Introduction to General Relativity

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Video: `https://drive.google.com/drive/folders/1Z2TKycQr-itlRpqlgjLqCprDoB6fB7rq?usp=sharing`

Contents

I. FIRST ENCOUNTER

A. Equivalence principle

- **Two kinds of mass:** inertial (m_{in}) and gravitational ($U_{gr} = m_{gr}U$))

  Two kinds of acceleration: dynamical ($\mathbf{a}$), inertial ($\mathbf{a}_{in}$)

$$m_{in}(\mathbf{a} - \mathbf{a}_{in}) = -m_{gr}\nabla U_{gr}(\mathbf{x}),$$

  Lorand Eötvös (1906-09): $m_{in} = m_{gr}$

- **Weak Equivalence Principle:** The world line of a small, free falling body is independent of its composition or structure,

$$\mathbf{a} - \mathbf{a}_{in} = -\nabla U_{gr}(\mathbf{x})$$

- **Strong Equivalence Principle:** Weak Equivalence Principle valid in the presence of other, non-gravitational force, $\mathbf{F}_{ext}$,

$$m[\mathbf{a} - \mathbf{a}_{in} + \nabla U_{gr}(\mathbf{x})] = \mathbf{F}_{ext}.$$

$$\nearrow \qquad \nearrow$$

- **Common origin** of inertial and gravitational forces?

  1. $U_{gr}(\mathbf{x}) = gz \Longleftrightarrow z \to z - \frac{g}{2}t^2$: weightless state in falling elevator
  2. Homogeneous gravitational force can be eliminated by suitable coordinates
  3. Inhomogeneous gravitational force $\Longrightarrow$ elmination can be made only locally

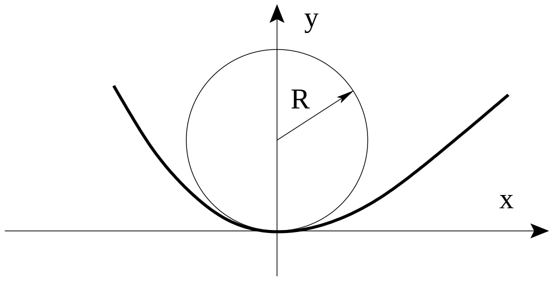
- **Gravitational force $\Leftrightarrow$ local aspects of the geometry of space and time**

  Three levels:

  1. In a small enough region of the space-time it is impossible to detect the presence of a gravitational field and the laws of physics reduce to those of Special Relativity in a suitable coordinate system.
  2. The space-time and the physical laws can be made locally Lorentz-invariant.
  3. Gravity as a gauge theory

- **Equivalence principle is violated** by $\mathcal{O}(\hbar)$ quantum effects. e.g. spin

- **Curvature:**



- The only characteristic classical quantity of the point particle, m, drops out from the gravitational dynamics
- What determines then the particle trajectory? What is the origin of the gravitational force?
- Ever since Riemann: the curvature (radius of the sphere touching the trajectory)

$$\begin{aligned} R^2 &= x^2 + (y-R)^2 = y^2 - 2yR + R^2 + x^2 \\ 0 &= y^2 - 2yR + x^2 \\ y &= \frac{1}{2}(2R \pm \sqrt{4R^2 - 4x^2}) \to R - R \underbrace{\sqrt{1 - \frac{x^2}{R^2}}}_{1-\frac{x^2}{2R^2}+\mathcal{O}\left(\frac{x^4}{R^4}\right)} = \frac{x^2}{2R} + \mathcal{O}\left(x^4\right) \\ R &= \frac{1}{\frac{d^2y}{dx^2}} \begin{cases} > 0 & \text{above} \\ < 0 & \text{below} \end{cases} \end{aligned}$$

- Free-fall:

$$\begin{aligned} x(t) &= vt, \quad y(t) = y_0 - \frac{g}{2}t^2 \quad \Longrightarrow \quad y(x) = y_0 - \frac{g}{2v^2}x^2 \\ R &= -\frac{v^2}{g} \end{aligned}$$

  Initial conditions in the dynamical law ???Z
- Einstein: trajectory in space-time
- Curvature of n-dimensional manifold:
  1. Embedd locally into an $n+1$ dimensional Euclidean space
  2. Linear approximation: tangent space
  3. Quadratic approximation: quadratic form, eigenvalues $\Longrightarrow$ curvature components

B. Comparison with electrodynamics

- **Similarity:**
  - Coulomb force: e and $\tilde{m} = m\sqrt{G}$ are coupling constants

$$\mathbf{F}_C = \mathbf{r}\frac{e_1 e_2}{r^3},$$

- Newton's gravitational law:

$$\mathbf{F}_g = -\mathbf{r}G\frac{m_1 m_2}{r^3} = -\mathbf{r}\frac{\tilde{m}_1\tilde{m}_2}{r^3}$$

- **Differences:**
  - $m > 0$
  - no screening
  - gravity remains long range interaction
  - modifies thermodynamics, e.g. black hole entropy
  - Carriers: $A_\mu(x)$ ($S = 1$) vs. $g_{\mu\nu}(x)$ ($S = 2 \implies m > 0$)
  - Time-dependen EM and gravitational forces are different
- **Unification:** gauge theory

II. CLASSICAL FIELD THEORIES

1. **Why fields?**

   - *Non-relativistic particles:*

$$\begin{aligned} m_a\frac{d^2\mathbf{x}_a(t)}{dt^2} &= \mathbf{F}_a(\mathbf{x}_1(t),\ldots,\mathbf{x}_N(t)) \\ \mathbf{x}_a(t_i) &= \mathbf{x}_{a,i}, \quad \frac{d\mathbf{x}_a(t)}{dt} = \mathbf{v}_{a,i} \end{aligned}$$

   - *Relativistic particles:* $\dot{x}(s) = \frac{dx}{ds}$

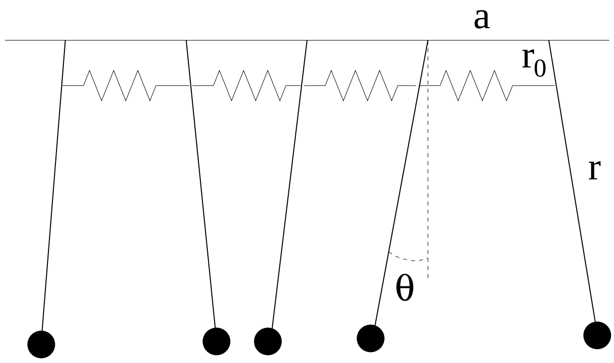
$$\begin{aligned} m_a\ddot{x}^\mu_a(s_a) &= F^\mu_a(x_1(s_1),\ldots,x_N(s_N)) \\ x^\mu_a(s_{a,i}) &= x^\mu_{a,i}, \quad \dot{x}^\mu_a(s_{a,i}) = u^\mu_{a,i}, \quad x^0_a(s_{a,i}) = t_i \end{aligned}$$

   - *Problem:*
     (a) Formal origin:
       i. $\dot{x}^2_a(s) = 1 \implies 0 = \dot{x}_a\ddot{x}_a = \dot{x}_a F_a \implies$ initial coniditions and E.O.M. mixed
       ii. No-go theorem: no covariant function satisfies $0 = \dot{x}_a F_a$
     (b) Physical origin: the instantaneous interaction spreads with infinitely large velocity
   - *Solution:*
     (a) Introduce dynamical degrees of freedom at each space point, $\phi(\mathbf{x})$, a field

(b) Let the particle interact with the field variable at the same space point

(c) Self-interaction of the field $\Longrightarrow$ signal propagates causally ($v_{prop} \leq c$)

2. **Mechanical toy model:** chain of pendulums



A. Variational principle

1. Single point on the real axis

1. **Problem:** Identification $x_{cl} \in \mathbb{R}$ in a reparametrization invariant manner

2. **Solution:**

- Find a function with vanishing derivative at x_{cl} only
- Impose

$$\frac{df(x)}{dx}_{|x=x_{cl}} = 0$$

- Reparametrization invariance: $x \to y$,

$$\frac{df(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{df(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0$$

3. **Variational principle:** infinitesimal variation $x \to x + \delta x$,

$$\begin{aligned} f(x_{cl} + \delta x) &= f(x_{cl}) + \delta f(x_{cl}) \\ &= f(x_{cl}) + \delta x \underbrace{f'(x_{cl})}_{0} + \frac{\delta x^2}{2} f''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \end{aligned}$$

Variation principle: ($\mathcal{O}(\delta x^n) = \mathcal{O}(\delta y^n)$)

$$\delta f(x_{cl}) = \mathcal{O}\left(\delta x^2\right),$$

1. **Problem:** Identification of a trajectory in a coordinate choice independent manner

2. **Variational principle:**

    - $x_{cl}(t)$ satisfying the auxiliary conditions $x_{cl}(t_i) = x_i$ $x_{cl}(t_f) = x_f$ is to be identified

    - Action:

$$S[x] = \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t))$$

$$\nearrow$$

Lagrangian $L(x(t), \dot{x}(t))$

    - Variation: $x(t) \to x(t) + \delta x(t)$, $\delta x(t_i) = \delta x(t_i) = 0$

    - E.O.M.:

$$\begin{aligned}
\delta S[x] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \delta \frac{d}{dt} x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\
&= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial x} + \frac{d}{dt} \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) \\
&\quad - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\
&= \int_{t_i}^{t_f} dt \delta x(t) \left[\frac{\partial L(x(t), \dot{x}(t))}{\partial x} - \frac{d}{dt} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \right] + \underbrace{\delta x(t)}_{0} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right)
\end{aligned}$$

    Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0$$

    - n-dimensional particle:

$$\boxed{\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} = 0}$$

3. **Choice of the Lagrangian:**

$$L = T - U = \frac{m}{2} \dot{\mathbf{x}}^2 - U(\mathbf{x}) \qquad \Longrightarrow \qquad m\ddot{\mathbf{x}} = -\nabla U(\mathbf{x})$$

4. **Generalized momentum:**

$$p = \frac{\partial L}{\partial \dot{x}} \qquad \Longrightarrow \qquad \dot{p} = \frac{\partial L}{\partial x}$$

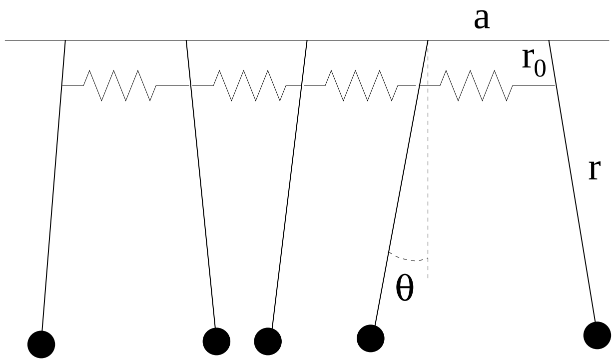
5. **Cyclic coordinate:**

$$\frac{\partial L}{\partial x_{cycl}} = 0$$

6. **Conservation law:** The generalized momentum of a cyclic coordinate is conserved

$$\boxed{\dot{p}_{cycl} = 0}$$

7. **Mechanical toy model for field theory:** chain of pendulums



   (a) *Lagrangian:*

$$L = \sum_n \left[\frac{mr^2}{2}\dot{\theta}_n^2 - \frac{kr_0^2}{2}(\theta_{n+1} - \theta_n)^2 - gr\cos\theta_n\right].$$

   (b) *Variable transformation:* $\theta_n(t) \to \Phi\theta_n(t) = \phi(t, x_n)$,

$$\Phi = r_0\sqrt{ak},\ c = a\tfrac{r_0}{r}\sqrt{\tfrac{k}{m}},\ \lambda = \tfrac{gr}{a}$$

$$L = a\sum_n \left[\frac{1}{2c^2}(\partial_t\phi_n)^2 - \frac{1}{2}\left(\frac{\phi_{n+1} - \phi_n}{a}\right)^2 - \lambda\cos\frac{\phi_n}{\Phi}\right]$$

   (c) *Continuum limit:* $a \to 0$

$$L = \int dx \left[\frac{1}{2c^2}(\partial_t\phi(x))^2 - \frac{1}{2}(\partial_x\phi(x))^2 - \lambda\cos\frac{\phi(x)}{\Phi}\right]$$

   (d) *Sine-Gordon model:*

$$S = \int dtdx \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \lambda\cos\frac{\phi(x)}{\Phi}\right].$$

3. *Scalar field*

1. **Problem:** Identification of the trajectories of an n-component field, $\phi_a(x)$, $a = 1, \ldots, n$

2. **Variation:**

$$\phi(x) \to \phi(x) + \delta\phi(x), \qquad \delta\phi(t_i, \mathbf{x}) = \delta\phi(t_f, \mathbf{x}) = 0.$$

3. **Action:**

$$S[\phi] = \int_V dtd^3xL(\phi, \partial\phi)$$

- Historical convention: dtd^3x rather than $d^4x = dx^0 d^3x$
- d^4x will be used below

$$S[\phi] = \frac{1}{c}\int_V d^4x L(\phi, \partial\phi)$$

4. **E.O.M.:**

$$\begin{aligned}
\delta S &= \int_V dtd^3x \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\delta\partial_\mu\phi\right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dtd^3x \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\partial_\mu\delta\phi\right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dtd^3x \left[\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \partial_\mu\left(\delta\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\delta\phi\right] + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_{\partial V} ds_\mu \delta\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi} + \int_V dtd^3x\delta\phi\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) + \mathcal{O}\left(\delta^2\phi\right)
\end{aligned}$$

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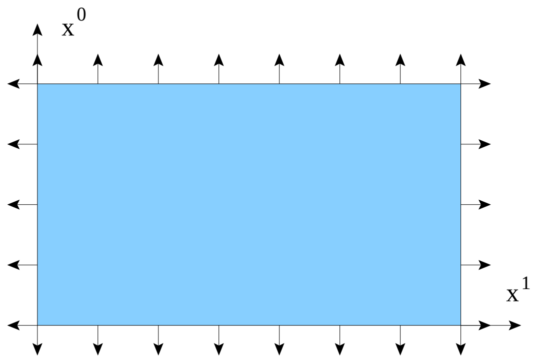
$\int_V dx\partial_\mu j^\mu = \int_{\partial V} ds_\mu j^\mu$ $\qquad\qquad$ 0 (local Lagrangian)



Euler-Lagrange equation:

$$\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\phi} = 0$$

5. **N.B.** Simple rule of covariant tensor calculus:

$$\begin{aligned}
\frac{\partial u_\mu v^\mu}{\partial u_\mu} &= v^\mu, \quad \frac{\partial u_\mu v^\mu}{\partial v^\mu} = u_\mu \\
\frac{\partial s(u_\mu)}{\partial u_\mu} &= s^\mu, \quad \frac{\partial s(v^\mu)}{\partial v^\mu} = s_\mu
\end{aligned}$$

6. N**-component field:** $\phi_a(x)$, $a = 1, \ldots, N$

$$\boxed{\frac{\partial L}{\partial\phi_a} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\phi_a} = 0}$$

7. **Current** associated to the field ϕ:

$$j^\mu_\phi = \frac{\partial L}{\partial\partial\phi_\mu}$$

8. **Conservation law:** Current of a cyclic field variable is conserved

$$\boxed{\partial_\mu j^\mu_{\phi_{cycl}} = 0}$$

9. **Example:** Relativistic scalar field theory, Compton wavelength $\lambda_C = \frac{\hbar}{mc}$, $U(\phi) = \mathcal{O}\left(\phi^3\right)$

$$L = \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{m^2c^2}{2\hbar^2}\phi^2}_{free} - \underbrace{U(\phi)}_{interaction}$$

$\hbar = c = 1$:

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - U(\phi) \qquad \Longrightarrow \qquad (\partial_\mu\partial^\mu + m^2)\phi = -U'(\phi)$$

B. Noether theorem

1. **Theorem:** There is a conserved current for each continuous symmetry

- *Symmetry:* invariance of the E.O.M. which is derived from the action

$$x^\mu \to x'^\mu, \quad \phi_a(x) \to \phi'_a(x) \qquad \Longrightarrow \qquad L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu \Lambda^\mu$$

- *External and internal spaces:*

$$\phi_a(x) : \underbrace{\mathbb{R}^4}_{\text{external space}} \to \underbrace{\mathbb{R}^n}_{\text{internal space}} .$$

- *Continuous symmetry:* $\exists$ infinitesimal symmetry transformations
  - External symmetry: $x^\mu \to x^\mu + \delta x^\mu$, e.g. Poincare group
  - Internal symmetry: $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, e.g. $\phi(x) \to e^{i\alpha}\phi(x)$ for a complex field
- *Conserved current:* $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \partial v j = -\int_{\partial V} d\mathbf{s} \cdot \mathbf{j}$$

2. **Continuous groups:**

- $\{\omega(\alpha)\}$:
  (a) continuous topology (infinitesimal neighborhoods)
  (b) multiplication law:

$$\omega(\alpha)\omega(\beta) = \omega(F(\alpha, \beta))$$

Convention: $\omega(0) = \mathbb{1}$

(c) Examples: translations, $\alpha = x^\mu$, $F(\alpha,\beta) = \alpha+\beta$

(d) Generators: $\tau^a = \frac{\partial\omega(0)}{\partial\alpha^a}$

(e) Infinitesimal group elements:

$$\omega(\epsilon) = 1\!\!1 + \sum_{n=1}^{n} \epsilon^a\tau^a + \mathcal{O}\left(\epsilon^2\right),$$

(f) Lie-algebra:

$$[\tau^a,\tau^b] = \sum_c f^{a,b,c}\tau^c.$$

Structure constants: $f^{a,b,c}$, uniquely determine the multiplication of infinitesimal group elements

(g) Exponential map:

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n(\frac{a}{n}+\mathcal{O}(n^{-2}))} = e^a \\ e^{\sum_a \alpha^a\tau^a} &= \lim_{n\to\infty}\left(1+\sum_a\frac{\alpha^a}{n}\tau^a\right)^n\end{aligned}$$

Any group element can be obtained in such a form in a connected group

$\nearrow$

group elements can be reached in small steps

(h) Example: Matrix groups

TABLE I: Real classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N)$	general linear group	$\det A \neq 0$[a]	N^2	$\{\tau$: real $N\times N$ matrices$\}$
$SL(N)$	special linear group	$\det A = 1$	N^2-1	$\mathrm{tr}\tau = 0$[b]
$O(N)$	orthogonal group	$A^{tr}A = 1\!\!1$[c]	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$
$SO(N)$	special orthogonal group	$A^{tr}A = 1\!\!1$, $\det A = 1$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$, $\mathrm{tr}\tau = 0$

[a] The matrix A is supposed to be an element of the group in question.
[b] $\det(1\!\!1 + \epsilon\tau) = 1 + \epsilon\mathrm{tr}\tau + \mathcal{O}\left(\epsilon^2\right)$
[c] $\det A^{tr}A = (\det A)^2 = 1$ and $\det A = \pm1$.

