satisfies the equation of motion

$$\begin{aligned} i\partial_t|\Psi(t)\rangle_i &= -H_0|\Psi(t)\rangle_i + e^{i(t-t_i)H_0}(H_0+H_1)|\Psi(t)\rangle_S \\ &= -H_0|\Psi(t)\rangle_i + e^{i(t-t_i)H_0}(H_0+H_1)e^{-i(t-t_i)H_0}e^{i(t-t_i)H_0}|\Psi(t)\rangle_S \\ &= H_{1i}(t)|\Psi(t)\rangle_i \end{aligned} \tag{E8}$$

is indeed a Schrödinger equation involving the interaction only. The operators follow the Heisenberg equation,

$$i\partial_t A_i(t) = [A_i, H_0]. \tag{E9}$$

2. Schrödinger equation with time dependent Hamiltonian

The interaction representation requires the solution of the Schrödinger equation with time dependent Hamiltonian,

$$i\partial_t|\Psi(t)\rangle = H(t)|\Psi(t)\rangle. \tag{E10}$$

To obtain it in a closed form one introduces the time-ordered product, a modified multiplication rule for operators depending on the time. For a chain of operators $A_1(t_1), \cdots, A_n(t_n)$ the time ordered product is defined by acting with the operators in the order of ascending time values. A sign factor is inserted, too, according to the parity of the permutation of the operators, needed to arrive to the time ordered form for fermionic operators. For two operators we have

$$T[A(t_A)B(t_B)] = \Theta(t_A - t_B)A(t_A)B(t_B) \pm \Theta(t_B - t_A)B(t_B)A(t_A), \tag{E11}$$

where

$$\Theta(t) = \begin{cases} 1 & t > 1 \\ 0 & t < 0 \end{cases} \tag{E12}$$

($\Theta(0)$ is ill defined) and the sign + or - is used for bosonic and fermion operators, respectively.

The solution of Eq. (E10) is

$$|\Psi(t)\rangle = U(t,t_i)|\Psi(t_i)\rangle \tag{E13}$$

where

$$U(t,t_i) = T[e^{-i\int_{t_i}^{t} dt' H(t')}]. \tag{E14}$$

To prove this result it is sufficient to show that the time evolution operator $U(t_2,t_1)$ must satisfies the equation of motion

$$i\partial_t U(t,t_i) = H(t)U(t,t_i). \tag{E15}$$

This can easily be done by writing

$$\begin{aligned} i\partial_t U(t,t_i) &= i\partial_t T[e^{-i\int_{t_i}^{t} dt' H(t')}] \\ &= i\partial_t \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_i}^{t} dt_1 \cdots \int_{t_i}^{t} dt_n T[H(t_1)\cdots H(t_n)]. \end{aligned} \tag{E16}$$

and noting that the derivative ∂_t generates n-times the same integrand,

$$\begin{aligned} i\partial_t U(t,t_i) &= \sum_{n=0}^{\infty} \frac{n(-i)^n}{n!} \int_{t_i}^{t} dt_1 \cdots \int_{t_i}^{t} dt_{n-1} T[H(t)H(t_1)\cdots H(t_{n-1})] \\ &= H(t) \sum_{n=0}^{\infty} \frac{(-i)^{n-1}}{(n-1)!} \int_{t_i}^{t} dt_1 \cdots \int_{t_i}^{t} dt_{n-1} T[H(t_1)\cdots H(t_{n-1})] \\ &= H(t)U(t,t_i). \end{aligned} \tag{E17}$$

Appendix F: Energy-momentum tensor of a hermitean scalar field theory

let us consider a free Hermitian scalar theory, defined by the Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2. \tag{F1}$$

The field operator is taken to be the general solution of the equation of motion,

$$\phi(x) = \int d\tilde{\boldsymbol{k}}[a(\boldsymbol{k})e^{-ikx} + a^\dagger(\boldsymbol{k})e^{ikx}]. \tag{F2}$$

The canonical energy-momentum tensor,

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right), \tag{F3}$$

yields the energy-momentum vector

$$P^\mu = \int d^3x \left[\partial_0\phi\partial^\mu\phi - g^{0\mu}\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)\right]. \tag{F4}$$