(i) Ado's theorem: any finite dimensional Lie-algebra is identical with a subspace of the generators of the matrix group $GL(N)$, with sufficiently large N

3. **Linear internal symmetries:**

- *Linear internal transformation:*

$$\delta x^\mu = 0, \quad \delta\phi_a(x) = \epsilon\tau_{ab}\phi_b(x).$$

TABLE II: Complex classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N,C)$	complex general linear group	$\det A \neq 0$	$2N^2$	$\{\tau :$ complex $N\times N$ matrices$\}$
$SL(N,C)$	complex special linear group	$\det A = 1$	$2N^2-2$	$\mathrm{tr}\tau = 0$
$U(N)$	unitary group	$A^\dagger A = 1\!\!1$ [a]	N^2	$\tau^\dagger = -\tau$
$SU(N)$	special unitary group	$A^\dagger A = 1\!\!1$, $\det A = 1$	N^2-1	$\tau^\dagger = -\tau$, $\mathrm{tr}\tau = 0$

[a] $\det A^\dagger A = (\det A)^* \det A = |\det A|^2 = 1$

- *Symmetry:*

$$L(\phi,\partial\phi) = L(\phi+\epsilon\tau\phi,\partial\phi+\epsilon\tau\partial\phi)+\mathcal{O}\left(\epsilon^2\right).$$

- *New field variable:* $\epsilon(x)$, $\phi(x)=\phi_{cl}(x)+\epsilon(x)\tau\phi_{cl}(x)$, $\frac{\delta S[\phi_{cl}]}{\delta\phi}=0$
- *Linearized Lagrangian* for $\epsilon(x)$ ($\epsilon=0$ is a solution!):

$$\begin{aligned} L(\epsilon,\partial\epsilon) &= L(\phi_{cl}+\epsilon\tau\phi,\partial\phi_{cl}+\partial\epsilon\tau\phi+\epsilon\tau\partial\phi) \\ &= \frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\phi}\epsilon\tau+\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\partial_\mu\phi}[\partial_\mu\epsilon\tau\phi+\epsilon\tau\partial_\mu\phi]+\mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

- *Symmetry:* $\frac{\partial L}{\partial\phi}\epsilon\tau\phi+\frac{\partial L}{\partial\partial_\mu\phi}\epsilon\tau\partial_\mu\phi=0 \implies \epsilon$ is a cyclic field $\implies$

$$\begin{aligned} 0 &= \frac{\partial L(\epsilon,\partial\epsilon)}{\partial\epsilon}-\partial_\mu\frac{\partial L(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon} \\ J^\mu_\epsilon &= \frac{\partial L(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon}=\frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\partial_\mu\phi}\tau\phi \\ \partial_\mu J^\mu_\epsilon &= 0 \end{aligned}$$

  (a) Independent conserved current for each independent direction in the symmetry group

  (b) Defined up to a multiplicative constant

- *Examples:*

  (a) n-component real scalar field: ϕ_a, $a=1,\cdots,n$, $G=O(n)$,

$$\begin{aligned} L &= \frac{1}{2}(\partial\phi)^2-V(\phi^2) \\ \delta\phi &= \epsilon^a\tau^a\phi \\ J^a_\mu &= -\partial_\mu\phi\tau^a\phi \end{aligned}$$

  (b) Single complex scalar field: $\phi=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$, $G=U(1)$, $\phi(x)\to e^{i\alpha}\phi(x)$

$$\begin{aligned} L &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1+\frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2-\frac{m^2}{2}(\phi_1^2+\phi_2^2)-V\left(\frac{1}{2}(\phi_1^2+\phi_2^2)\right) \\ &= \partial_\mu\phi^*\partial^\mu\phi+\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^\dagger\phi-V(\phi^\dagger\phi) \end{aligned}$$

  Field variable:

i. $\binom{\phi}{\phi^*}$:

$$\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} : \begin{pmatrix}\phi\\ \phi^*\end{pmatrix} \to \begin{pmatrix}e^{i\alpha}\phi\\ e^{-i\alpha}\phi^*\end{pmatrix}, \quad \delta\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = i\alpha\begin{pmatrix}\phi\\ -\phi^*\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi\\ \phi^*\end{pmatrix}, \quad \tau = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$J = -\frac{\partial L}{\partial\partial_\mu\phi}\tau\phi = -i\left(\frac{\partial L}{\partial\partial_\mu\phi}\phi - \frac{\partial L}{\partial\partial_\mu\phi^*}\phi^*\right) = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) = i\phi^*\overleftrightarrow{\partial}_\mu\phi$$

ii. $\binom{\phi_1}{\phi_2}$:

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} : \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}[\cos\alpha\phi_1 - \sin\alpha\phi_2 + i(\cos\alpha\phi_2 + \sin\alpha\phi_1)]$$

$$\delta\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \alpha\begin{pmatrix}-\phi_2\\ \phi_1\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned}
J &= -\frac{\partial L(\phi,\partial\phi_1)}{\partial\partial_\mu\phi}\tau\phi = -\left(-\frac{\partial L}{\partial\partial_\mu\phi_1}\phi_2 + \frac{\partial L}{\partial\partial_\mu\phi_2}\phi_1\right) = \partial_\mu\phi_1\phi_2 - \partial_\mu\phi_2\phi_1\\
&= \frac{i}{2}[\partial_\mu(\phi_1+i\phi_2)^*(\phi_1+i\phi_2) - (\phi_1+i\phi_2)^*\partial_\mu(\phi_1+i\phi_2)] = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi)
\end{aligned}$$

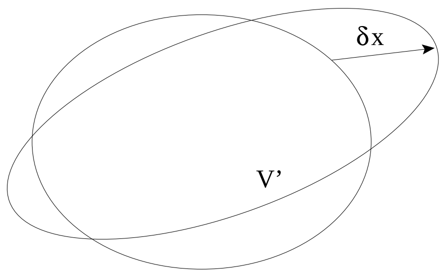
(c) n-component complex scalar field: ϕ_a, $a = 1, \cdots, n$, $G = U(n)$

$$\begin{aligned}
L &= \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi)\\
\delta\phi &= \epsilon^a\tau^a\phi, \quad \delta\phi^\dagger = \epsilon^a(\phi\tau^a)^\dagger = -\epsilon^a\phi^\dagger\tau^a\\
J^a_\mu &= -\partial_\mu\phi^\dagger\tau^a\phi + \partial_\mu\phi(\tau^a)^{\rm tr}\phi^\dagger = -\partial_\mu\phi^\dagger\tau^a\phi + \phi^\dagger\tau^a\partial_\mu\phi = \phi^\dagger\tau^a\overleftrightarrow{\partial}_\mu\phi
\end{aligned}$$

4. **External Symmetries:**

- *Space-time symmetries:*
  (a) Nonrelativistic dynamics: translations of the space-time, rotations of the space and boosts ($4 + 3 + 3 = 10$ dimensional Galilean group)
  (b) Relativistic dynamics: translations and Lorentz transformations of the space-time ($4+3+3 = 10$ dimensional Poincaré group
- *Translations:*
  - Action is rewritten in terms of $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$
  - $x \to x' + \epsilon$, $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$, $\delta\phi(x) = -\epsilon^\mu\partial_\mu\phi(x)$, $S \to S' = S$

$$\begin{aligned}
0 &= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{V'-V} dxL(\phi(x), \partial\phi(x))\\
&= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{\partial V} dS_\nu\epsilon^\nu L(\phi(x), \partial\phi(x))\\
&= -\int_V dx\epsilon^\nu\partial_\nu\phi\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) \quad \leftarrow \quad 0 \quad \text{E.O.M.}\\
&\int_{\partial V} dS_\mu\left[-\epsilon^\nu\partial_\nu\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi} + \epsilon^\mu L(\phi,\partial\phi)\right]
\end{aligned}$$



- Translation in direction ν:

$$0 = \int_{\partial V} dS_\mu \left[-\partial_\nu \phi \frac{\partial L(\phi, \partial\phi)}{\partial \partial_\mu \phi} + g^\mu_\nu L(\phi, \partial\phi) \right]$$

- V is arbitrary: The energy momentum tensor

$$T^{\mu\nu} = \frac{\partial L}{\partial \partial_\nu \phi} \partial^\mu \phi - g^{\mu\nu} L$$

is conserved

$$\partial_\mu T^{\mu\nu} = 0$$

- "Charge" of the translation ϵ^ν: energy momentum

$$P^\mu = \int d^3x T^{0\mu}$$

- Parameterization:

$$\begin{aligned} T^{\mu\nu} &= \begin{pmatrix} \epsilon & c\mathbf{p} \\ \frac{1}{c}\mathbf{s} & \sigma \end{pmatrix} \\ \epsilon &= \text{energy density} \\ \mathbf{p} &= \text{momentum density} \\ \mathbf{S} &= \text{energy flux density} \\ \sigma^{jk} &= \text{momentum flux } p^k \text{ in the direction } j \end{aligned}$$

(c is restored).

- *Lorentz symmetry* $\Longrightarrow$ six conserved currents, angular momentum and generators of Lorentz boosts
- *Conservation of angular momentum* $\Longrightarrow$ $T^{\mu\nu} = T^{\nu\mu}$ for bosonic field theories

III. GAUGE THEORIES

Four fundamental interactions:

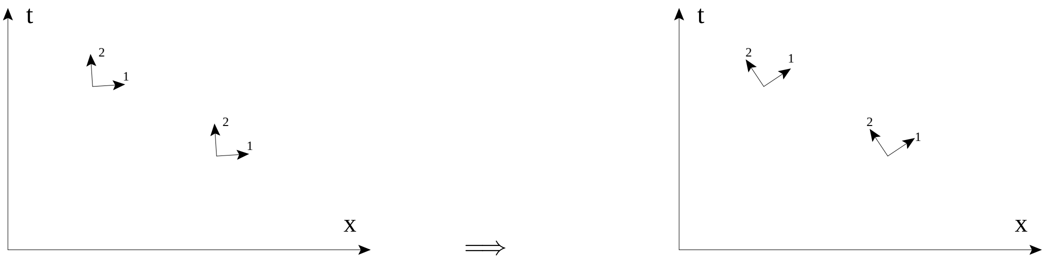
Gravitational, strong, weak, electromagnetic are described by gauge theories

A. Conflict with causality

1. **Global symmetry:** same transformation at each point of the space-time

   - *External symmetry:* Poincaré group, $x \to x^\omega = \Lambda x + a$
   - *Internal symmetry:* $\omega \in SO(N)$ ($\phi^a \in \mathbb{R}^N$) or $\omega \in SU(N)$ ($\phi^a \in \mathbb{C}^N$)

$$\phi(x) \to \phi^\omega(x) = \omega\phi(x)$$



2. **Yang and Mills (1954):** global symmetry ϟ causality

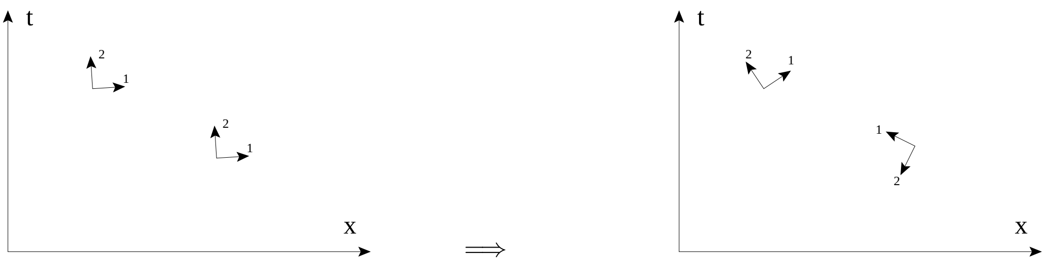
   - Symmetry transformation: change of basis
   - Global symmetry $\implies$ same change of basis everywhere and always
   - Two spatially separated identical experiments compared by using the same basis
   - Use of different bases yield inequivalent results
   - How can an experiment detect that different basis was used at the other?

3. **Local symmetry:**

$$\psi(x) \to \psi^\omega(x) = \omega(x)\psi(x)$$

$$\nearrow$$

homogeneous transformation rule



4. **Gauge (local) symmetry is a redundancy** of the dynamics:

$$\boxed{\text{(classical) observables are gauge invariant}}$$

5. **Gauging:** Lift a global symmetry to a local one

- Globally symmetrical theory: $\phi^\omega(x) = \omega\phi(x)$

$$L(\phi^\omega, \partial\phi^\omega) = L(\phi, \partial\phi), \quad \omega \in G$$

- Example: N component complex scalar field, ϕ_a, $a = 1, \cdots, N$, $G = U(N)$

$$\begin{aligned} L &= (\partial_\mu\phi)^\dagger\partial^\mu\phi - V(\phi^\dagger\phi) \\ L^\omega &= (\partial_\mu\omega\phi)^\dagger\partial^\mu\omega\phi - V((\omega\phi)^\dagger\omega\phi) \\ &= (\omega\partial_\mu\phi)^\dagger\omega\partial^\mu\phi - V((\omega\phi)^\dagger\omega\phi) = L \end{aligned}$$

- Key equation of global symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega\phi(x) = \omega\partial_\mu\phi(x)$$

$$\nearrow$$

homogeneous transformation rule

- Local symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega(x)\phi(x) = \omega(x)\partial_\mu\phi(x) + \underbrace{(\partial_\mu\omega(x))\phi(x)}_{x-\text{dependend basis change}} \neq \omega(x)\partial_\mu\phi(x)$$

$$\nearrow$$

inhomogeneous transformation rule

- Locally invariant Lagrangian: $L(\phi, \partial\phi) \to L(\phi, D\phi)$

$$\nearrow$$

covariant derivative: homogeneous transformation rule

$$\swarrow$$

$$D_\mu\phi(x) \to D^\omega_\mu\phi^\omega(x) = \omega D_\mu\phi(x)$$

B. Covariant derivative

1. **Key equation** of global symmetry:

$$\partial_\mu\phi(x) \to \partial_\mu\phi^\omega(x) = \partial_\mu\omega\phi(x) = \omega\partial_\mu\phi(x)$$

2. **Local symmetry:**

$$\partial_\mu \phi(x) \to \partial_\mu \phi^\omega(x) = \partial_\mu \omega(x)\phi(x) = \omega(x)\partial_\mu \phi(x) + \underbrace{(\partial_\mu \omega(x))\phi(x)}_{x-\text{dependend basis change}}$$

3. **Partial derivative:**

$$\partial_\mu \phi(x) = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon n_\mu) - \phi(x)}{\epsilon}$$

compares ϕ in different bases

4. **Change of basis:**

$$\omega(y \leftarrow x) = \mathbb{1} - \Delta x^\mu A_\mu(x) + \mathcal{O}\left(\Delta^2 x\right)$$

$$\nearrow$$

generators of the gauge group $A_\mu(x) = A^a_\mu(x)\tau^a$

5. **Covariant derivative:**

$$\begin{aligned} D_\mu \phi(x) &= \lim_{\epsilon \to 0} \frac{e^{\epsilon n \cdot A(x+\epsilon n)}\phi(x + \epsilon n_\mu) - \phi(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{[1 + \epsilon n \cdot A(x + \epsilon n)]\phi(x + \epsilon n_\mu) - \phi(x)}{\epsilon} \\ &= (\partial_\mu + A_\mu)\phi(x) \end{aligned}$$

$$\boxed{D_\mu = \partial_\mu + A_\mu}$$

N.B. Action of A_μ depends on the representation of G: $A_\mu = \sum_a A^a_\mu \tau_a$

e.g. $G = SO(3)$, scalar: $A_\mu = 0$, vector: $A_\mu = \sum_j A^j_\mu S_j$

6. **Transformation of** $A_\mu(x)$ during the gauge transformations:

$$\begin{aligned} \psi(x) &\to \psi^\omega(x) = \omega(x)\psi(x) \qquad \Longrightarrow \qquad D_\mu\psi \to D^\omega_\mu \psi^\omega \\ D^\omega_\mu \psi^\omega &= (\partial_\mu + A^\omega_\mu)\psi^\omega, \qquad\qquad \omega D_\mu \psi = \omega(\partial_\mu + A_\mu)\psi \\ \omega(\partial_\mu + A_\mu)\psi &= (\partial_\mu + A^\omega_\mu)\psi^\omega = (\partial_\mu\omega)\psi + \omega\partial_\mu\psi + A^\omega_\mu \omega\psi \\ A^\omega_\mu &= -(\partial_\mu\omega)\omega^{-1} + \omega A_\mu \omega^{-1}. \end{aligned}$$

Useful identity:

$$\begin{aligned} \mathbb{1} &= \omega(x)\omega^{-1}(x) \\ 0 &= (\partial_\mu\omega)\omega^{-1} + \omega\partial_\mu\omega^{-1} \end{aligned}$$

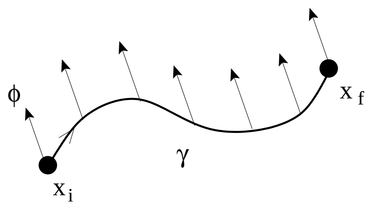
$$\boxed{A_\mu \to A^\omega_\mu = \omega(\partial_\mu + A_\mu)\omega^{-1}}$$

7. **Gauging:**

$$L(\phi, \partial\phi) \to L(\phi, D\phi) = L(\phi, (\partial + A)\phi)$$

C. Parallel transport

1. **Constant vector** $\phi(x)$ along a path $\gamma^\mu : [0,1] \to \mathbb{R}^4$, $\gamma^\mu(0) = x_i^\mu$, $\gamma^\mu(1) = x_f^\mu$



2. **Along the path:**

$$\frac{d\gamma^\mu}{ds} D_\mu \phi(\gamma(s)) = 0$$

3. **End points:**

$$\phi(y) = W_\gamma(y,x)\phi(x)$$

$\nearrow$

path γ dependence?

4. **Evolution equation:**

$$\frac{d\gamma^\mu}{d\tau} D_{y^\mu} W_\gamma(y,x) = 0$$

5. **Infinitesimal path:** $A_\mu(x)$ is constant in the line segment $[x, x+\Delta]$,

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n &= \lim_{n\to\infty} e^{n\ln(1+\frac{a}{n})} = \lim_{n\to\infty} e^{n(\frac{a}{n}+\mathcal{O}(n^{-2}))} = e^a \\
W(x+\Delta x, x) &= \mathbb{1} - \Delta x^\mu A_\mu(x) + \mathcal{O}\left(\Delta x^2\right) \to e^{-\Delta x^\mu A_\mu(x)}
\end{aligned}$$

6. **Homogeneous transformation rule:**

$$\begin{aligned}
\psi^\dagger(y)W_\gamma(y,x)\phi(x) &= \left[\psi^\dagger(y)W_\gamma(y,x)\phi(x)\right]^\omega \\
&= \psi^\dagger(y)\omega^\dagger(y)W_\gamma^\omega(y,x)\omega(x)\phi(x) \\
W_\gamma(y,x) &= \omega^\dagger(y)W_\gamma^\omega(y,x)\omega(x) \\
W_\gamma^\omega(y,x) &= \omega(y)W_\gamma(y,x)\omega^\dagger(x)
\end{aligned}$$

$\nearrow$

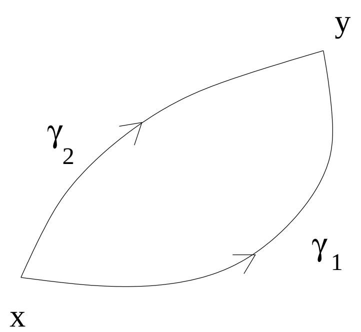
homogeneous transformation rule

7. **Path independence:**

- *Path independence* $\Leftrightarrow W_\gamma(x,x) = \mathbb{1} \Leftrightarrow A_\mu(x)$ *is pure gauge*
- *P*roof:

  (a) Multiplication of paths:

$$\begin{aligned}
\gamma_1(0) &= x,\ \gamma_1(1) = y,\ \gamma_2(0) = y,\ \gamma_2(1) = z \implies \gamma_2\gamma_1(s) = \begin{cases} \gamma_1(2s) & 0 < s < \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} < s < 1 \end{cases} \\
W_{\gamma_2\gamma_1}(z,x) &= W_{\gamma_2}(z,y)W_{\gamma_1}(y,x) \\
\gamma^{-1}(s) &= \gamma(1-s) \\
W_{\gamma^{-1}}(x,y) &= W_\gamma^{-1}(y,x)
\end{aligned}$$



$$W_\gamma(x,x) = W_{\gamma_2^{-1}}(x,y)W_{\gamma_1}(x,y) = W_{\gamma_2}^{-1}(x,y)W_{\gamma_1}(x,y)$$

  (b) Path independence $\implies$ $W(x,x) = \mathbb{1}$

  (c) $W(x,x) = \mathbb{1}$ $\implies$ path independence

  (d) Path independence $\implies$ $A_\mu(x)$ is pure gauge ($A_\mu^\omega = \omega(\partial_\mu + \underbrace{A_\mu}_{0})\omega^{-1} = \omega\partial_\mu\omega^{-1}$)

  Reference point x_0: $\omega(x) = W^{-1}(x,x_0)$

$$W'(x_2,x_1) = W^{-1}(x_2,x_0)W(x_2,x_1)W(x_1,x_0) = W(x_0,x_2)W(x_2,x_0) = \mathbb{1} \quad \implies \quad A_\mu(x) = 0$$

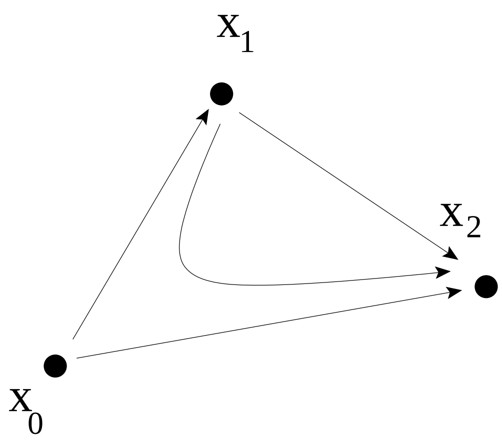
  (e) $A_\mu(x)$ is pure gauge $\implies$ Path independence

$$A_\mu^\omega = \omega\partial_\mu\omega^{-1}, \quad \mathbb{1} = \omega^{-1}(y)W(y,x)\omega(x) \quad \implies \quad W(y,x) = \omega(y)\omega^{-1}(x)$$

D. Field strength tensor

1. **Gauging:** field $\implies$ particle $\implies$ dynamics

$$L(\phi, \partial_\mu\phi) \to L(\phi, (\partial_\mu + A_\mu)\phi) \to L(\phi, D_\mu\phi) + L_A$$



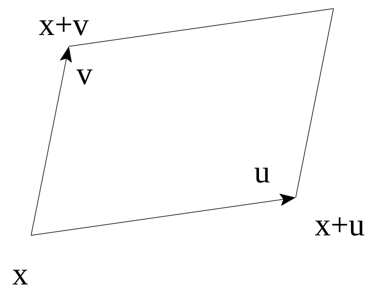
2. L_A:

   (a) quadratic in the velocities, $L_A = \mathcal{O}\left((\partial_0 A_\mu)^2\right)$

   (b) Lorentz invariant

   (c) gauge invariant

3. **Local measure of the path dependence of parallel transport (curvature):**



$$
\begin{aligned}
\delta\phi^a &= -F^a_{\ b\mu\nu} u^\mu v^\nu \phi^b \\
U_\Box &= e^{vA(x)} e^{uA(x+v)} e^{-vA(x+u)} e^{-uA(x)} \\
&\approx \left(\mathbb{1} + vA(x) + \frac{1}{2}[vA(x)]^2\right)\left(\mathbb{1} + uA(x+v) + \frac{1}{2}[uA(x+v)]^2\right) \\
&\quad \times \left(\mathbb{1} - vA(x+u) + \frac{1}{2}[vA(x+u)]^2\right)\left(\mathbb{1} - uA(x) + \frac{1}{2}[uA(x)]^2\right) \\
&\approx \mathbb{1} + vA(x) + uA(x+v) - vA(x+u) - uA(x) \\
&\quad + \frac{1}{2}[vA(x)]^2 + \frac{1}{2}[uA(x+v)]^2 + \frac{1}{2}[vA(x+u)]^2 + \frac{1}{2}[uA(x)]^2 \\
&\quad + vA(x)uA(x+v) - vA(x)vA(x+u) - vA(x)uA(x) - uA(x+v)vA(x+u) - uA(x+v)uA(x) \\
&\quad + vA(x+u)uA(x) \\
&\approx \mathbb{1} + (v\partial)uA - (u\partial)vA + \frac{1}{2}(vA)^2 + \frac{1}{2}(uA)^2 + \frac{1}{2}(vA)^2 + \frac{1}{2}(uA)^2 \\
&\quad + vAuA - vAvA - vAuA - uAvA - uAuA + vAuA \\
&\approx \mathbb{1} + (v\partial)uA - (u\partial)vA - (uA)(vA) + (vA)(uA) \\
&= \mathbb{1} - u^\mu v^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])
\end{aligned}
$$

Alternative form:

$$\begin{aligned}
[D_\mu, D_\nu]\phi &= [\partial_\mu + A_\mu, \partial_\nu + A_\nu]\phi \\
&= \underbrace{[\partial_\mu, \partial_\nu]}_{0}\phi + \underbrace{[\partial_\mu, A_\nu]\phi}_{\partial_\mu A_\nu\phi - A_\nu\partial_\mu\phi = (\partial_\mu A_\nu)\phi} + \underbrace{[A_\mu, \partial_\nu]\phi}_{-[\partial_\nu, A_\mu]\phi} + [A_\mu, A_\nu]\phi \\
&= (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\phi
\end{aligned}$$

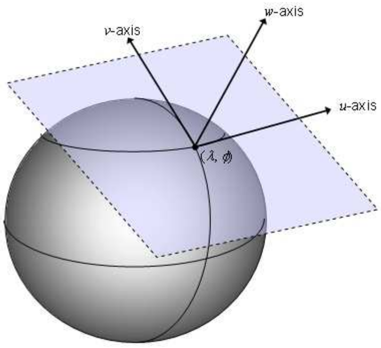
$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = [D_\mu, D_\nu]}$$

Generator valued field:

$$\begin{aligned}
F_{\mu\nu} &= F^a_{\mu\nu}\frac{\tau^a}{2i} \\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu
\end{aligned}$$

4. **Relation to curvature:** $S_2 = \{\mathbf{n}|\mathbf{n}^2 = 1\}$

- *Tangent space:* $T_{\mathbf{n}} = \{\dot{\mathbf{x}}(s)_{|s=0}|\mathbf{x}(0) = \mathbf{n}\}$ (velocity space)



$$L = \mathbf{n} \otimes \mathbf{n}, \quad T = \mathbb{1} - L \quad \leftarrow \quad \text{projection onto the tangent space } T_{\mathbf{n}} = R^2$$

- *Parallel transport:*

$$\dot{\gamma}^\mu D_\mu u^\phi = 0 \iff \frac{d}{ds}Tu = 0$$

- *Parallel transport along a spherical triangle:*

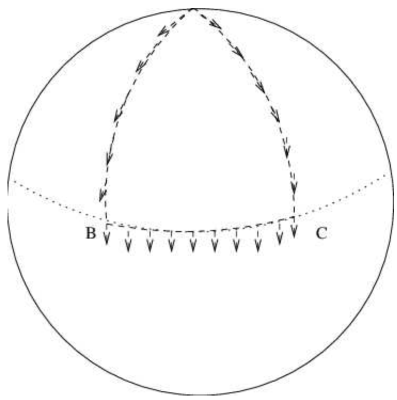
5. **Transformation rule under gauge transformations:**

$$\begin{aligned}
W_\square &= \mathbb{1} - u^\mu v^\nu F_{\mu\nu}(x) \to \omega(x)[\mathbb{1} - u^\mu v^\nu F_{\mu\nu}(x)]\omega^{-1}(x) \\
F_{\mu\nu}(x) &\to \omega(x)F_{\mu\nu}(x)\omega^{-1}(x)
\end{aligned}$$

6. **Pure gauge** $(A_\mu = \omega\partial_\mu\omega^{-1}) \Leftrightarrow F = 0$ (connected space-time)

7. **Yang-Mills action:** (unique in space-time with no boundary)

$$L_{YM} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu})^2 = -\frac{1}{4g^2}(F^a_{\mu\nu})^2$$



E. Classical electrodynamics

1. **Action:**

$$S = -cm\int ds - \frac{e}{c}\int ds\dot{x}^\mu(x(s))A_\mu(x(s)) - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$= -cm\int ds - \frac{e}{c^2}\int d^4xj^\mu(x)A_\mu(x) - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x)$$

- *Electric current:*

$$j^\mu(x) = c\sum_a\int ds\delta(x - x_a(s))\dot{x}^\mu$$
$$= c\sum_a\int ds\delta(\mathbf{x} - \mathbf{x}_a(s))\delta(x^0 - x^0_a(s))\dot{x}^\mu$$
$$= c\sum_a\delta(\mathbf{x} - \mathbf{x}_a(s))\frac{\dot{x}^\mu}{|\dot{x}^0|}$$
$$= \underbrace{\sum_a\delta(\mathbf{x} - \mathbf{x}_a(s))}_{\rho(\mathbf{x})}\frac{dx^\mu}{dt}$$
$$= (c\rho, \mathbf{j}) = (c\rho, \rho\mathbf{v}) = \rho\frac{ds}{dt}\dot{x}^\mu$$

- Current conservation:

$$\partial_\mu j^\mu = \partial_t\rho + \nabla\cdot\mathbf{j}$$
$$= \sum_a[-\mathbf{v}_a(t)\nabla\delta(\mathbf{x} - \mathbf{x}_a(t)) + \nabla\delta(\mathbf{x} - \mathbf{x}_a(t))\mathbf{v}_a(t)] = 0$$

- *Gauge invariance:* $A_\mu \to A_\mu + \partial_\mu\alpha$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha = F_{\mu\nu}$

$$S = -cm\int ds - \frac{e}{c^2}\int d^4xj^\mu(x)A_\mu(x) - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$\to -cm\int ds - \frac{e}{c^2}\int d^4xj^\mu(x)[A_\mu(x) + \partial_\mu\alpha(x)] - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x)$$
$$= -cm\int ds - \frac{e}{c^2}\int d^4x[j^\mu(x)A_\mu(x) - \partial_\mu j^\mu(x)\alpha(x)] - \frac{1}{16\pi c}\int d^4xF_{\mu\nu}(x)F^{\mu\nu}(x) = S$$

2. **Maxwell's equation:**

$$\begin{aligned}
S_A &= -\frac{e}{c^2}\int d^4x j^\mu A_\mu - \frac{1}{16\pi c}\int d^4x(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
c\frac{\delta}{\delta A_\nu}: \quad \frac{e}{c}j^\nu &= \frac{1}{4\pi}\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \frac{1}{4\pi}\partial_\mu F^{\mu\nu}
\end{aligned}$$

3. **Mechanical E.O.M.:** $x(s) \to x(\tau)$ to avoid $\dot{x}^2(s) = 1$

$$\begin{aligned}
S_{ch} &= -\int d\tau \left[mc\sqrt{\dot{x}^\mu(\tau)g_{\mu\nu}\dot{x}^\nu(\tau)} + \frac{e}{c}\dot{x}^\mu(\tau)A_\mu(x(\tau))\right] \\
\frac{\delta}{\delta x^\mu}: \quad 0 &= -\frac{e}{c}\dot{x}^\nu(\tau)\partial_\mu A_\nu(x(\tau)) - \frac{d}{d\tau}\left[-mc\frac{\dot{x}^\mu(\tau)}{\sqrt{\dot{x}^\mu(\tau)g_{\mu\nu}\dot{x}^\nu(\tau)}} - \frac{e}{c}A_\mu(x(\tau))\right] \\
&= mc\frac{\ddot{x}^\mu(\tau)}{\sqrt{\dot{x}^2(\tau)}} - \frac{e}{c}\dot{x}^\nu(\tau)[\partial_\mu A_\nu(x(\tau)) - \partial_\nu A_\mu(x(\tau))] + mc\frac{\dot{x}^\mu(\tau)}{[\dot{x}^2(\tau)]^{3/2}}\ddot{x}^\mu(\tau)\dot{x}_\mu(\tau) \\
\tau \to s: \quad mc\ddot{x}^\mu(s) &= \frac{e}{c}F_{\mu\nu}\dot{x}^\nu(s)
\end{aligned}$$

IV. GRAVITY

A. Classical field theory on curved space-time

1. **Fields over the space-time:** $\phi(x) : E \to I$

$$\nearrow \qquad\qquad \nwarrow$$

"where" "what" happens

2. **External space (where):**

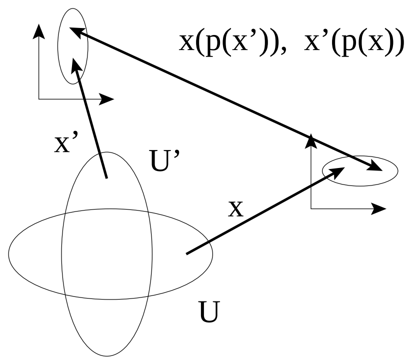
- meter rods, clocks $\Longrightarrow$ coordintes
- E.O.M. $\Longrightarrow$ coordinate singularities in fields
- To unfold the coordinate singularities $\Longrightarrow$ local coordinate patches
  - (a) Maps: $\forall\, p \in E\ \exists$ open set $M_p \subset E$, $p \in M_p$
  - (b) Coordinates: $\forall\, M\ \exists\, x_M : M \to V \subset \mathcal{R}^d$, $x_M^{-1} = p(x) : V \to M$, $d = 4$ for gravity
  - (c) Coordinate transformations: If $p \in M, M'\ \exists\, x(p)$, $x'(p)$, $x'(x) = x'(p(x))$ is infinitely many times differentiable.