The insertion of the field operator (F2) into this equation leads to

$$\begin{aligned}
P^\mu &= \int d^3x d\tilde{\boldsymbol{k}} d\tilde{\boldsymbol{\ell}} \Big[-\omega_{\boldsymbol{k}} \ell^\mu [-a(\boldsymbol{k}) e^{-ikx} + a^\dagger(\boldsymbol{k}) e^{ikx}][-a(\boldsymbol{\ell}) e^{-i\ell x} + a^\dagger(\boldsymbol{\ell}) e^{i\ell x}] \\
&\quad - g^{0\mu} \Big(-\frac{k^\nu \ell_\nu}{2} [-a(\boldsymbol{k}) e^{-ikx} + a^\dagger(\boldsymbol{k}) e^{ikx}][-a(\boldsymbol{\ell}) e^{-i\ell x} + a^\dagger(\boldsymbol{\ell}) e^{i\ell x}] \\
&\quad - \frac{m^2}{2} [a(\boldsymbol{k}) e^{-ikx} + a^\dagger(\boldsymbol{k}) e^{ikx}][a(\boldsymbol{\ell}) e^{-i\ell x} + a^\dagger(\boldsymbol{\ell}) e^{i\ell x}] \Big) \Big] \\
&= \int d^3x d\tilde{\boldsymbol{k}} d\tilde{\boldsymbol{\ell}} \Big[\Big(-\omega_{\boldsymbol{k}} \ell^\mu + \frac{g^{0\mu}}{2} (k^\nu \ell_\nu + m^2) \Big) e^{-i(k+\ell)x} a(\boldsymbol{k}) a(\boldsymbol{\ell}) \\
&\quad + \Big(-\omega_{\boldsymbol{k}} \ell^\mu + \frac{g^{0\mu}}{2} (k^\nu \ell_\nu + m^2) \Big) e^{i(k+\ell)x} a^\dagger(\boldsymbol{k}) a^\dagger(\boldsymbol{\ell}) \\
&\quad + \Big(\omega_{\boldsymbol{k}} \ell^\mu + \frac{g^{0\mu}}{2} (-k^\nu \ell_\nu + m^2) \Big) e^{i(-k+\ell)x} a^\dagger(\boldsymbol{k}) a(\boldsymbol{\ell}) \\
&\quad + \Big(\omega_{\boldsymbol{k}} \ell^\mu + \frac{g^{0\mu}}{2} (-k^\nu \ell_\nu + m^2) \Big) e^{-i(k-\ell)x} a(\boldsymbol{k}) a^\dagger(\boldsymbol{\ell}) \Big]
\end{aligned} \tag{F5}$$

The expression for the energy is

$$\begin{aligned}
P^0 &= \int d\tilde{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} \Big[\Big(-\omega_{\boldsymbol{k}}^2 + \frac{1}{2} (\omega_{\boldsymbol{k}}^2 + \boldsymbol{k}^2 + m^2) \Big) e^{-2i\omega_{\boldsymbol{k}} x^0} a(\boldsymbol{k}) a(-\boldsymbol{k}) \\
&\quad + \Big(-\omega_{\boldsymbol{k}}^2 + \frac{1}{2} (\omega_{\boldsymbol{k}}^2 + \boldsymbol{k}^2 + m^2) \Big) e^{2i\omega_{\boldsymbol{k}} x^0} a^\dagger(\boldsymbol{k}) a^\dagger(-\boldsymbol{k}) \\
&\quad + \Big(\omega_{\boldsymbol{k}}^2 + \frac{1}{2} (-\omega_{\boldsymbol{k}}^2 + \boldsymbol{k}^2 + m^2) \Big) \omega_{\boldsymbol{k}}^2 a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + \Big(-\omega_{\boldsymbol{k}}^2 + \frac{1}{2} (-\omega_{\boldsymbol{k}}^2 + \boldsymbol{k}^2 + m^2) \Big) a(\boldsymbol{k}) a^\dagger(\boldsymbol{k}) \Big] \\
&= \frac{1}{2} \int d\tilde{\boldsymbol{k}} \omega_{\boldsymbol{k}} [a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + a(\boldsymbol{k}) a^\dagger(\boldsymbol{k})]
\end{aligned} \tag{F6}$$

and the tree-momentum,

$$\boldsymbol{P} = \int d\tilde{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}} [\omega_{\boldsymbol{k}} \boldsymbol{k} e^{-2i\omega_{\boldsymbol{k}} x^0} a(\boldsymbol{k}) a(-\boldsymbol{k}) + \omega_{\boldsymbol{k}} \boldsymbol{k} e^{2i\omega_{\boldsymbol{k}} x^0} a^\dagger(\boldsymbol{k}) a^\dagger(-\boldsymbol{k}) + \omega_{\boldsymbol{k}} \boldsymbol{k} [a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + a(\boldsymbol{k}) a^\dagger(\boldsymbol{k})]], \tag{F7}$$

reduces to

$$\boldsymbol{P} = \frac{1}{2} \int d\tilde{\boldsymbol{k}} \boldsymbol{k} [a^\dagger(\boldsymbol{k}) a(\boldsymbol{k}) + a(\boldsymbol{k}) a^\dagger(\boldsymbol{k})] \tag{F8}$$

after realizing that the integral $\int d\tilde{\boldsymbol{k}} \boldsymbol{k}$ is vanishing because its integrand is the product of a space-inversion symmetric integral measure $d\tilde{\boldsymbol{k}}$ and of a space-inversion asymmetric factor $\boldsymbol{k}$.

Appendix G: Strong Wick theorem

The strong form of Wick theorem, referring to operators rather than their vacuum expectation values is proved in this Appendix.