3. **Internal space (what):**

(a) *Non-gravitational events:* $p \to \mathcal{R}^{d_{NG}}$

(b) *Gravitational events:* directions, vectors, tensors, tangent space $p \to T_p = \mathcal{R}^4$

T_p : {equivalence classes of world lines$|x(s) \sim x'(s) \leftrightarrow x(0) = x'(0) = p, \dot{x}(0) = \dot{x}'(0)$}



(c) *External space* $\longleftrightarrow$ *Internal space:* movement of free point particle from p: $x^\mu(p,u,s)$

i. initial conds.: $x^\mu(p,u,0) = x^\mu(p)$ $\dot{x}^\mu(p,u,0) = u^\mu$

ii. $x^\mu(p,u,s)$: parallel transport of the velocity

iii. $X^\mu(u) = x^\mu(p,u,1) : T_p \to M_p$

iv. $\exists\, W \subset T_p$, $X^\mu_{|W}$ is invertible, $\exists\, U \subset M_p$, $X^{-1}_{|U} : U \to T_p$

4. **Standard map:**

(a) *Standard coordinates:* $X^\mu(p)$

(b) *Standard basis in* T_p*:*

$$e_\mu = \frac{\partial X^{-1}(p(x))}{\partial x^\mu}, \qquad x = X: \quad e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{d-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

5. **Change of coordinates:** $x \to x' = x'(x)$, change of basis in T_p

(a) Covariant basis:

$$e_\mu = \frac{\partial X^{-1}(p(x))}{\partial x^\mu} = \frac{\partial X'^{-1}(p(x'))}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = e'_\nu \frac{\partial x'^\nu}{\partial x^\mu}$$

(b) Contravariant basis: $e^\mu f_\mu$ remains invariant

$$\begin{aligned} e^\mu &= \frac{\partial x^\mu}{\partial x'^\nu} e'^\nu \\ e^\mu f_\mu &= \frac{\partial x^\mu}{\partial x'^\nu} e'^\nu f'_\rho \frac{\partial x'^\rho}{\partial x^\mu} = \frac{\partial x'^\rho}{\partial x'^\nu} e'^\nu f'_\rho = e'^\nu f'_\nu \end{aligned}$$

(c) Formal analogy, without physical content:

$$\partial_\mu = \partial'_\nu \frac{\partial x'^\nu}{\partial x^\mu}, \quad \delta x^\mu = \frac{\partial x^\mu}{\partial x'^\nu}\delta x'^\nu$$

$$\boxed{u_\mu = u'_\nu \frac{\partial x'^\nu}{\partial x^\mu}, \quad u^\mu = \frac{\partial x^\mu}{\partial x'^\nu} u'^\nu}$$

B. Geometrical structure inferred from observations

1. **Metric structure:** measuring distances and time

$$ds^2(x) = dx^\mu g_{\mu\nu}(x) dx^\nu$$

Equivalence principle: signature $+, -, -, -$

2. **Affine structure:** Locally (equivalence principle) conserved vectors of free particles (velocity, angular momentum)

$$\dot{\gamma}^\mu D_\mu \phi = \dot{\gamma}^\mu (\partial_\mu + \Gamma_\mu)\phi = 0$$

3. **Torsion:** $\mathcal{O}(\hbar)$ spin effect, ignored

C. Gauge group

1. **Internal space of gravity:**
   - *What happens?* ballistic motion, $I = \{\text{particles pass the observer}\}$
   - *Gauge theory* with $I_p = T_p$

$$\boxed{\text{Gravity: } E \text{ and } I \text{ are related}}$$

2. **Space-time diffeomorphism:**
   - *Special relativity:*
     - Translation invariance $\implies$ relative coordinates
     - Boost invariance $\implies$ relative velocity
   - *General relativity:*
     - Space-time diffeomorphism invariance $\implies$ relative higher order derivatives

3. **Space-time diffeomorphism as gauge symmetry:**
   - *Coordinate tetrad vectors:* $e_\mu(x)$ instead of the coordinates $x^\mu(p)$
     - Holonomic tetrads:

$$\boxed{\{e_\mu(x)\} \text{ is compatible with a coordinate system} \iff \partial_\mu e_\nu = \partial_\nu e_\mu}$$

Necessary: $\partial_\mu e_\nu = \partial_\mu \partial_\nu X^{-1}(p(\bar{x})) = \partial_\nu \partial_\mu X^{-1}(p(\bar{x})) = \partial_\nu e_\mu$

Sufficient: local existence and unicity of integral curves $\partial_\mu X^{-1}(p(\bar{x})) = e_\mu(\bar{x})$

– Diffeomorphism preserves holonomy

$$e'_\mu = e_\nu \frac{\partial x^\nu}{\partial x'^\mu} \quad \Longrightarrow \quad \partial'_\kappa e'_\mu = e_\nu \frac{\partial^2 x^\nu}{\partial x'^\mu \partial x'^\kappa} = \partial'_\mu e'_\kappa$$

- *Coordinate transformation:* $x^\mu \to x'^\mu(x) \quad \Longleftrightarrow \quad e'_\mu = e_\nu \frac{\partial x^\nu}{\partial x'^\mu}$
- *Gauge group:* $G = GL(4)$
  - Gauge transformation:

$$e_\mu(x) \to e'_\mu(x) = \omega_\mu{}^\nu(x) e_\nu(x), \quad \omega_\mu{}^\nu = \frac{\partial x^\nu}{\partial x'^\mu}, \quad \det[\omega] \neq 0$$

  - Gauge field: $(A_\mu)^\nu{}_\rho = \Gamma^\nu_{\rho\mu}$ affine connection
- *Internal space:* Tangent space T_x (relating external and internal spaces)
- *Non-gauge field:* metric tensor, $g_{\mu\nu}$

4. **Poincaré group as gauge symmetry:**

- *Equivalence Principle:* local Lorentz frames Which one to use? A freedom of the Poincaré group
- *Two bases in the internal space:* $\dot{x}^\mu \in T_x^{world}$, $\dot{\xi}^a \in T_p^{Lorentz}$

$$\dot{x}^\mu = \dot{\xi}^a e^\mu_a, \quad \dot{x}^a = \dot{\xi}^\mu e^a_\mu$$

- *Local Lorentz transformations:*

$$e^a(x) \to e'^a(x) = \omega^a{}_b(x) e^b(x).$$

- *Local translations:* space-time diffeomorphism

$$x^\mu(x) \to x'^\mu(x) = x^\mu(x) + \delta x^\mu(x), \quad \xi^a(x) \to \xi'^a(x) = \xi^a(x) + \delta\xi^a(x), \quad \delta\xi^a(x) = e^a_\mu \delta x^\mu(x).$$

- *Fermions:* need Lorentz basis

D. Gauge theory of diffeomorphism

1. **Covariant derivative:**

- *Gauge field:* affin connection, $(\Gamma_\rho)^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}$

(a) Covariant field:

$$\begin{aligned} D_\nu v^\mu &= \partial_\nu v^\mu + \Gamma^\mu{}_{\rho\nu} v^\rho \\ (D_\nu v)^\mu &= (\partial_\nu v + \Gamma_\nu v)^\mu \end{aligned}$$

(b) *Leibniz's rule:*

$$
\begin{aligned}
D(uv) &= (Du)v + u(Dv) \\
\text{e.g. } D_\mu(fu^\mu) &= (D_\mu f)u^\mu + (fD_\mu u^\mu) = \partial_\mu f u^\mu + fD_\mu u^\mu
\end{aligned}
$$

↑

scalar representation of $GL(4)$

(c) *Contravariant field:*

$$
\begin{aligned}
D_\mu(u^\nu v_\nu) &= (\partial_\mu u + \Gamma_\mu u)^\nu v_\nu + u^\nu(\partial_\mu v - v\Gamma_\mu)_\nu = \partial_\mu(u^\nu v_\nu) \\
(D_\nu v)_\mu &= (\partial_\nu v - v\Gamma_\nu)_\mu
\end{aligned}
$$

(d) *Mixed tensor field:* acting on each index, eg.

$$D_\nu v^\mu_\rho = \partial_\nu v^\mu_\rho + \Gamma^\mu_{\kappa\nu} v^\kappa_\rho - v^\mu_\kappa \Gamma^\kappa_{\rho\nu}.$$

It is easy to check that such an extension reproduces Leibniz rule,

$$D_\mu(u^\rho v^\sigma) = (D_\mu u^\rho)v^\sigma + u^\rho D_\mu(v^\sigma).$$

- **Gauge transformation:**

(a) $A_\mu \to A'_\mu = \omega(\partial_\mu + A_\mu)\omega^{-1}$

$$
\begin{aligned}
\delta x^\mu \to \delta x'^\mu &= \omega^\mu{}_\kappa \delta x^\kappa, \quad \omega^\mu{}_\kappa = \frac{\partial x'^\mu}{\partial x^\kappa}, \quad (\omega^{-1})^\mu{}_\kappa = \frac{\partial x^\mu}{\partial x'^\kappa} \\
\Gamma^\mu{}_{\nu\rho} \to \Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho}\omega^\mu{}_\kappa(\delta^\kappa_\lambda \partial_\sigma + \Gamma^\kappa{}_{\lambda\sigma})(\omega^{-1})^\lambda{}_\nu
\end{aligned}
$$

↗

transformation of a covariant index Γ_ρ: $\partial_\rho \to \partial'_\rho = \frac{\partial x^\sigma}{\partial x'^\rho}\partial_\sigma$

$$
\begin{aligned}
\Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\delta^\kappa_\lambda \frac{\partial}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\
&= \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial x^\kappa}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\
&= \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial^2 x^\kappa}{\partial x'^\nu \partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu},
\end{aligned}
$$

(b) $\omega\omega^{-1} = \mathbb{1}$, $\partial_\mu\omega\omega^{-1} + \omega\partial_\mu\omega^{-1} = 0$, $A'_\mu = -\partial_\mu\omega\omega^{-1} + \omega A_\mu \omega^{-1}$

$$
\begin{aligned}
\Gamma'^\mu{}_{\nu\rho} &= \frac{\partial x^\sigma}{\partial x'^\rho}\left[-\partial_\sigma \omega^\mu{}_\kappa \delta^\kappa_\lambda (\omega^{-1})^\lambda{}_\nu + \omega^\mu{}_\kappa \Gamma^\kappa{}_{\lambda\sigma}(\omega^{-1})^\lambda{}_\nu\right] \\
&= -\frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial}{\partial x^\sigma}\frac{\partial x'^\mu}{\partial x^\kappa}\delta^\kappa_\lambda \frac{\partial x^\lambda}{\partial x'^\nu} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\
&= -\frac{\partial x^\kappa}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\kappa \partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu} \\
&= -\frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x'^\rho} + \frac{\partial x^\sigma}{\partial x'^\rho}\frac{\partial x'^\mu}{\partial x^\kappa}\Gamma^\kappa{}_{\lambda\sigma}\frac{\partial x^\lambda}{\partial x'^\nu}
\end{aligned}
$$

(c) Symmetric part: inhomogeneous transformation rules, not a tensor

(d) Antisymmetric part: homogeneous transformation rules, torsion tensor,

$$S^{\rho}_{\nu\mu} = \frac{1}{2}(\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu})$$

- **Harmonic gauge:** (harmonic coordinate system)

$$\Gamma^{\rho} = g^{\mu\nu}\Gamma^{\rho}{}_{\mu\nu} = 0$$

(a) *Origin:*

$$\Box x^{\mu} = g^{\rho\nu} D_{\rho} D_{\nu} x^{\mu} = g^{\nu\rho} D_{\rho}\partial_{\nu} x^{\mu} = g^{\nu\rho}(\partial_{\rho}\partial_{\nu} x^{\mu} - \Gamma^{\kappa}_{\nu\rho}\partial_{\kappa} x^{\mu}) = -g^{\nu\rho}\Gamma^{\mu}_{\nu\rho} = -\Gamma^{\mu}$$

$$\uparrow$$

x^{μ} is a scalar field ($\notin T_p$)

(b) *Gauge transformation:*

$$\begin{aligned}
\Gamma^{\mu} \to g'^{\nu\rho}\Gamma'^{\mu}{}_{\nu\rho} &= g^{\tau\sigma}\frac{\partial x'^{\nu}}{\partial x^{\tau}}\frac{\partial x'^{\rho}}{\partial x^{\sigma}}\left(-\frac{\partial^2 x'^{\mu}}{\partial x'^{\nu}\partial x'^{\rho}} + \frac{\partial x^{\sigma'}}{\partial x'^{\rho}}\frac{\partial x'^{\mu}}{\partial x^{\kappa}}\Gamma^{\kappa}{}_{\lambda\sigma'}\frac{\partial x^{\lambda}}{\partial x'^{\nu}}\right) \\
&= -g^{\tau\sigma}\frac{\partial^2 x'^{\mu}}{\partial x^{\tau}\partial x^{\sigma}} + g^{\tau\sigma}\frac{\partial x'^{\mu}}{\partial x^{\kappa}}\Gamma^{\kappa}{}_{\tau\sigma} \\
&= -g^{\tau\sigma}\frac{\partial^2 x'^{\mu}}{\partial x^{\tau}\partial x^{\sigma}} + \frac{\partial x^{\mu}}{\partial x'^{\kappa}}\Gamma^{\kappa} \\
\Longrightarrow \quad g^{\tau\sigma}\frac{\partial^2 x'^{\mu}}{\partial x^{\tau}\partial x^{\sigma}} &= \frac{\partial x^{\mu}}{\partial x'^{\kappa}}\Gamma^{\kappa}
\end{aligned}$$

(c) *Always can be reached:* by solving this equation for $x'^{\mu}(x)$ for given Γ^{κ}

- **Equivalence Principle:**

(a) *Gauge theory:* at x_0

$$\begin{aligned}
A^{\omega}_{\mu}(x) &= \omega(x)(\partial_{\mu} + A_{\mu}(x))\omega^{-1}(x), \qquad \omega(x) = e^{(x^{\mu}-x^{\mu}_0)A_{\mu}(x_0)} \\
&= [\mathbb{1} + (x^{\mu} - x^{\mu}_0)A_{\mu}(x_0)](\partial_{\mu} + A_{\mu}(x))[\mathbb{1} - (x^{\mu} - x^{\mu}_0)A_{\mu}(x_0)] + \mathcal{O}\left((x - x_0)^2\right) \\
&= \mathcal{O}\left(x - x_0\right)
\end{aligned}$$

(b) *Gravity:* $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$ (global rotations and rescaling $\Longrightarrow$ Minkowski metric)

new coordinates: $x \to x'$

$$\begin{aligned}
x^{\mu} - x^{\mu}_0 &= x'^{\mu} - x'^{\mu}_0 - \frac{1}{2}\Gamma^{\mu}{}_{\nu\rho}(x_0)(x'^{\nu} - x'^{\nu}_0)(x'^{\rho} - x'^{\rho}_0) \\
\frac{\partial x^{\kappa}}{\partial x'^{\mu}} &= \delta^{\kappa}_{\mu} - \frac{1}{2}\Gamma^{\kappa}{}_{\mu\rho}(x_0)(x'^{\rho} - x'^{\rho}_0) - \frac{1}{2}\Gamma^{\kappa}{}_{\nu\mu}(x_0)(x'^{\nu} - x'^{\nu}_0) \\
\frac{\partial x^{\kappa}}{\partial x'^{\mu}}_{|x'=x'_0} &= \delta^{\kappa}_{\mu}, \qquad \frac{\partial^2 x^{\kappa}}{\partial x'^{\nu}\partial x'^{\rho}}_{|x'=x'_0} = -\Gamma^{\kappa}{}_{\nu\rho}(x_0)
\end{aligned}$$

$$\begin{aligned}
\Gamma'^{\mu}_{\nu\rho}(x') &= \frac{\partial x'^{\mu}}{\partial x^{\kappa}}\frac{\partial^2 x^{\kappa}}{\partial x'^{\nu}\partial x'^{\rho}} + \frac{\partial x^{\sigma}}{\partial x'^{\rho}}\frac{\partial x'^{\mu}}{\partial x^{\kappa}}\Gamma^{\kappa}{}_{\lambda\sigma}\frac{\partial x^{\lambda}}{\partial x'^{\nu}} \\
&= -\frac{\partial x'^{\mu}}{\partial x^{\kappa}}\Gamma^{\kappa}{}_{\nu\rho}(x_0) + \frac{\partial x^{\sigma}}{\partial x'^{\rho}}\frac{\partial x'^{\mu}}{\partial x^{\kappa}}\Gamma^{\kappa}{}_{\lambda\sigma}(x)\frac{\partial x^{\lambda}}{\partial x'^{\nu}} \\
\Gamma'^{\mu}_{\nu\rho}(x'_0) &= -\Gamma^{\mu}{}_{\nu\rho}(x_0) + \Gamma^{\mu}{}_{\nu\rho}(x_0) = 0
\end{aligned}$$

2. **Field strength tensor:**

- *Gauge theory:*

$$F_{\mu\nu} = [D_\mu, D_\nu] = [\partial_\mu + \Gamma_\mu, \partial_\nu + \Gamma_\nu] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = -F_{\nu\mu}$$

- *Curvature tensor:*

$$\boxed{R^\mu{}_{\nu\rho\sigma} = (F_{\rho\sigma})^\mu{}_\nu = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho}}$$

- *Useful identity for symmetrical connection:*

$$R^\rho{}_{\kappa\mu\nu} + R^\rho{}_{\mu\nu\kappa} + R^\rho{}_{\nu\kappa\mu} = 0.$$

- *Bianchi identity:*
  - Commutators:

$$0 = [A, [B, C]] + [B, [C, A]] + [C, [A, B]],$$

  - Covariant derivative:

$$\begin{aligned} 0 &= [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] \\ &= [D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] \\ &= D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu}, \end{aligned}$$

  - Curvature:

$$0 = D_\mu R^\sigma{}_{\kappa\nu\rho} + D_\nu R^\sigma{}_{\kappa\rho\mu} + D_\rho R^\sigma{}_{\kappa\mu\nu},$$

- *Ricci tensor:*
  - Gauge theory: internal space external space directions are independent

$$L_A = -\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2g^2} F^a{}_{b\mu\nu} F^b{}_a{}^{\mu\nu}$$

  - Gravity: internal space external space directions are related, simpler contraction possibility

$$\begin{aligned} R^\mu{}_{\nu\rho\sigma} &= (F_{\rho\sigma})^\mu{}_\nu = \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho} \\ R_{\nu\sigma} &= R^\rho{}_{\nu\rho\sigma} \\ &= \partial_\rho \Gamma^\rho{}_{\nu\sigma} - \partial_\sigma \Gamma^\rho{}_{\nu\rho} + \Gamma^\rho{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\rho{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho} \\ &= -R^\rho{}_{\nu\sigma\rho} \end{aligned}$$

– An alternative:

$$\begin{aligned} R'_{\rho\sigma} &= R^{\mu}{}_{\mu\rho\sigma} \\ &= \partial_\rho \Gamma^{\mu}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\mu\rho} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\mu\rho} \\ &= \partial_\rho \Gamma^{\mu}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\mu\rho} \\ &= -R'_{\sigma\rho} \end{aligned}$$

- *Lagrangian for gravity:* $R = g^{\mu\nu} R_{\mu\nu}$, $R' = g^{\mu\nu} R'_{\mu\nu} = 0$ (only if $T = 0$)

E. Metric admissibility

1. **Geometric relation between the affine and the metric structure**

   $T = 0 \implies \{{}^{\rho}_{\mu\nu}\} = \frac{1}{2}(\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\mu\nu}) = \Gamma^{\rho}_{\mu\nu}$ (Christoffel symbol)

2. **Parallel transport of two vector fields:** $u^\mu(x)$, $v^\mu(x)$ along $\gamma(s)$

   - *Parallel transport:*

   $$\dot\gamma(s) D_\mu u = \dot\gamma(s) D_\mu v = 0.$$

   - *Scalar product preserved:*

   $$\begin{aligned} \dot\gamma(s) D_\mu u^\nu g_{\nu\rho} v^\rho &= (\dot\gamma(s) D_\mu u^\nu) g_{\nu\rho} v^\rho + u^\nu (\dot\gamma(s) D_\mu g_{\nu\rho}) v^\rho + u^\nu g_{\nu\rho} (\dot\gamma(s) D_\mu v^\rho) \\ &= u^\nu v^\rho \dot\gamma(s) D_\mu g_{\nu\rho} = 0 \qquad \Longrightarrow \qquad D_\mu g_{\nu\rho} = 0 \end{aligned}$$

3. **Metric admissibility:**

   $$Dg = 0$$

4. **Solution for the Chistoffel symbols:** $\Gamma_{\rho\mu\nu} = g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu}$

   $$\begin{aligned} D_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - g_{\kappa\nu}\Gamma^{\kappa}_{\mu\rho} - g_{\mu\kappa}\Gamma^{\kappa}_{\nu\rho} \\ D_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - g_{\kappa\rho}\Gamma^{\kappa}_{\nu\mu} - g_{\nu\kappa}\Gamma^{\kappa}_{\rho\mu} \\ D_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - g_{\kappa\mu}\Gamma^{\kappa}_{\rho\nu} - g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu} \\ 0 = D_\mu g_{\nu\rho} + D_\nu g_{\rho\mu} - D_\rho g_{\mu\nu} &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \underbrace{- g_{\kappa\rho}\Gamma^{\kappa}_{\nu\mu} - g_{\rho\kappa}\Gamma^{\kappa}_{\mu\nu}}_{-\Gamma_{\rho\mu\nu} - \Gamma_{\rho\nu\mu}} \\ \Gamma_{\rho\mu\nu} + \Gamma_{\rho\nu\mu} &= \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \end{aligned}$$

   $$\boxed{\{{}^{\rho}_{\mu\nu}\} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}$$

5. **Example:** S_2, $x^\mu = (\theta, \phi)$

   - *Invariant length:*

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

   - *Metric tensor:*

$$g_{\mu\nu} = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \quad g^{\mu\nu} = \frac{1}{r^2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

   - *Christoffel symbol:*

$$\left\{ {\theta \atop \phi\phi} \right\} = -\sin\theta\cos\theta, \quad \left\{ {\phi \atop \theta\phi} \right\} = \left\{ {\phi \atop \phi\theta} \right\} = \cot\theta$$

   - *Curvature:*

$$\begin{aligned} R^\theta_{\phi\theta\phi} &= \partial_\theta \left\{ {\theta \atop \phi\phi} \right\} - \partial_\phi \left\{ {\theta \atop \theta\phi} \right\} + \left\{ {\theta \atop \theta\mu} \right\} \left\{ {\mu \atop \phi\phi} \right\} - \left\{ {\theta \atop \phi\mu} \right\} \left\{ {\mu \atop \theta\phi} \right\} \\ &= -\partial_\theta \sin\theta\cos\theta + \sin\theta\cos\theta\cot\theta = \sin^2\theta \\ R^\phi_{\theta\theta\phi} &= g^{\phi\phi} R_{\phi\theta\theta\phi} = -g^{\phi\phi} R_{\theta\phi\theta\phi} = -g^{\phi\phi} g_{\theta\theta} R^\theta_{\phi\theta\phi} = -1 \end{aligned}$$

   - *Ricci tensor:*

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

   - *Scalar curvature:* $R = \frac{2}{r^2}$

6. **Metric admissible curvature tensor:**

$$R_{\rho\kappa\mu\nu} = -R_{\kappa\rho\mu\nu} = R_{\mu\nu\rho\kappa}$$

   - *Number of independent components:* $256 \to 20$.
   - $R^\mu_{\nu\rho\sigma} = 0$ *for flat space only*

7. **Double contracted Bianchy identity:**

$$\begin{aligned} 0 &= D_\mu R^\sigma{}_{\kappa\nu\rho} + D_\nu R^\sigma{}_{\kappa\rho\mu} + D_\rho R^\sigma{}_{\kappa\mu\nu} && \sigma \leftrightarrow \nu \\ 0 &= D_\mu R_{\kappa\rho} + D_\nu R^\nu{}_{\kappa\rho\mu} - D_\rho R_{\kappa\mu} && \kappa \leftrightarrow \mu \\ 0 &= 2 D_\mu R^\mu{}_\rho - D_\rho R \end{aligned}$$

8. **Einstein tensor:**

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ D_\mu G^\mu{}_\nu &= D_\mu \left(R^\mu{}_\nu - \frac{1}{2} g^\mu_\nu R \right) = 0 \end{aligned}$$

F. Technical details

1. **Gauge invariance of the action:**

$$S = \int d^4 x L$$
$$d^4 x \rightarrow d^4 x' \left|\det \frac{\partial x}{\partial x'}\right| \neq d^4 x'$$

2. **Invariant integral:** $g = \det g_{\mu\nu}$

$$\begin{aligned} g_{\mu\nu} &= \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma} \\ g &= g' \left(\det \frac{\partial x'}{\partial x}\right)^2 \\ d_{\text{inv}} x &= dx\sqrt{-g} \rightarrow dx' \left|\det \frac{\partial x}{\partial x'}\right| \sqrt{-g'} \left|\det \frac{\partial x'}{\partial x}\right| = dx' \sqrt{-g'} \end{aligned}$$

3. **Minor matrix:**

$$(M_A)_{j,k} = (-1)^{j+k} d_{k,j},$$

$\nearrow$

determinant of the matrix obtained by omitting the j-th row and the k-th column of A.

- *Determinant of A:*

$$\begin{aligned} \det[A] &= \sum_j A_{k,j}(-1)^{j+k} d_{k,j} \quad \leftarrow \quad \text{expansion along a row} \\ &= \sum_k A_{k,j}(-1)^{j+k} d_{k,j} \quad \leftarrow \quad \text{expansion along a culomn} \end{aligned}$$

- *Two identical rows:* $k \neq \ell$, $A_{k,j} = A_{\ell,j}$

$$\sum_j A_{k,j}(-1)^{j+k} d_{\ell,j} = \det[A] = 0$$

- *Summary:*

$$\sum_j A_{k,j} M_{A\ j,\ell} = \det[A]\delta_{k,\ell} \quad \rightarrow \quad AM_A = \det[A]\mathbb{1} \quad \rightarrow \quad A^{-1} = \frac{M_A}{\det[A]}$$

4. **Variation of the metric tensor:**

$$\begin{aligned} \frac{\partial \det[A]}{\partial A_{k,j}} &= (-1)^{j+k} d_{k,j} = (M_A)_{j,k} \\ \delta g &= \frac{\partial g}{\partial g_{\sigma\mu}} \delta g_{\sigma\mu} = \underbrace{g g^{\sigma\mu}}_{M_g} \delta g_{\mu\sigma} \\ \partial_\nu g &= g g^{\sigma\mu} \partial_\nu g_{\sigma\mu} \end{aligned}$$

5. **Divergence of a vector field:** (no torsion)

$$\begin{aligned}
\Gamma^\mu_{\nu\mu} &= \frac{1}{2}g^{\sigma\mu}(\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\nu\mu}) = \frac{1}{2}g^{\sigma\mu}\partial_\nu g_{\sigma\mu} = \frac{\partial_\nu g}{2g} = \frac{\partial_\nu\sqrt{-g}}{\sqrt{-g}} \\
D_\mu v^\mu &= \partial_\mu v^\mu + \Gamma^\mu_{\nu\mu}v^\nu = \partial_\mu v^\mu + \frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}}v^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu) \\
\int dx\sqrt{-g}D_\mu v^\mu &= \int dx\partial_\mu(\sqrt{-g}v^\mu) = \int ds_\mu\sqrt{-g}v^\mu \\
D_\mu j^\mu &= 0 \qquad \Longrightarrow \qquad \partial_\mu(\sqrt{-g}j^\mu) = 0, \qquad Q = \int d^3x\sqrt{-g}j^0
\end{aligned}$$

6. **Metric admissibility:**

$$\begin{aligned}
D_\mu D^\mu &= g^{\mu\nu}D_\mu D_\nu = D_\mu g^{\mu\nu}D_\nu = D_\mu D_\nu g^{\mu\nu} \\
D_\mu D^\mu\phi &= D_\mu\partial^\mu\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi)
\end{aligned}$$

G. Dynamics

1. **Einstein-Hilbert action:**

$$\boxed{S_E = -\frac{1}{16\pi G}\int dx\sqrt{-g}(R + 2\Lambda) = -\frac{1}{16\pi G}\int dx\sqrt{-g}(g^{\mu\nu}R_{\mu\nu} + 2\Lambda)}$$

2. **Independent fields:**

   (a) *Affine connection:*

tensor

↘

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \left\{{\rho \atop \mu\nu}\right\} + C^\rho_{\mu\nu} \\
\tilde{D}_\mu v^\nu &= \partial_\mu v^\nu + \left\{{\nu \atop \rho\mu}\right\}v^\rho
\end{aligned}$$

   (b) *Metric tensor:* $g^{\mu\nu}$

   not $g_{\mu\nu}$(!):

$$\begin{aligned}
\delta^\mu_\rho &= g^{\mu\nu}g_{\nu\rho} \\
0 &= \delta g^{\mu\nu}g_{\nu\rho} + g^{\mu\nu}\delta g_{\nu\rho} \\
\delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}
\end{aligned}$$

3. **Variation of the field strength tensor:** $C \to C + \delta C$

   (a) *Gauge theory:*

$$\begin{aligned}
F_{\mu\nu} &= [D_\mu, D_\nu] \\
\delta F_{\mu\nu} &= [D_\mu + \delta C_\mu, D_\nu + \delta C_\nu] - [D_\mu, D_\nu]
\end{aligned}$$

$$
\begin{aligned}
&= (D_\mu + \delta C_\mu)(D_\nu + \delta C_\nu) - (D_\nu + \delta C_\nu)(D_\mu + \delta C_\mu) - D_\mu D_\nu + D_\nu D_\mu \\
&= D_\mu \delta C_\nu + \delta C_\mu D_\nu - D_\nu \delta C_\mu - \delta C_\nu D_\mu + \mathcal{O}\left(\delta C^2\right) \\
&= (D_\mu \delta C_\nu) - (D_\nu \delta C_\mu) + \mathcal{O}\left(\delta C^2\right)
\end{aligned}
$$

(b) *Gravity:*

$$
\begin{aligned}
\delta R^\mu{}_{\nu\rho\sigma} &= D_\rho \delta C^\mu{}_{\nu\sigma} - D_\sigma \delta C^\mu{}_{\nu\rho} \\
\delta R_{\nu\sigma} &= D_\rho \delta C^\rho{}_{\nu\sigma} - D_\sigma \delta C^\rho{}_{\nu\rho}
\end{aligned}
$$

(c) *Covariance:*

$$
g^{\nu\sigma}\delta R_{\nu\sigma} = K'_\kappa{}^{\nu\rho}\delta C^\kappa{}_{\nu\rho} + \partial_\mu v^\mu = K_\kappa{}^{\nu\rho}\delta C^\kappa{}_{\nu\rho} + \tilde{D}_\rho v^\rho
$$

$\nearrow$ scalar $\quad\uparrow$ tensor $\quad\nwarrow$ vector

(d) *Non-singular K:* $K(C)$ is linear in C and $K(C) = 0 \implies C = 0$

4. **Variation of the integrand:** $\delta g = g g^{\sigma\mu}\delta g_{\mu\sigma}$

$\downarrow$

$$
\begin{aligned}
\delta\sqrt{-g} &= -\frac{\delta g}{2\sqrt{-g}} = -\frac{g}{2\sqrt{-g}} g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g} g^{\mu\nu} g_{\mu\rho} g_{\nu\sigma}\delta g^{\rho\sigma} = -\frac{1}{2}\sqrt{-g} g_{\nu\sigma}\delta g^{\nu\sigma} \\
\delta[\sqrt{-g}(R_{\mu\nu}g^{\mu\nu} + 2\Lambda)] &= \delta\sqrt{-g}(R_{\mu\nu}g^{\mu\nu} + 2\Lambda) + \sqrt{-g}\delta R_{\mu\nu} g^{\mu\nu} + \sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g} g_{\nu\sigma}\delta g^{\nu\sigma}(R + 2\Lambda) + \sqrt{-g}\tilde{D}_\rho v^\rho + \sqrt{-g}\delta C^\kappa{}_{\nu\rho} K_\kappa{}^{\nu\rho} + \sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu} \\
&= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}\delta C^\kappa{}_{\nu\rho} K_\kappa{}^{\nu\rho} + \sqrt{-g}\tilde{D}_\rho v^\rho
\end{aligned}
$$

$\uparrow$ Einstein equation $\quad\nearrow$ metric admissibility $\quad\uparrow$ surfacce term, ignored

$\searrow$

$$
\boxed{G_{\mu\nu} - \Lambda g_{\mu\nu} = 0}
$$

V. COUPLING TO MATTER

A. Point particle in an external gravitational field

1. **Equivalence Principle:** no gravitational field at a selected space-time point

$$
\frac{d\dot{\xi}^a(s)}{ds} = 0
$$

- *Filling up the space-time with the solutions:* $u^\mu(x) = \dot{x}^\mu$

$$\boxed{u^\nu D_\nu u^\mu = \dot{u}^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu = 0}$$

- *Non-gravitational force:*

$$\begin{aligned}
\dot{u}^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu &= \frac{F^\mu}{mc} \\
mc\dot{u}^\mu &= F^\mu + F^\mu_{gr} \\
F^\mu_{gr} &= -mu^\mu \Gamma^\mu{}_{\rho\nu} u^\nu
\end{aligned}$$