Theorem 1: The identity

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \frac{\lambda^3}{3!} [A, [A, [A, B]]] + \cdots \tag{G1}$$

holds for any pair of operators A and B and $\lambda \in \mathbb{R}$.

Proof: Let us define

$$f(\lambda) = e^{\lambda A} B e^{-\lambda A} \tag{G2}$$

and consider its McLauren series,

$$\begin{aligned}\frac{df(\lambda)}{d\lambda} &= [A, f(\lambda)] \\ \frac{d^2 f(\lambda)}{d\lambda^2} &= [A, \frac{df(\lambda)}{d\lambda}] \\ &= [A, [A, f(\lambda)]] \end{aligned} \tag{G3}$$

where we recognize the structure of the right hand side of Eq. (G1). The desired identity can easily be recovered by setting $\lambda = 0$.

Theorem 2: (Baker-Haussdorf simplified) If two operators A and B commute with their commutator, $[A, [A, B]] = [B, [A, B]] = 0$ then

$$a^A a^B = a^{A+B+\frac{1}{2}[A,B]}. \tag{G4}$$

Proof: Let us consider the operator

$$f(\lambda) = e^{\lambda A} e^{\lambda B} \tag{G5}$$

for $\lambda \in \mathbb{R}$ whose derivative is

$$\begin{aligned}\frac{df(\lambda)}{d\lambda} &= A e^{\lambda A} e^{\lambda B} + e^{\lambda A} B e^{\lambda B} \\ &= (A + e^{\lambda A} B e^{-\lambda A}) f(\lambda). \end{aligned} \tag{G6}$$

By means of the identity (G1) we have

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] \tag{G7}$$

and

$$\frac{df(\lambda)}{d\lambda} = (A + B + \lambda [A, B]) f(\lambda). \tag{G8}$$

The solution of this differential equation corresponding to the initial condition $f(0) = \mathbb{1}$ is

$$f(\lambda) = a^{\lambda(A+B)+\frac{\lambda}{2}[A,B]} \tag{G9}$$

because $[A, B]$ can be treated as a c-number in an equation involving A and B. The theorem follows from the choice $\lambda = 1$.

Theorem 3: Let us consider a field operator which is the sum of a positive and negative energy part, $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$, and the c-number source $j(x)$. Wick theorem states that the relation

$$T[e^{-i\int dx j(x)\phi(x)}] =: e^{-i\int dx j(x)\phi(x)} : e^{-\frac{1}{2}\int dx dy j(x)\langle 0|T[\phi(x)\phi(y)]|0\rangle j(y)} \tag{G10}$$

holds.

Proof: The previous theorem will be useful because the commutator of free fields is c-number. We approximate the left hand side of (G10) by

$$T[e^{-i\int dx j(x)\phi(x)}] = e^{-i\Delta t \int d^3x j(t_n,\boldsymbol{x})\phi(t_n,\boldsymbol{x})} \cdots e^{-i\Delta t \int d^3x j(t_1,\boldsymbol{x})\phi(t_1,\boldsymbol{x})}(1 + \mathcal{O}(1/n)) \tag{G11}$$

where $t_j = j\Delta t$, $\Delta t = t/n$ and write

$$T[e^{-i\int dx j(x)\phi(x)}] = e^{-i\Delta t \sum_{j=1}^n \int d^3x j(t_j,\boldsymbol{x})\phi(t_j,\boldsymbol{x}) - \frac{\Delta t^2}{2}\sum_{j>k=1}^n [\int d^3x j(t_j,\boldsymbol{x})\phi(t_j,\boldsymbol{x}), \int d^3y j(t_k,\boldsymbol{y})\phi(t_k,\boldsymbol{y})]}(1 + \mathcal{O}(1/n)) \tag{G12}$$

according to the previous theorem. The limit $n \to \infty$ gives

$$\begin{aligned} T[e^{-i\int dx j(x)\phi(x)}] &= e^{-i\int dx j(x)\phi(x) - \frac{1}{2}\int dx dy j(x) j(y)\Theta(x^0 - y^0)[\phi(x)\phi(y)]} \\ &= e^{-i\int dx j(x)\phi(x)} e^{-\frac{1}{2}\int dx dy j(x) j(y)\Theta(x^0 - y^0)[\phi(x),\phi(y)]} \end{aligned} \tag{G13}$$