Lorentz force of electrodynamics:

$$\begin{aligned}
mc\ddot{x}^\mu &= F^\mu + F^\mu_{ed} \\
F^\mu_{ed} &= \frac{e}{c} F^\mu{}_\nu u^\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\end{aligned}$$

2. **Spin precession:**

- *Rest frame:* $S^\mu = (0, \mathbf{S})$, $S_\mu \dot{x}^\mu = 0$
- *E.O.M.:*

$$0 = \frac{dS^\mu}{ds} \to \dot{S}^\mu + \Gamma^\mu{}_{\rho\nu} S^\rho \dot{x}^\nu$$

- *Auxiliary condition:*

$$S_\mu \dot{x}^\mu = 0$$

- *External force without torque:*

$$\begin{aligned}
\frac{d\mathbf{S}}{dt} &= 0 \quad \leftarrow \quad \text{co} - \text{moving frame}: \quad \dot{S} = a\dot{x} \\
0 &= \frac{d}{ds}(S_\mu \dot{x}^\mu) = a\dot{x}_\mu \dot{x}^\mu + S_\mu \ddot{x}^\mu \quad \Longrightarrow \quad a = -S_\mu \ddot{x}^\mu = -S_\mu \frac{F^\mu}{mc}
\end{aligned}$$

- *Fermi-Walker transport:*

$$\boxed{\dot{S}^\mu = -S_\nu \frac{F^\nu}{mc} \dot{x}^\mu}$$

3. **Variational equation of motion:**

- *Action:* $\dot{x}^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau}$ ($\tau \neq s$ to leave the variation of all components of x^μ independent)

$$S = -mc \int \sqrt{\dot{x}^\mu g_{\mu\nu}(x) \dot{x}^\nu} d\tau$$

- *Euler-Lagrange equation:*

$$\begin{aligned}
\frac{\partial L}{\partial x^\rho} &= -mc\frac{\dot x^\mu \partial_\rho g_{\mu\nu}\dot x^\nu}{2\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu}}, \\
\frac{\partial L}{\partial \frac{dx^\rho}{d\tau}} &= -mc\frac{g_{\rho\nu}\dot x^\nu}{\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu}} \\
0 &= \frac{\partial L}{\partial x^\rho} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot x^\rho} \\
&= -\frac{\dot x^\mu \partial_\rho g_{\mu\nu}\dot x^\nu}{2\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu}} + \frac{d}{d\tau}\frac{g_{\rho\nu}\dot x^\nu}{\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu}} \\
&= \frac{1}{\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\nu}}\left[-\frac{1}{2}\dot x^\mu \partial_\rho g_{\mu\nu}\dot x^\nu + \dot x^\kappa \partial_\kappa g_{\rho\nu}\dot x^\nu + g_{\rho\nu}\ddot x^\nu + g_{\rho\nu}\dot x^\nu \frac{d}{d\tau}\frac{1}{\sqrt{\dot x^\mu g_{\mu\lambda}\dot x^\lambda}}\right]
\end{aligned}$$

$\uparrow$

symmetrize in κ and ν the factor $\partial_\kappa g_{\rho\nu}$

$$\begin{aligned}
0 &= -\frac{1}{2}\dot x^\mu \partial_\rho g_{\mu\nu}\dot x^\nu + \frac{1}{2}\dot x^\kappa(\partial_\kappa g_{\rho\nu} + \partial_\nu g_{\rho\kappa})\dot x^\nu + g_{\rho\nu}\ddot x^\nu + g_{\rho\nu}\frac{dx^\nu}{d\tau}\frac{d}{d\tau}\frac{1}{\sqrt{\dot x^\mu g_{\mu\lambda}\dot x^\lambda}} \\
&= g_{\rho\sigma}(\ddot x^\sigma + \Gamma^\sigma_{\nu\kappa}\dot x^\nu \dot x^\kappa + \dot x^\sigma \dot f), \quad f(\tau) = \frac{1}{\sqrt{\dot x^\mu g_{\mu\lambda}\dot x^\lambda}}
\end{aligned}$$

$\nearrow$

Lagrange multiplier for the constraint $\dot x^2 = 1$

- *Geodesics, $\tau = s$:*

$$u^\nu D_\nu u^\mu = \dot u^\mu + u^\rho \Gamma^\mu{}_{\rho\nu} u^\nu = 0$$

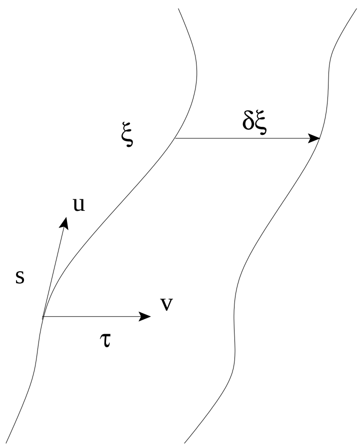
4. **Geodesic deviation:**

- *Newton equation:*

$$\begin{aligned}
m\ddot\xi^j &= -\nabla^j U \\
\xi(t_i) &\to \xi(t_i) + \delta\xi(t_i) \\
m\delta\ddot\xi^j &= -\delta\xi^k \nabla^k \nabla^j U
\end{aligned}$$

- *Geodesics:*
  - Fill up the space-time with geodesics $\xi(s)$, $u(x) = \frac{d\xi(x)}{ds} = \dot\xi(x)$, $\frac{d\phi}{ds} = u^\nu D_\nu \phi = \dot\phi$
  - The space-time strip, bounded by $\xi(s)$ and $\xi(s) + \delta\xi(s)$: $\xi^\mu(s,\tau)$
  - Holonomic coordinate axis vectors: $u = \partial_s \xi(s,\tau)$, $v = \partial_\tau \xi(s,\tau)$

$$\begin{aligned}
\partial_s v &= \partial_\tau u \\
u^\nu D_\nu v^\mu &= v^\nu D_\nu u^\mu
\end{aligned}$$



– Deviation equation for $\delta\xi = \epsilon\partial_\tau \xi = \epsilon v$:

$$\begin{aligned}
\ddot{v} &= u^\mu D_\mu (u^\nu D_\nu v) \\
&= u^\mu D_\mu (v^\nu D_\nu u) \quad \leftarrow \quad \text{holonomy} \\
&= u^\mu (D_\mu v^\nu) D_\nu u + u^\mu v^\nu D_\mu D_\nu u \quad \leftarrow \quad \text{Leibnitz rule} \\
&= v^\mu (D_\mu u^\nu) D_\nu u + u^\mu v^\nu [D_\mu, D_\nu] u + u^\mu v^\nu D_\nu D_\mu u \quad \leftarrow \quad \text{holonomy} \ + \ \text{alg. id.} \\
&= v^\mu (D_\mu u^\nu) D_\nu u + u^\mu v^\nu [D_\mu, D_\nu] u + v^\nu D_\nu (u^\mu D_\mu u) - v^\nu (D_\nu u^\mu) D_\mu u \quad \leftarrow \quad \text{Leibnitz rule} \\
&= u^\mu v^\nu [D_\mu, D_\nu] u + v^\nu D_\nu (u^\mu D_\mu u) \quad \leftarrow \quad \text{alg. cancellation} \\
\ddot{v}^\rho &= R^\rho{}_{\kappa\mu\nu} u^\kappa u^\mu v^\nu \quad \leftarrow \quad u^\mu D_\mu u = \ddot{\xi} = 0
\end{aligned}$$

- *Electrodynamics:*

$$\begin{aligned}
mc\ddot{x}^\mu_n(s) &= \frac{e}{c} F_{\mu\nu} \dot{x}^\nu_n(s) \\
mc(\ddot{x}^\mu + \delta\ddot{x}^\mu) &= \frac{e}{c} F^\mu{}_\nu (x + \delta x)(\dot{x}^\nu + \delta\dot{x}^\nu) \\
mc\delta\ddot{x}^\mu &= \frac{e}{c} \delta x^\rho \partial_\rho F^\mu{}_\nu \dot{x}^\nu + \frac{e}{c} F^\mu{}_\nu \delta\dot{x}^\nu
\end{aligned}$$

5. **Newtonian limit:** slow test particle, $\frac{dx^\mu}{ds} \approx (1,0,0,0)$, and weak, static gravitational field

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \gamma_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - \gamma^{\mu\nu} \\
\ddot{x}^\mu \approx -\Gamma^\mu_{00} &= \frac{1}{2} \frac{\partial \gamma_{00}}{\partial x_\mu} \quad \leftarrow \quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \\
\ddot{\mathbf{x}} &= -\nabla \phi
\end{aligned}$$

$$\boxed{\phi_N = \frac{\gamma_{00}}{2}}$$

B. Interacting matter-gravity system

1. **Gravitational E.O.M.:**

$$\begin{aligned} S &= S_E + S_M = -\frac{1}{16\pi G}\underbrace{\int dx\sqrt{-g}(g^{\mu\nu}R_{\mu\nu}+2\Lambda)}_{\text{E.O.M.: } G-\Lambda g=0} + S[g,x(s),\phi,\ldots] \\ G_{\mu\nu}-\Lambda g_{\mu\nu} &= 8\pi G M_{\mu\nu}(x) \\ M_{\mu\nu}(x) &= \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}(x)} \end{aligned}$$

E-M conservation: (inhomogeneous geometry!)

$$0 = D_\mu(G^{\mu\nu}-\Lambda g^{\mu\nu}) = D_\mu M^{\mu\nu} = 0$$

2. **Mechanical E.O.M.:**

$$\begin{aligned} S_M &= -mc\int d\tau\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} \\ \delta S_M &= -\frac{1}{2}mc\int d\tau\frac{\dot{x}^\mu\delta g_{\mu\nu}\dot{x}^\mu}{\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu}} \\ &= -\frac{1}{2}mc\int ds\dot{x}^\mu\dot{x}^\nu\delta g_{\mu\nu} \qquad (\tau\to s) \\ &= -\frac{1}{2}\int dx\int ds\frac{p^\mu(s(t))p^\nu(s(t))}{mc}\delta(x-x(s))\delta g_{\mu\nu}(x) \\ \sqrt{-g}M^{\mu\nu}(x) &= \int ds\frac{p^\mu(s(t))p^\nu(s(t))}{mc}\delta(x-x(s)) \end{aligned}$$

Canonical E-M tensor:

$$\begin{aligned} T^{\mu 0}(t,\mathbf{x}) &= p^\mu(s(t))\delta(\mathbf{x}-\mathbf{x}(t)) \\ T^{\mu\nu}(x) &= \frac{p^\mu(s(t))p^\nu(s(t))}{p^0(s(t))}\delta(\mathbf{x}-\mathbf{x}(t)) \end{aligned}$$

Covariant generalization:

$$\begin{aligned} T^{\mu\nu}(x) &= \int ds\frac{p^\mu(s(t))p^\nu(s(t))}{p^0(s(t))}c\frac{dt}{ds}\delta(x-x(s)) \\ &= \int ds\frac{p^\mu(s(t))p^\nu(s(t))}{mc}\delta(x-x(s)), \end{aligned}$$

$$\boxed{\sqrt{-g}M = T}$$

3. **Ideal fluid:** homogeneous and isotropic fluid in the rest frame ($\dot{x}^\mu = (1,0,0,0)$)

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Covariant generalization:

$$T^{\mu\nu} = (p+\epsilon)\dot{x}^\mu \dot{x}^\nu - p g^{\mu\nu},$$

Short enough mean free path and times:

$$T^{\mu\nu}(x) = (p(x)+\epsilon(x))u^\mu(x)u^\nu(x) - p(x)g^{\mu\nu} \qquad (u^\mu(x) = \dot{x}^\mu(x))$$

4. **Classical scalar field:**

$$\begin{aligned} S_M &= \int dx\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - V(\phi(x))\right] \\ M_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}L + 2\underbrace{\frac{\partial L}{\partial g^{\mu\nu}}}_{\frac{1}{2}g_{\nu\kappa}\frac{\partial L}{\partial\partial_\kappa\phi}\partial_\mu\phi} \\ \sqrt{-g}M_{\mu\nu} &= \frac{\partial L}{\partial\partial^\mu\phi}\partial_\nu\phi - g_{\mu\nu}L = T_{\mu\nu} \end{aligned}$$

VI. SCHWARZSCHILD SOLUTION

A. Metric

1. **Symmetry:** translations in time and rotations inspace

2. **Metric:**

$$ds^2 = f(r)dt^2 - \sum_{j,k=1}^{3} x^j h_{jk}(r)x^k$$

3. **Gauge choice for r:** A = area of the surface of a sphere of radius r

$$r = \sqrt{\frac{A}{4\pi}}$$

4. **Geometry:**

   (a) *Metric:*

$$ds^2 = f(r)dt^2 - h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

   (b) *Christoffel symbols:* non-vanishing components only,

$$\begin{aligned} \Gamma^t{}_{rt} &= \frac{f'}{2f}, \quad \Gamma^r{}_{tt} = \frac{f'}{2h} \\ \Gamma^r{}_{rr} &= \frac{h'}{2h}, \quad \Gamma^r{}_{\theta\theta} = -\frac{r}{h}, \quad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{h}, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\phi{}_{r\phi} = \frac{1}{r} \\ \Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta, \quad \Gamma^\phi{}_{\theta\phi} = \cot\theta \end{aligned}$$

(c) *Ricci tensor:* diagonal

$$\begin{aligned} R_{tt} &= -\frac{f''}{2h} + \frac{f'}{4h}\left(\frac{f'}{f} + \frac{h'}{h}\right) - \frac{f'}{rh} \\ R_{rr} &= \frac{f''}{2f} - \frac{f'}{4f}\left(\frac{f'}{f} + \frac{h'}{h}\right) - \frac{h'}{rh} \\ R_{\theta\theta} &= -1 + \frac{r}{2h}\left(\frac{f'}{f} - \frac{h'}{h}\right) + \frac{1}{h} \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta. \end{aligned}$$

(d) *Scalar curvature:*

$$R_{|r\neq 0} = 0$$

5. **Einstein equation:** for $r \neq 0$

$$\begin{aligned} 0 &= R_{tt} = R_{rr} = R_{\theta\theta} \\ 0 &= \frac{R_{tt}}{f} + \frac{R_{rr}}{h} = -\frac{1}{rh}\left(\frac{f'}{f} + \frac{h'}{h}\right) \quad \Longrightarrow \quad 0 = \frac{f'}{f} + \frac{h'}{h} = \frac{d}{dr}\ln fh \quad \Longrightarrow \quad hf = A \end{aligned}$$

$r \to \infty$: flat space $\Longrightarrow$ $f = h = 1$ $\Longrightarrow$ $A = 1$

$$\begin{aligned} 0 &= R_{\theta\theta} = -1 + rf\frac{f'}{f} + f = rf' + f - 1 \quad \Longrightarrow \quad \frac{drf}{dr} = 1 \quad \Longrightarrow \quad f = 1 + \frac{B}{r}, \quad B = -\frac{2GM}{c^2} = -r_s \\ ds^2 &= \left(1 - \frac{r_s}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

Dimensionless coordinates: $t \to tr_s$ and $r \to rr_s$

$$\boxed{ds^2 = \left(1 - \frac{1}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{1}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

6. **Properties:**

(a) *Weak field limit:* $r \gg r_s$, $x^0 = tc$

$$\phi_N = \frac{\gamma_{00}c^2}{2} = -\frac{c^2 r_s}{2r} = -\frac{GM}{r}$$

(b) *Gravitational red shift:* (between stationary observers)

i. signals at t_{e1} and t_{e2} emitted at r_e and received at t_{r1} and t_{r2} at r_r

$$\text{time independent metric} \quad \Longrightarrow \quad t_{r1} - t_{e1} = t_{r2} - t_{e2} \quad \Longrightarrow \quad t_{r2} - t_{r1} = t_{e2} - t_{e1}$$

ii. proper time is measured by a clock travelling with the object

$$\Delta s_e = \sqrt{1 - \frac{1}{r_e}}(t_{e2} - t_{e1}), \quad \Delta s_r = \sqrt{1 - \frac{1}{r_r}}(t_{r2} - t_{r1})$$

iii. redshift:

$$\frac{\Delta s_e}{\Delta s_r} = \frac{\nu_r}{\nu_e} = \sqrt{\frac{1-\frac{1}{r_e}}{1-\frac{1}{r_r}}}$$

(c) *Singularities:*

i. Coordinate singularity at $r = r_s$

$$r_s \approx \begin{cases} 2.8\frac{M}{M_{sun}} \text{ km} \\ 2.4\frac{M}{M_{proton}} \cdot 10^{-52} \text{ cm} \end{cases}$$

$$R_{t\theta t\theta} = R_{t\phi t\phi} = -R_{r\theta r\theta} = -R_{r\phi r\phi} = \frac{1}{2}R_{\theta\phi\theta\phi} = -\frac{1}{2}R_{trtr} = \frac{1}{r^3}$$

$\Longrightarrow$ Regular force during the free fall

ii. True singularity at $r = 0$

(d) *Singular forces on a stationary extended objects:*

$$\begin{aligned} u^\mu &= \left(\frac{1}{\sqrt{1-\frac{1}{r}}}, 0, 0, 0\right) \\ a^\mu &= u^\nu D_\nu u^\mu = u^\nu \partial_\nu u^\mu + \Gamma^\mu{}_{\rho\nu} u^\rho u^\nu = \frac{\Gamma^\mu{}_{tt}}{1-\frac{1}{r}} = (0, a^r, 0, 0), \quad a^r = \frac{1}{2r^2} = \phi'_N(r) \end{aligned}$$

Gauge invariant acceleration: is obtained by multiplying it by $\sqrt{g_{rr}}$,

$$\sqrt{-a^\mu g_{\mu\nu} a^\nu} = a^r \sqrt{-g_{rr}} = \frac{1}{2r^2\sqrt{1-\frac{1}{r}}},$$

diverging tidal (r-dependent) forces at $r = 1$

(e) *Causal singularity at r_s:*

i. light cones flip at r_s:

$$ds^2 = \left(1-\frac{1}{r}\right) dt^2 - \frac{dr^2}{1-\frac{1}{r}} \qquad \Longrightarrow \qquad \frac{dt}{dr} = \pm\frac{1}{1-\frac{1}{r}}$$

ii. the role of time and radius is exchanged

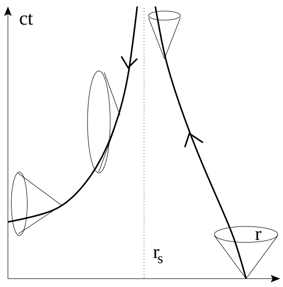
inward outward

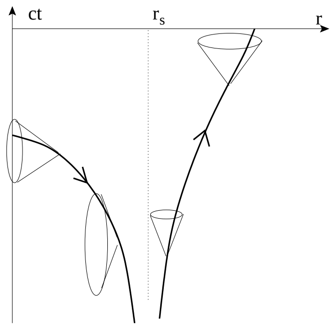
iii. Irreversible dynamics within the Schwarzschild sphere $\Longrightarrow$ Impossible to leave it?