Since

$$
\begin{aligned}
:e^{-i\int dxj(x)\phi(x)}: &= e^{-i\int dxj(x)\phi^{(-)}(x)}e^{-i\int dxj(x)\phi^{(+)}(x)} \\
&= e^{-i\int dxj(x)\phi(x)-\frac{1}{2}\int dxdy[\phi^{(-)}(x),\phi^{(+)}(y)]}
\end{aligned}
\tag{G14}
$$

we have

$$
\begin{aligned}
T[e^{-i\int dxj(x)\phi(x)}] &= e^{-i\int dxj(x)\phi(x)-\frac{1}{2}\int dxdyj(x)j(y)\Theta(x^0-y^0)[\phi(x)\phi(y)]} \\
&= e^{-i\int dxj(x)\phi(x)}e^{-\frac{1}{2}\int dxdyj(x)j(y)\Theta(x^0-y^0)[\phi(x),\phi(y)]},
\end{aligned}
\tag{G15}
$$

yielding

$$
T[e^{-i\int dxj(x)\phi(x)}] =: e^{-i\int dxj(x)\phi(x)}: e^{\frac{1}{2}\int dxdyj(x)j(y)([\phi^{(-)}(x),\phi^{(+)}(y)]-\Theta(x^0-y^0)[\phi(x),\phi(y)])}. \tag{G16}
$$

The second exponent, made up by the commutator of free fields, is actually a c-number therefore,

$$
\begin{aligned}
[\phi^{(-)}(x),\phi^{(+)}(y)]-\Theta(x^0-y^0)[\phi(x),\phi(y)] &= \langle 0|[\phi^{(-)}(x),\phi^{(+)}(y)]-\Theta(x^0-y^0)[\phi(x),\phi(y)]|0\rangle \\
&= \langle 0|\phi^{(-)}(x)\phi^{(+)}(y)-\phi^{(+)}(y)\phi^{(-)}(x)-\Theta(x^0-y^0)[\phi(x),\phi(y)]|0\rangle
\end{aligned}
\tag{G17}
$$

The positive frequency part contains the operator $a(\boldsymbol{p})$ only and annihilates the vacuum, thus

$$
\begin{aligned}
[\phi^{(-)}(x),\phi^{(+)}(y)]-\Theta(x^0-y^0)[\phi(x),\phi(y)] &= \langle 0|-\phi^{(+)}(y)\phi^{(-)}(x)-\Theta(x^0-y^0)[\phi(x),\phi(y)]|0\rangle \\
&= \langle 0|-\phi(y)\phi(x)-\Theta(x^0-y^0)[\phi(x),\phi(y)]|0\rangle \\
&= \langle 0|(\Theta(x^0-y^0)-1)\phi(y)\phi(x)-\Theta(x^0-y^0)\phi(x)\phi(y)|0\rangle \\
&= -\langle 0|T[\phi(x)\phi(y)]|0\rangle.
\end{aligned}
\tag{G18}
$$

Remark: A more useful way of expressing Wick theorem,

$$
\begin{aligned}
T[\phi(x_1)\cdots\phi(x_{2n})] &= :\phi(x_1)\cdots\phi(x_n):+\sum_{j<k}T[\phi(x_j)\phi(x_k)]:\phi(x_1)\cdots\hat{\phi}(x_j)\cdots\hat{\phi}(x_k)\cdots\phi(x_{2n}): \\
&\quad+\sum_{j_1\le j_2,j_1\le k_1,j_2\le k_2}T[\phi(x_{j_1})\phi(x_{k_1})]T[\phi(x_{j_2})\phi(x_{k_2})] \\
&\quad\times:\phi(x_1)\cdots\hat{\phi}(x_{j_1})\cdots\hat{\phi}(x_{k_1})\cdots\hat{\phi}(x_{j_2})\cdots\hat{\phi}(x_{k_2})\cdots\phi(x_{2n}):+\cdots
\end{aligned}
\tag{G19}
$$

is obtained by expanding both sides of Eq. (G10) in the source j and by identifying the coefficients of the same power on the two sides.

The weak form of the theorem can be obtained by taking the vacuum expectation value of Eq. (G10) when the only surviving term is the last one without normal ordering part ($\langle 0|:A:|0\rangle=0$) is

$$
\langle 0|T[\phi(x_1)\cdots\phi(x_{2n})]|0\rangle=\frac{1}{n!2^n}\sum_{\pi\in S_{2n}}T[\phi(x_{\pi(1)})\phi(x_{\pi(2)})]\cdots T[\phi(x_{\pi(2n-1)})\phi(x_{\pi(2n)})] \tag{G20}
$$

for even n where the symbols marked by a hat are omitted in the product.

Appendix H: Electron-electron scattering

We briefly present the calculation of the electron-electron scattering cross section in leading order of the perturbation expansion The process considered first is $e(\boldsymbol{p}_1,\alpha_1)+e(\boldsymbol{p}_2,\alpha_2)\to e(\boldsymbol{q}_1,\beta_1)+e(\boldsymbol{q}_2,\beta_2)$ where particles three-momentum and helicity are indicated. The transition amplitude of the process is

$$
\langle f|S|i\rangle=\langle 0|T[c_{o\beta_1}(\boldsymbol{q}_1)c_{o\beta_2}(\boldsymbol{q}_2)e^{-ie\int dz\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z)}c^\dagger_{i\alpha_2}(\boldsymbol{p}_2)c^\dagger_{i\alpha_1}(\boldsymbol{p}_1)]|0\rangle \tag{H1}
$$

in terms of asymptotic in and out operators for electrons.