(f) *Realistic, non-singular mass distribution $\rho(r) \neq \delta(r)$:* naked horizon only if $\rho(r) = 0$ for $r \geq r_s$

(g) *Characteristic lengths of a particle:*

i. Compton wavelength $\lambda_C = \frac{\hbar}{mc}$, maximal localization





ii. Schwarzschild radius $r_s = \frac{2Gm}{c^2}$

iii. $r_s \ll \lambda_C$: quantum particle, vacuum polarization, known physics

iv. $\lambda_C \ll r_s$: black hole ???

v. $2\lambda_C = r_s$: Planck mass, $m_{Pl} = \sqrt{\frac{\hbar c}{G}} \sim 2.1 \times 10^{-5}$g, Planck length: $\lambda_C = \frac{\hbar}{m_{Pl}c} \sim 1.6 \times 10^{-33}$cm

(h) *Role of spherical symmetry:*

i. The solution remains the same when time independence is not assumed

ii. The spherically symmetric solutions of the vacuum Einstein equation are static

iii. Trivial in the Newtonian theory (no gravitational dynamical degrees of freedom)

iv. No s-waves in gravitational radiation field

B. Geodesics

1. **Lagrangian:**

$$\begin{aligned} L &= -mc\sqrt{\dot{x}^\mu g_{\mu\nu}(x)\dot{x}^\nu} \\ &= -mc\sqrt{\left(1 - \frac{1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{1}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)} \end{aligned}$$

2. **E.O.M. for θ:**

$$r^2 \sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{ds} r^2\dot{\theta} \qquad \Longrightarrow \qquad \theta = 0, \frac{\pi}{2}$$

$\theta = \frac{\pi}{2}$, planar motion

3. **Cyclic coordinates:** t and ϕ

$$\begin{aligned} -\frac{1}{mc}\frac{\partial L}{\partial \dot{t}} &= \left(1 - \frac{1}{r}\right)\dot{t} = E, \\ \frac{1}{mc}\frac{\partial L}{\partial \dot{\phi}} &= r^2\dot{\phi} = \ell, \end{aligned}$$

4. **Radial equation:**

(a) Energy conservation $\leftarrow \dot{x}^2 = \kappa (=1)$

$$\kappa = \left(1-\frac{1}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{1}{r}} - r^2\dot{\phi}^2 = \left(1-\frac{1}{r}\right)\frac{E^2}{(1-\frac{1}{r})^2} - \frac{\dot{r}^2}{1-\frac{1}{r}} - r^2\frac{\ell^2}{r^4} = \frac{E^2-\dot{r}^2}{1-\frac{1}{r}} - \frac{\ell^2}{r^2}$$

$$E^2 - \dot{r}^2 = V(r), \quad E^2 = \dot{r}^2 + V(r), \quad V(r) = \left(1-\frac{1}{r}\right)\left(\kappa + \frac{\ell^2}{r^2}\right) = 2\phi_N(r) + \mathcal{O}\left(r^{-3}\right)$$

↗

relativistic effects

(b) Effective potential:

$$\frac{dV(r)}{d\frac{1}{r}} = -3\frac{\ell^2}{r^2} + 2\frac{\ell^2}{r} - 1 = 0 \quad \Longrightarrow \quad D = 4\ell^4 - 12\ell^2 = 4\ell^2(\ell^2-3)$$

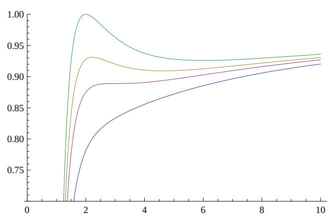


FIG. 1: $\ell^2 = 1.5, \sqrt{3}, 1.85, 2.0$

(c) Radial motion:

i. $\ell \le \sqrt{3}$: falling into the center

ii. $\ell > \sqrt{3}$: stable orbits $r_{min} \le r \le r_{max}$ in certain range of E

$$r_{max} = \frac{3}{1+\sqrt{1-\frac{3}{\ell^2}}}, \quad r_{min} = \frac{3}{1-\sqrt{1-\frac{3}{\ell^2}}},$$

(d) *Perihelion motion of Mercury:* $\mathcal{O}\left(r^{-3}\right)$ relativistic term in the effective potential

5. *Light:*

(a) $m = 0$ coupling to gravity?

(b) Fermi-Walker transport of the polarization direction $\Longrightarrow$ coupling to gravity

(c) geometric optics, Fermat's principle, $\kappa = 0$

(d) Deflection of light around the Sun

C. Space-like hyper-surfaces

1. Visualisation of a $t = t_0$ and $\theta = \frac{\pi}{2}$ hyper-surfaces:

$$-ds^2 = \frac{dr^2}{1-\frac{1}{r}} + r^2 d\phi^2.$$

2. Embedded into a three-dimensional Euclidean space with cylindrical coordinates (z, r, ϕ):

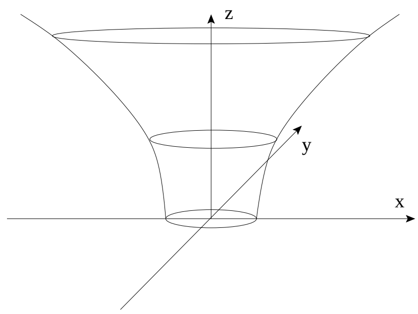
$$-ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$

on a surface $z = z(r)$:

$$-ds^2 = \left[1 + z'^2(r)\right] dr^2 + r^2 d\phi^2$$

3. Matching:

$$1 + z'^2(r) = \frac{1}{1-\frac{1}{r}} \implies z(r) = \int_1^r dr' \sqrt{\frac{1}{1-\frac{1}{r}} - 1} = \int_1^r dr' \sqrt{\frac{\frac{1}{r}}{1-\frac{1}{r}}} = \int_1^r \frac{dr'}{\sqrt{r'-1}} = 2\sqrt{r-1} \ (r > 1)$$



D. Around the Schwarzschild-horizon

$$ds^2 = \left(1 - \frac{1}{r}\right) dt^2 - \frac{dr^2}{1-\frac{1}{r}} - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\nearrow$ too slow time $\qquad$ $\nwarrow$ too detailed radius

1. **Falling through the horizon:** radial motion, $\ell = 0$

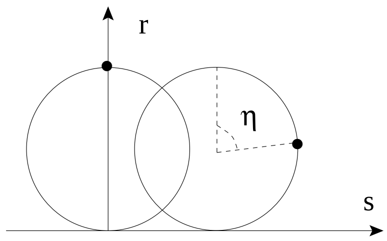
   (a) *E.O.M.:*

$$\begin{aligned} E^2 &= \dot{r}^2 + V(r), \quad V(r) = \left(1 - \frac{1}{r}\right)\left(\kappa + \frac{\ell^2}{r^2}\right) \\ \dot{r}^2 &= \frac{1}{r} + E^2 - 1 \end{aligned}$$

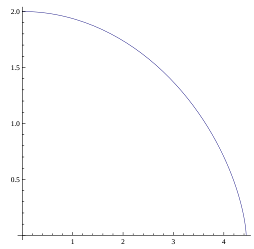
   (b) *Initial conditions:* $r(s_0) = r_0 > 1$, $\dot{r}(s_0) = 0$

$$\dot{r} = \sqrt{\frac{1}{r} + E^2 - 1} = \sqrt{\frac{1}{r} - \frac{1}{r_0}}$$

$$
\begin{aligned}
ds &= \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{r_0}}} \\
r &= \frac{r_0}{2}(1 + \cos\eta) \\
s &= \frac{r_0^{3/2}}{2}(\eta + \sin\eta)
\end{aligned}
$$



Geometrical origin



Free fall from $r = 2$

(c) *No irregularity at the horizon:* $r = 1$

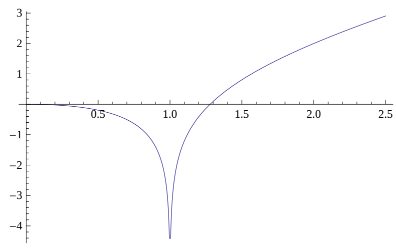
(d) *Proper time of falling into the center:* $\eta = \pi$, $s = \frac{\pi}{2} r_0^{3/2}$

2. **Stretching the horizon:** tortoise coordinate

- *Origin of the singularity:*

$$\dot{r} = \frac{dr}{dt}\dot{t} = \frac{dr}{dt}\frac{E}{1 - \frac{1}{r}},$$

- *Zoom onto $r = 1$: $r \to r^*$*



$$
\begin{aligned}
dr^* &= \frac{dr}{1 - \frac{1}{r}}, \quad \frac{dr^*}{dr} = \frac{1}{1 - \frac{1}{r}}, \quad r^* = r + \ln|r - 1| \quad \left(\frac{dr^*}{dr} = 1 + \frac{1}{r-1} = \frac{r}{r-1}\right) \\
\dot{r} &= \frac{dr}{dt}\frac{dr^*}{dr}E = \frac{dr^*}{dt}E
\end{aligned}
$$

- *E.O.M.:*

$$
\begin{aligned}
\dot{r}^2 &= \frac{1}{r} + E^2 - 1 \\
E^2\left[1 - \left(\frac{dr^*}{dt}\right)^2\right] &= 1 - \frac{1}{r}
\end{aligned}
$$

- *Approaching the horizon:* $r \searrow 1 \implies \frac{dr^*}{dt} \to -1 \implies t \sim -r^* \to \infty$

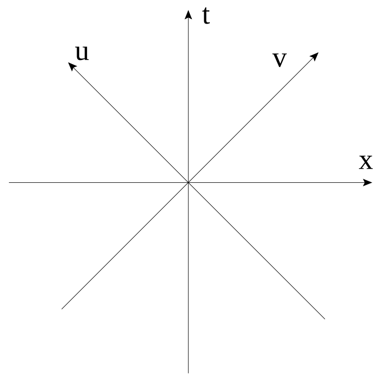
  takes infinitely long coordinate time

3. **Szekeres-Kruskall coordinate system:**

- *Light cone coordinates:*

  (a) Minkowski space-time:

$$u = t - r, \quad v = t + r, \quad ds^2 = dt^2 - dr^2 = dudv \quad \implies \quad \text{light cone}: \ du = 0 \ \text{ or } \ dv = 0$$



  (b) Schwarzschild geometry:

$$\begin{aligned} u^* &= t - r^*, \quad v^* = t + r^* \\ \frac{dr^*}{dr} &= \frac{1}{1-\frac{1}{r}} \implies dr^2 = \left(1-\frac{1}{r}\right)^2 dr^{*2} \\ ds^2 &= \left(1-\frac{1}{r}\right) dt^2 - \frac{dr^2}{1-\frac{1}{r}} = \left(1-\frac{1}{r}\right)(dt^2 - dr^{2*}) = \left(1-\frac{1}{r}\right) du^* dv^* \end{aligned}$$

  is still singular but already has a clear causal structure

- *Yet another singular rescaling to avoid the singularity at $r = 1$:*

$$\begin{aligned} (I)\ (r>1) \quad u' &= -e^{-\frac{u^*}{2}} = -\sqrt{r-1}e^{\frac{r-t}{2}}, \quad v' = e^{\frac{v^*}{2}} = \sqrt{r-1}e^{\frac{r+t}{2}}, \\ u^* &= t - r - \ln|r-1|, \quad v^* = t + r + \ln|r-1| \\ du' &= \frac{du^*}{2}e^{-\frac{u^*}{2}}, \quad dv' = \frac{dv^*}{2}e^{\frac{v^*}{2}} \\ du'dv' &= \frac{du^*dv^*}{4}e^{\frac{v^*-u^*}{2}} = \frac{du^*dv^*}{4}e^{r^*} = \frac{e^r}{4}(r-1)du^*dv^* \ \leftarrow \ r^* = r + \ln|r-1| \\ du^*dv^* &= \frac{4}{r-1}e^{-r}du'dv' = \frac{4}{r}\frac{1}{1-\frac{1}{r}}e^{-r}du'dv' \\ (II)\ (r<1) \quad u' &= e^{-\frac{u^*}{2}} = \sqrt{1-r}e^{\frac{r-t}{2}}, \quad v' = e^{\frac{v^*}{2}} = \sqrt{1-r}e^{\frac{r+t}{2}} \\ ds^2 &= \left(1-\frac{1}{r}\right) du^*dv^* = \frac{4}{r}e^{-r}du'dv' \end{aligned}$$

  regular at $r = 1$

- *Return to "time" and "space" coordinates:*

$$\begin{aligned} u^* &= t - r^*, \quad v^* = t + r^* \quad \rho = \frac{v' - u'}{2}, \quad \tau = \frac{v' + u'}{2} \\ (I) \quad u' &= -\sqrt{r-1}e^{\frac{r-t}{2}}, \quad v' = \sqrt{r-1}e^{\frac{r+t}{2}}, \\ \rho &= \sqrt{r-1}e^{\frac{r}{2}}\cosh\frac{t}{2}, \quad \tau = \sqrt{r-1}e^{\frac{r}{2}}\sinh\frac{t}{2} \\ (II) \quad u' &= e^{-\frac{u^*}{2}} = \sqrt{1-r}e^{\frac{r-t}{2}}, \quad v' = e^{\frac{v^*}{2}} = \sqrt{1-r}e^{\frac{r+t}{2}} \\ \rho &= \sqrt{1-r}e^{\frac{r}{2}}\sinh\frac{t}{2}, \quad \tau = \sqrt{1-r}e^{\frac{r}{2}}\cosh\frac{t}{2} \end{aligned}$$

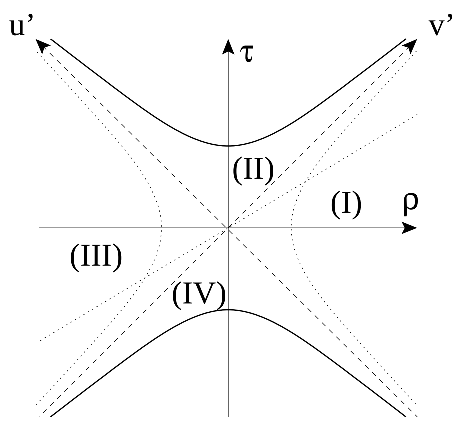
- *Analytic continuation for $v' < 0$:*

$$\begin{aligned} (III) \quad u' &= e^{-\frac{u^*}{2}} = \sqrt{r-1}e^{\frac{r-t}{2}}, \quad v' = -e^{\frac{v^*}{2}} = -\sqrt{r-1}e^{\frac{r+t}{2}} \\ \rho &= -\sqrt{r-1}e^{\frac{r}{2}}\cosh\frac{t}{2}, \quad \tau = -\sqrt{r-1}e^{\frac{r}{2}}\sinh\frac{r}{2} \\ (IV) \quad u' &= -e^{-\frac{u^*}{2}} = \sqrt{1-r}e^{\frac{r-t}{2}}, \quad v' = -e^{\frac{v^*}{2}} = -\sqrt{1-r}e^{\frac{r+t}{2}} \\ \rho &= -\sqrt{1-r}e^{\frac{r}{2}}\sinh\frac{t}{2}, \quad \tau = -\sqrt{1-r}e^{\frac{r}{2}}\cosh\frac{t}{2}. \end{aligned}$$

- *Metric:*

$$\boxed{ds^2 = \frac{4}{r}e^{-r}(d\tau^2 - d\rho^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2)}$$

$$(r-1)e^r = \rho^2 - \tau^2 \ (I),\ (II),\ (III) \text{ and } (IV), \quad t = \begin{cases} 2\text{arcth}\frac{\tau}{\rho} & (I) \text{ and } (III), \\ 2\text{arcth}\frac{\rho}{\tau} & (II) \text{ and } (IV), \end{cases}$$