It is advantageous to separate off simpler scattering processes from the transition probability by assuming that all particle changes its four-momentum during the collision. In fact, if a particle preserves its four-momentum it makes a forward scattering and can be considered as a spectator, it traverses the collision zone without interaction and

the actual collision process involves less particles. The formal way such a forward scattering contributions can be eliminated is offered by the reduction formulae and the basic idea is the following. The creation and annihilation operators are expressed by the help of asymptotic field operator (88),

$$c_\alpha(x^0,\boldsymbol{k}) = \int d^3x \bar{u}^{(\alpha)}(\boldsymbol{k}) e^{ikx}\gamma^0\psi(x) \tag{H2}$$

where the time when the annihilation operator is considered is written explicitly and the orthogonality of the spinors, Eq. (D67) was used. To have a more suitable form we use the identity

$$f(t_f) = f(t_i) + \int_{t_i}^{t_f} dt \frac{df(t)}{dt} \tag{H3}$$

to write

$$\begin{aligned} c_{o\alpha}(\boldsymbol{k}) &= c_{i\alpha}(\boldsymbol{k}) + \frac{1}{\sqrt{Z}}\int dx \bar{u}^{(\alpha)}(\boldsymbol{k})\partial_0[e^{ikx}\gamma^0\psi(x)] \\ &= c_{i\alpha}(\boldsymbol{k}) + \frac{1}{\sqrt{Z}}\int dx \bar{u}^{(\alpha)}(\boldsymbol{k})[(\partial_0 e^{ikx})\gamma^0\psi(x) + e^{ikx}\gamma^0\partial_0\psi(x)] \end{aligned} \tag{H4}$$

Since the spinor $\bar{u}^{(\alpha)}(\boldsymbol{k})e^{ikx}$ satisfies the Dirac equation $\bar{u}^{(\alpha)}(\boldsymbol{k})e^{ikx}(-i\overleftarrow{\partial}-m)=0$ where the arrow shows the direction the partial derivatives act we find

$$\begin{aligned} c_{o\alpha}(\boldsymbol{k}) &= c_{i\alpha}(\boldsymbol{k}) + \frac{1}{\sqrt{Z}}\int dx \bar{u}^{(\alpha)}(\boldsymbol{k})\{[(-\boldsymbol{\gamma}\boldsymbol{\nabla} + im)e^{ikx}]\psi(x) + e^{ikx}\gamma^0\partial_0\psi(x)\} \\ &= c_{i\alpha}(\boldsymbol{k}) - \frac{i}{\sqrt{Z}}\int dx \bar{u}^{(\alpha)}(\boldsymbol{k})e^{ikx}(i\not{\partial} - m)\psi(x) \end{aligned} \tag{H5}$$

where the partial integration was carried out in the second equation. The hermitian conjugate of this equation is

$$c^\dagger_{i\alpha}(\boldsymbol{k}) = c^\dagger_{o\alpha}(\boldsymbol{k}) - \frac{i}{\sqrt{Z}}\int dx \bar{\psi}(x)(-i\overleftarrow{\not{\partial}} - m)u^{(\alpha)}(\boldsymbol{k})e^{-ikx}. \tag{H6}$$

The creation and annihilation operators written in this manner can now be inserted into Eq. (H1). The contribution of the in operator on the right hand side of Eq. (H5) removes an electron from the initial state. This particle does not participate in the collision process, being eliminated already at the initial time. This is the contribution of a collision where the particle in question preserves its state and makes a forward scattering. In a similar manner, the contribution of the out creation operator in the right hand side of Eq. (H6) stand for the forward scattering of a particle which is inserted at the final time, after the collision process. We consider genuine scattering process without forward scattering and omit such terms. We set $Z=1$ in leading order and find

$$\begin{aligned} \langle f|S|i\rangle &= \frac{(-i^4)}{Z^2}\int dy_1 dy_2 dx_1 dx_2 e^{-ix_1p_1 - ix_2p_2 + iy_1q_1 + iy_2q_2}\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)(i\not{\partial}_{y_1} - m)\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)(i\not{\partial}_{y_2} - m) \\ &\quad \langle 0|T[\psi(y_1)\psi(y_2)e^{-ie\int dz\bar{\psi}(z)\gamma^\mu\psi(z)A_\mu(z)}\bar{\psi}(x_2)\bar{\psi}(x_1)]|0\rangle^c(-i\overleftarrow{\not{\partial}}_{x_2} - m)u^{(\alpha_2)}(\boldsymbol{p}_2)(-i\overleftarrow{\not{\partial}}_{x_1} - m)u^{(\alpha_1)}(\boldsymbol{p}_1) \end{aligned} \tag{H7}$$

where the index c on the vacuum expectation value reminds us to take the connected (non forward scattering part) of the amplitude. The leading order contributions to the vacuum expectation value are given in Eq. (158).

The Dirac operators on the right hand side of Eq. (H7), the inverses of the free electron propagator remove the external legs of the graphs and one finds

$$\begin{aligned} \langle f|S|i\rangle &= (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)(-ie)^4\Bigg[-\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\mu u^{(\alpha_1)}(\boldsymbol{p}_1)\frac{-i}{(p_1-q_1)^2+i\epsilon}\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma_\mu u^{(\alpha_2)}(\boldsymbol{p}_2) \\ &\quad +\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma^\mu u^{(\alpha_1)}(\boldsymbol{p}_1)\frac{-i}{p_1-q_2)^2+i\epsilon}\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\mu u^{(\alpha_2)}(\boldsymbol{p}_2)\Bigg] \end{aligned} \tag{H8}$$

where the photon propagator is given by Eq. (128) in Feynman gauge $\lambda=1$. The transition amplitude appearing in Eqs. (99) is the expression in the square bracket in (H8).

Let us simplify matters and consider unpolarized collision where one averages over the initial and sums over final polarization. The transition amplitude square for such a collision process is therefore

$$|\mathcal{T}|^2 = \frac{e^4}{4}\sum_{\alpha_1,\alpha_2,\beta_1,\beta_2}\left[\frac{\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\mu u^{(\alpha_1)}(\boldsymbol{p}_1)\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma_\mu u^{(\alpha_2)}(\boldsymbol{p}_2)}{(p_1-q_1)^2} - \frac{\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma^\mu u^{(\alpha_1)}(\boldsymbol{p}_1)\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\mu u^{(\alpha_2)}(\boldsymbol{p}_2)}{(p_1-q_2)^2}\right] \times\left[\frac{\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\nu u^{(\alpha_1)}(\boldsymbol{p}_1)\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma_\nu u^{(\alpha_2)}(\boldsymbol{p}_2)}{(p_1-q_1)^2} - \frac{\bar{u}^{(\beta_2)}(\boldsymbol{q}_2)\gamma^\nu u^{(\alpha_1)}(\boldsymbol{p}_1)\bar{u}^{(\beta_1)}(\boldsymbol{q}_1)\gamma^\nu u^{(\alpha_2)}(\boldsymbol{p}_2)}{(p_1-q_2)^2}\right]^* \tag{H9}$$

where the $i\epsilon$ was dropped in the photon propagator being irrelevant in the absence of integration over the photon energy-momentum. The projector operator (D68) with $u^{(\alpha)}(k) = u^{(\alpha)}(\omega_{\boldsymbol{k}}, \boldsymbol{k})$ allows us to write this expression as

$$|\mathcal{T}|^2 = \frac{e^4}{64m^4}\left\{\frac{\mathrm{tr}[\gamma_\mu(\not{p}_1+m)\gamma_\nu(\not{q}_1+m)]\mathrm{tr}[\gamma^\mu(\not{p}_2+m)\gamma^\nu(\not{q}_2+m)]}{[(p_1-q_1)^2]^2} - \frac{\mathrm{tr}[\gamma_\mu(\not{p}_1+m)\gamma_\nu(\not{q}_2+m)]\mathrm{tr}[\gamma^\mu(\not{p}_2+m)\gamma^\nu(\not{q}_1+m)]}{(p_1-q_1)^2(p_1-q_2)^2} + (q_1\leftrightarrow q_2)\right\}. \tag{H10}$$

The Dirac algebra trace can easily be calculated by means of Eq. (D8),

$$\begin{aligned}
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= -\mathrm{tr}(\gamma^\nu\gamma^\mu) + 2g^{\mu\nu}\mathrm{tr}1\\
\mathrm{tr}(\gamma^\mu\gamma^\nu) &= g^{\mu\nu}\mathrm{tr}1 \to -\delta^{\mu\nu}\mathrm{tr}1\\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= \mathrm{tr}(\gamma^\nu\gamma^\mu)2g^{\rho\sigma} - \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho)\\
&= 2g^{\mu\nu}g^{\rho\sigma}\mathrm{tr}1 + \mathrm{tr}(\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho) - 2g^{\sigma\nu}\mathrm{tr}(\gamma^\mu\gamma^\rho)\\
&= 2g^{\mu\nu}g^{\rho\sigma}\mathrm{tr}1 - \mathrm{tr}(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho) + 2g^{\mu\sigma}\mathrm{tr}(\gamma^\nu\gamma^\rho) - 2g^{\sigma\nu}g^{\mu\rho}\mathrm{tr}1\\
\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\sigma\nu}g^{\mu\rho})\mathrm{tr}1
\end{aligned} \tag{H11}$$

giving

$$\begin{aligned}
\mathrm{tr}[\gamma^\mu(p\bar{+}m)\gamma^\nu(q\bar{+}m)] &= 4m^2g^{\mu\nu} + p_\alpha q_\beta \mathrm{tr}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\\
&= 4(m^2-pq)g^{\mu\nu} + 4p^\mu q^\nu + 4p^\nu q^\mu
\end{aligned} \tag{H12}$$

where $\mathrm{tr}\mathbb{1} = 4$ has been used, as well. The transition amplitude square $|\mathcal{T}|^2$ thus becomes

$$\frac{1}{2m^4}\left\{\frac{(p_1p_2)^2+(q_1q_2)^2+2m^2(p_1q_2-p_1p_2)}{[(p_1-q_1)^2]^2} + \frac{(p_1p_2)^2+(q_1q_1)^2+2m^2(p_1q_1-p_1p_2)}{[(p_1-q_2)^2]^2} - \frac{(p_1p_2)^2-2m^2(p_1p_2)}{(p_1-q_1)^2(p_1-q_2)^2}\right\}. \tag{H13}$$

The simplest parameterization of the cross section (109)

$$d\sigma = \frac{1}{4\sqrt{(p_1p_2)^2-m^4}}\int_\Delta \frac{d^3q_1d^3q_2}{(2\pi)^6 4\omega_{q_1}\omega_{q_2}}(2\pi)^4\delta(p_1+p_2-q_1-q_2)|\mathcal{T}|^2 \tag{H14}$$

is to use the particle momentum p and the scattering angle θ as kinematical parameters in the center of mass frame where $p = |\boldsymbol{p}_j| = |\boldsymbol{q}_k|$, $p_j^0 = q_j^0 = \omega_p$, $p^2 = \boldsymbol{p}^2$, $p_1p_2 = \omega_p^2 + p^2$, $p_1q_1 = m^2 + 2p^2\sin^2\theta/2$, $p_1q_2 = \omega_p^2 + p^2\cos\theta$ and the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(m^2+2p^2)^2}{4p^4(m^2+p^2)}\left[\frac{4}{\sin^4\theta} - \frac{3}{\sin^2\theta} + \left(1+\frac{4}{\sin^2\theta}\right)\frac{p^4}{(m^2+2p^2)^2}\right] \tag{H15}$$

with $\alpha = e^2/4\pi$.

Appendix I: Non-interacting particles in the functional Schrödinger representation

We construct here the wave functional for the vacuum state and the one-particle excitations in the absence of interactions, $U(\phi) = 0$. To simplify the expressions we shall use the notation $f\cdot g = \int dx f(x)g(x)$ and $\omega_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2}$.

The vacuum state functional is sought as a Gaussian,

$$\Psi_0[\phi] = e^{-\frac{1}{2}\int dxdy\phi(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{y})\phi(\boldsymbol{x})} = e^{-\frac{1}{2}\phi\cdot K\cdot\phi}. \tag{I1}$$

The derivatives

$$\begin{aligned}\frac{\delta\Psi_0[\phi]}{\delta\phi} &= -K\cdot\phi e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\\ \frac{\delta^2\Psi_0[\phi]}{\delta\phi_{\boldsymbol{x}}\delta\phi_{\boldsymbol{y}}} &= [-K_{\boldsymbol{xy}}+(K\cdot\phi)_{\boldsymbol{x}}(K\cdot\phi)_{\boldsymbol{y}}]e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\end{aligned} \tag{I2}$$

give the Schrödinger equation

$$H\Psi_0[\phi]=\frac{1}{2}[\mathrm{Tr}K-\phi\cdot K\cdot K\cdot\phi-\phi\cdot\Delta\cdot\phi+m^2\phi\cdot\phi]e^{-\frac{1}{2}\phi\cdot K\cdot\phi}=E_0e^{-\frac{1}{2}\phi\cdot K\cdot\phi}, \tag{I3}$$

together with

$$K(\boldsymbol{x},\boldsymbol{y})=(\sqrt{-\Delta+m^2})_{\boldsymbol{xy}}=\int\frac{d^3p}{(2\pi)^3}e^{i(\boldsymbol{x}-\boldsymbol{x})\boldsymbol{p}}\omega_{\boldsymbol{p}} \tag{I4}$$

and

$$E_0=\frac{1}{2}\mathrm{Tr}K=\int\frac{d^3p}{(2\pi)^3}\frac{\omega_{\boldsymbol{p}}}{2}. \tag{I5}$$

The one-particle states are supposed to have the form

$$\Psi[\phi]=\int d^3xd^3y\phi(\boldsymbol{x})K(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{x})\Psi_0[\phi]=\phi\cdot K\cdot fe^{-\frac{1}{2}\phi\cdot K\cdot\phi}. \tag{I6}$$

The first two derivatives are

$$\begin{aligned}\frac{\delta\Psi[\phi]}{\delta\phi} &= K\cdot(f-\phi\phi\cdot K\cdot f)e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\\ \frac{\delta^2\Psi[\phi]}{\delta\phi_{\boldsymbol{x}}\delta\phi_{\boldsymbol{y}}} &= \{[-(K\cdot f)_{\boldsymbol{x}}+(K\cdot\phi)_{\boldsymbol{x}}(\phi\cdot K\cdot f)](K\cdot\phi)_{\boldsymbol{y}}-(\phi\cdot K\cdot f)K_{\boldsymbol{xy}}-(K\cdot\phi)_{\boldsymbol{x}}(K\cdot f)_{\boldsymbol{y}}\}e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\\ &= \{-(K\cdot f)_{\boldsymbol{x}}(K\cdot\phi)_{\boldsymbol{y}}-(K\cdot\phi)_{\boldsymbol{x}}(K\cdot f)_{\boldsymbol{y}}+(K\cdot\phi)_{\boldsymbol{x}}(\phi\cdot K\cdot f)(K\cdot\phi)_{\boldsymbol{y}}-(\phi\cdot K\cdot f)K_{\boldsymbol{xy}}\}e^{-\frac{1}{2}\phi\cdot K\cdot\phi}.\end{aligned} \tag{I7}$$

and the functional Schrödinger equation reads

$$\begin{aligned}H\Psi[\phi] &= \frac{1}{2}\{2f\cdot K\cdot K\cdot\phi+[-\phi\cdot K\cdot K\cdot\phi+\mathrm{Tr}K-\phi\cdot\Delta\cdot\phi+m^2\phi\cdot\phi](\phi\cdot K\cdot f)\}e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\\ &= [f\cdot K\cdot K\cdot\phi+E_0(\phi\cdot K\cdot f)]e^{-\frac{1}{2}\phi\cdot K\cdot\phi}\\ &= E_1\phi\cdot K\cdot fe^{-\frac{1}{2}\phi\cdot K\cdot\phi}.\end{aligned} \tag{I8}$$

Our ansatz is a solution if

$$E_1\phi\cdot K\cdot f=f\cdot K\cdot K\cdot\phi+E_0(\phi\cdot K\cdot f) \tag{I9}$$

or

$$K\cdot f=\Delta Ef \tag{I10}$$

where $E_1=E_0+\Delta E$. This eigenvalue condition can be written as

$$\Delta E\tilde{f}^{(\boldsymbol{P})}(\boldsymbol{q})=\omega_{\boldsymbol{q}}\tilde{f}^{(\boldsymbol{P})}(\boldsymbol{q}) \tag{I11}$$

where

$$\tilde{f}^{(\boldsymbol{P})}(\boldsymbol{p})=\int d^3xe^{-i\boldsymbol{xp}}f^{(\boldsymbol{P})}(\boldsymbol{y}). \tag{I12}$$

The solution of Eq. (I11) is $\tilde{f}^{(\boldsymbol{P})}(\boldsymbol{p})=c(\boldsymbol{P})\delta(\boldsymbol{P}-\boldsymbol{p})$ giving $\Delta E=\omega_{\boldsymbol{P}}$.

One can easily construct the creation and destruction operators, too. It fact, it is easy to verify that the operator

$$a(\boldsymbol{p}) = f^{(\boldsymbol{p})*} \cdot K \cdot \phi + f^{(\boldsymbol{p})*} \cdot \frac{\delta}{\delta\phi} = f^{(\boldsymbol{p})*} \cdot \left(\omega_{\boldsymbol{p}}\phi + \frac{\delta}{\delta\phi}\right) \tag{I13}$$

satisfies the commutation relation

$$\begin{aligned}
[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})] &= \left[f^{(\boldsymbol{p})*} \cdot \left(\omega_{\boldsymbol{p}}\phi + \frac{\delta}{\delta\phi}\right), f^{(\boldsymbol{q})} \cdot \left(\omega_{\boldsymbol{q}}\phi - \frac{\delta}{\delta\phi}\right)\right] \\
&= -\left[f^{(\boldsymbol{p})*} \cdot \omega_{\boldsymbol{p}}\phi, f^{(\boldsymbol{q})} \cdot \frac{\delta}{\delta\phi}\right] + \left[f^{(\boldsymbol{p})*} \cdot \frac{\delta}{\delta\phi}, f^{(\boldsymbol{q})} \cdot \omega_{\boldsymbol{q}}\phi\right] \\
&= \omega_{\boldsymbol{p}} f^{(\boldsymbol{p})*} \cdot f^{(\boldsymbol{p})} + \omega_{\boldsymbol{q}} f^{(\boldsymbol{p})*} \cdot f^{(\boldsymbol{q})} = \delta(\boldsymbol{p} - \boldsymbol{q})
\end{aligned} \tag{I14}$$

as soon as the orthogonality condition

$$f^{(\boldsymbol{p})*} \cdot f^{(\boldsymbol{q})} = \frac{\delta(\boldsymbol{p} - \boldsymbol{q})}{2\omega_{\boldsymbol{p}}} \tag{I15}$$

is assured.

The wave-functional of states with several particles can easily be constructed by acting with $a^\dagger(\boldsymbol{p})$ on $\Psi_0[\phi]$.

Appendix J: References

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