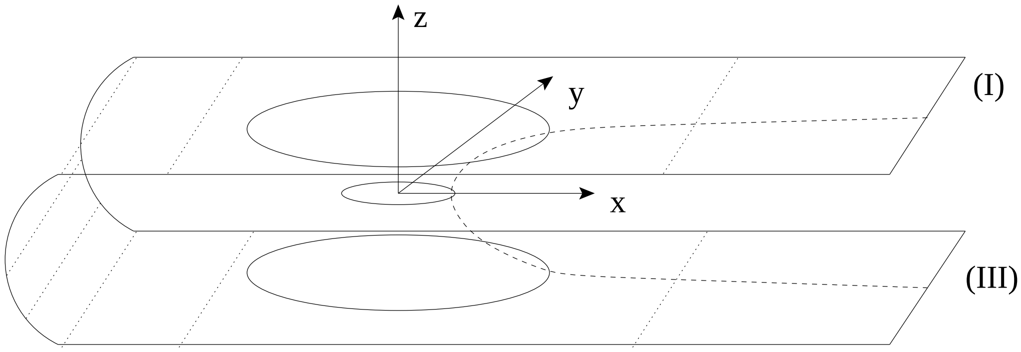
4. **Einstein-Rose bridge:**

- *Embedding the surface* (ρ, ϕ) into an Euclidean space with coordinates (z, u, ϕ) as $z(\rho)$

$$\begin{aligned} -ds^2 &= dz^2 + d\rho^2 + r^2 d\phi^2 = [1 + z'^2(\rho)]d\rho^2 + r^2 d\phi^2 \\ &= \frac{4}{r}e^{-r}d\rho^2 + r^2 d\phi^2 \end{aligned}$$

$$
\begin{aligned}
1 + z'^2(\rho) &= \frac{4}{r} e^{-r} \\
z(r) &= \int_1^r dr' \frac{d\rho}{dr} \sqrt{\frac{4}{r'} e^{-r'} - 1}
\end{aligned}
$$

Reduplication of the Universe at each point particle



- *Time inversion:*

  (a) Classical physics: $T : x(t) \to Tx(t) = x(-t)$

  (b) Pure states: $T : |\psi(t)\rangle \to T|\psi(t)\rangle = |\psi(-t)\rangle = |\psi(t)\rangle^* = \langle\psi(t)|^{\text{tr}}$,

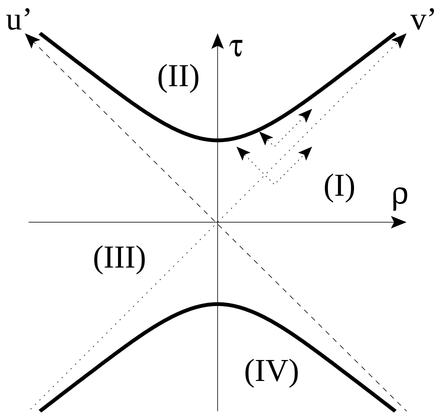
  $$\bar{A}(t) = \langle\psi(t)|A|\psi(t)\rangle = (|\psi(t)\rangle)^\dagger A|\psi(t)\rangle = (T|\psi(t)\rangle)^{\text{tr}} A|\psi(t)\rangle$$

  uncorrelated quantum fluctuations

  (c) Mixed states: $\sum_n p_n |\psi_n\rangle\langle\psi_n|$ correlated quantum fluctuations

  (d) Mixed states of black holes?

5. **Causal structure:** no way to leave (II) and (IV)



VII. HOMOGENEOUS AND ISOTROPIC COSMOLOGY

A. Robertson-Walker metric

1. **Curvature tensor in maximally symmetric spatial geometry:**

$$g_{\mu\nu} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & h \end{pmatrix}$$

   (a) A transformation of second order antisymmetric contravariant spatial tensors:

$$\begin{aligned} \tilde{R}^j_{k\ell m} \to \tilde{R}^{jk}_{\ell m} &= \tilde{R}^j_{n\ell m} h^{kn} \\ T^{\ell k} &\to \tilde{R}^{k\ell}_{mn} T^{mn} \end{aligned}$$

   (b) $\tilde{R}^{k\ell}_{mn} = \tilde{R}^{mn}_{k\ell} \implies \tilde{R}$ is diagonalizable

   (c) Isotropy: degeneracy in d spatial dimensions

$$\begin{aligned} \tilde{R}^{jk}_{\ell m} &= K(\delta^j_\ell \delta^k_m - \delta^k_\ell \delta^j_m) \\ \tilde{R} &= \tilde{R}^{jk}_{jk} = Kd(d-1) = k|\tilde{R}|, \quad k = -1, 0, 1 \\ \tilde{R}^k_m &= \tilde{R}^{jk}_{jm} = K(d-1)\delta^k_m = \frac{\tilde{R}}{d}\delta^k_m \end{aligned}$$

2. **Embedding into R^4:** $k = \mathrm{sign} K$

   (a) $k = 1$:

$$\begin{aligned} a^2 &= x^2 + y^2 + z^2 + w^2 \\ 0 &= xdx + ydy + zdz + wdw \\ ds^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{a^2 - x^2 - y^2 - z^2} \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{r^2 dr^2}{a^2 - r^2} \\ &= \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned}$$

   (b) $k = -1$:

$$\begin{aligned} -a^2 &= x^2 + y^2 + z^2 - w^2 \\ 0 &= xdx + ydy + zdz - wdw \\ ds^2 &= dx^2 + dy^2 + dz^2 - dw^2 \\ &= dx^2 + dy^2 + dz^2 - \frac{(xdx + ydy + zdz)^2}{a^2 + x^2 + y^2 + z^2} \\ &= dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{r^2 dr^2}{a^2 + r^2} \\ &= \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned}$$

3. **Invariant length:** $r \to a(\tau)r$

$$
\begin{aligned}
ds^2 &= d\tau^2 - a^2(\tau)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\
&= d\tau^2 - a^2(\tau)\left[d\chi^2 + h^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad \underbrace{\frac{dr}{\sqrt{1-kr^2}} = d\chi}_{\frac{dr}{d\chi}=\sqrt{1-kr^2}}, \; r = h(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ \sinh\chi & k=-1 \end{cases} \\
& \qquad r_{ph} = a(\tau)h(\chi) \quad \text{cosmic distance} \\
&= a^2(\eta)\left[d\eta^2 - d\chi^2 - h^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad \eta = \int \frac{d\tau}{a(\tau)}
\end{aligned}
$$

4. **Geometry:** $\mu, \nu = (\tau, r, \theta, \phi)$

$$
\begin{aligned}
g_{\mu\nu} &= \begin{pmatrix} 1 & 0 \\ 0 & -a^2(t)\tilde{g}_{ij} \end{pmatrix}, \quad \tilde{g}_{ij} = \begin{pmatrix} \frac{1}{1-kr^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad \sqrt{-g} = a^3(\tau)\frac{r^2\sin\theta}{\sqrt{1-kr^2}} \\
\Gamma^\tau_{ij} &= \dot{a}a\tilde{g}_{ij}, \qquad \Gamma^i_{\tau j} = \frac{\dot{a}}{a}\tilde{g}^i_j, \qquad \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} = \frac{1}{2}\tilde{g}^{i\ell}(\partial_j\tilde{g}_{k\ell} + \partial_k\tilde{g}_{\ell j} - \partial_\ell\tilde{g}_{jk}), \qquad \left(\dot{a} = \frac{da}{d\tau}\right) \\
\tilde{R}_{jm} &= 2k\tilde{g}_{jm} \\
R_{\mu\nu} &= \begin{pmatrix} R_{00} & 0 \\ 0 & \tilde{R}_{jk} + \tilde{g}_{jk}(a\ddot{a} + 2\dot{a}^2) \end{pmatrix} = \begin{pmatrix} -3\frac{\ddot{a}}{a} & 0 \\ 0 & \tilde{g}_{jk}(a\ddot{a} + 2\dot{a}^2 + 2k) \end{pmatrix} \\
R &= R_{00} - \frac{1}{a^2}\tilde{g}^{jk}R_{jk} = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).
\end{aligned}
$$

B. Equation of motion

1. **Energy-momentum conservation:**

   (a) *Divergence of a tensor:*

$$
\begin{aligned}
D_\mu v^\mu &= \partial_\mu v^\mu + \Gamma^\mu_{\nu\mu}v^\nu = \partial_\mu v^\mu + \frac{\partial_\mu\sqrt{-g}}{\sqrt{-g}}v^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}v^\mu) \\
D_\nu A^{\mu\nu} &= \partial_\nu A^{\mu\nu} + \Gamma^\nu_{\rho\nu}A^{\mu\rho} + \Gamma^\mu_{\rho\nu}A^{\rho\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}A^{\mu\nu}) + \Gamma^\mu_{\rho\nu}A^{\rho\nu}
\end{aligned}
$$

   (b) *Energy-momentum tensor:*

$$
T^{\mu\nu} = (\rho c^2 + p)u^\mu u^\nu - pg^{\mu\nu}
$$

- Scale invariance (EM radiation):
  - Generator of scale transformations $x^\mu \to e^\lambda x^\mu$: T^μ_μ
  - $T^\mu_\mu = \rho c^2 - 3p = 0$
- Matter at rest (cosmic dust): $p = 0$

- Vacuum: $T^{\mu\nu} \sim g^{\mu\nu}$, $p = -\rho c^2$

always $\rho c^2 + 3p \geq 0$

(c) *Continuity equation:*

$$0 = -\partial_\nu p g^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\nu[\sqrt{-g}(\rho c^2 + p)u^\mu u^\nu] + \Gamma^\mu_{\rho\nu}(\rho c^2 + p)u^\rho u^\nu$$

$u^\mu = (1,0,0,0) \implies \mu = 0$ is non-trivial

$$\dot{p} = \frac{1}{\sqrt{-g}}\frac{d}{d\tau}[\sqrt{-g}(\rho c^2 + p)]$$

i. Cosmic dust:

$$\begin{aligned} a^3\dot{p} &= \frac{d}{d\tau}[a^3(\rho c^2 + p)] \implies 0 = \frac{d}{d\tau}(a^3\rho c^2) \\ \rho &\sim \frac{1}{a^3} \end{aligned}$$

ii. Radiation:

$$\begin{aligned} \underbrace{a^3\dot{p}}_{\frac{a^3}{3}\dot{\rho}c^2} &= \frac{d}{d\tau}[a^3\underbrace{(\rho c^2 + p)}_{\frac{4}{3}\rho c^2}] \\ 0 &= \frac{4}{3}\frac{d}{d\tau}(a^3\rho c^2) - \frac{a^3}{3}\dot{\rho}c^2 = 4\dot{a}a^2\rho c^2 + \frac{4}{3}a^3\dot{\rho}c^2 - \frac{a^3}{3}\dot{\rho}c^2 = 4\dot{a}a^2\rho c^2 + a^3\dot{\rho}c^2 = \frac{1}{a}\frac{d}{d\tau}(a^4\rho c^2) \\ \rho &\sim \frac{1}{a^4} \end{aligned}$$

iii. Faster expansion in the first (radiation dominated) phase of the Universe

2. **Einstein equations:**

$$\begin{aligned} R_{00} - \frac{1}{2}R - \Lambda &= 3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 8\pi G T_{\tau\tau} = 8\pi G\rho c^2 \\ \frac{1}{a^2\tilde{g}_{rr}}\left[R_{rr} - \frac{1}{2}g_{rr}(R + 2\Lambda)\right] &= \frac{1}{a^2\tilde{g}_{rr}}\left[R_{rr} + \frac{a^2\tilde{g}_{rr}}{2}(R + 2\Lambda)\right] \\ &= \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) + \Lambda \\ &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda \\ &= \frac{8\pi G}{a^2\tilde{g}_{rr}}T_{rr} = 8\pi G p \end{aligned}$$

3. **Friedmann equation $\frac{1}{2}[\frac{1}{3}(00) + (rr)]$:**

$$\boxed{\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4}{3}\pi G(3p + \rho c^2)}$$

- *Cosmological constant:* negative pressure and modification of the Newtonian limit. $\Lambda = 0$ in the rest
- *Absence of static solution:* $\ddot{a} < 0$, for $3p + \rho c^2 > 0$

- Rate of change of spatial physical distances, $\ell_{ph} = \tilde{\ell}a$ with fixed $\tilde{\ell}$

$$v = \frac{d\ell_{ph}}{d\tau} = \ell_{ph}\frac{\dot{a}}{a} = H\ell_{ph}, \quad H = \frac{\dot{a}}{a} \sim 70km/s/Mpc \leftarrow \text{Hubble's constant}$$

- Hubble's law:

$$\boxed{v = H\ell_{ph}}$$

N.B. $v > c$ for large separation (not in contradiction with special relativity) $\implies$ horizons

- *Evolution of the universe:*
  (a) Expanding at the present, $\dot{a} > 0$
  (b) Deacceleration: $\ddot{a} < 0$, expansion slows down
  (c) For constant expansion rate:

$$a(\tau) = \dot{a}(\tau_{obs})\tau$$

Approximative age of the Universe:

$$\tau_{obs} = \frac{a(\tau)}{\dot{a}(\tau_{obs})} = \frac{1}{H}$$

Slowing expansion rate $\implies$ the true age is shorter

  (d) Big Bang singularity at $\tau = 0$
  (e) Singularity theorems of general relativity: singularity even without assuming homogeneity and isotropy
  (f) Early Universe is in the quantum regime until $a \sim \ell_{Planck}$
  (g) k-dependence:
    i. Flat or open universe: $k = 0$ or $k = -1$ expansion forever

$$\begin{aligned} R_{00} - \frac{1}{2}R &= 3\frac{\dot{a}^2 + k}{a^2} = 8\pi G T_{\tau\tau} = 8\pi G\rho c^2 \\ \dot{a}^2 &= \frac{8\pi G}{3}a^2\rho c^2 - k > 0 \end{aligned}$$

    ii. Closed universe: $k = 1$,
      A. deacceleration continues

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(3p + \rho c^2) \quad \leftarrow \quad \text{Friedmann eq. for } \Lambda = 0$$

      B. contraction after

$$a_{max} = \sqrt{\frac{3}{8\pi G\rho c^2}} \quad \leftarrow \quad \dot{a} = 0$$

C. big crunch at $a = 0$

iii. Critical density at $k = 0$:

$$\dot{a}^2 = \frac{8\pi G}{3} a^2 \rho c^2, \quad \frac{a}{\dot{a}(\tau_{obs})} = \frac{1}{H}, \quad \rightarrow \quad \rho_c c^2 = \frac{3H^2}{8\pi G}$$

(h) Observations: $\rho_{matter} + \rho_\Lambda \approx \rho_c$, $\rho_{matter} \approx 0.27\rho_c$, $\rho_\Lambda \approx 0.73\rho_c$

C. Doppler effect

1. **Relativistic:** Monochromatic plane wave, $k^\mu = (\frac{\omega_0}{c}, \mathbf{k})$, $k^2 = \frac{m^2c^2}{\hbar^2}$

   Source is moving with velocity $\mathbf{v}$ with respect to the observer in flat space-time

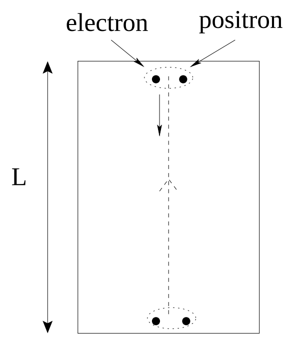
   In the observer's reference frame:

$$\begin{aligned} k^2 &= \frac{\omega_0^2}{c^2} - \mathbf{k}^2 \implies |\mathbf{k}| = \sqrt{\frac{\omega_0^2}{c^2} - k^2} \implies \mathbf{vk} = v\frac{\omega_0}{c}\sqrt{1 - \frac{m^2c^4}{\hbar^2\omega_0^2}} \cos\theta \\ \omega &= \frac{\omega_0 - \mathbf{vk}}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \omega_0 \frac{1 - \frac{v}{c}\sqrt{1 - \frac{m^2c^4}{\hbar^2\omega_0^2}}\cos\theta}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} \end{aligned}$$

   Broadening of spectral lines of distant stars

2. **Time independent gravitational field:**

   - e^-e^+ pair at rest at $\mathbf{x}_1$ and falls to $\mathbf{x}_2$



$$E_1 = 2mc^2, \quad E_2 = E_1 + 2m\Delta U \implies E_2 = E_1\left(1 + \frac{\Delta U}{c^2}\right)$$

   - e^-e^+ pair annihilates at $\mathbf{x}_2$ and the photon propagates back to $\mathbf{x}_1$

$$\begin{aligned} E_2 &= \hbar\omega_2, \quad E_1 = \hbar\omega_1 \\ \frac{\omega_2}{\omega_1} &= \frac{E_2}{E_1} = 1 + \frac{\Delta U}{c^2} \\ \frac{\Delta\omega}{\omega_1} &= \frac{\Delta U}{c^2} \end{aligned}$$

- Sun:

$$r_1 = \infty, \quad U(r) = \frac{GM}{r}, \quad \frac{\Delta\omega}{\omega} = \frac{GM}{rc^2} \sim 2 \cdot 10^{-6} \text{ at the surface}$$

3. **Time dependent gravitational field:**

- A light signal is emitted at $p_0 = (ct_e, \mathbf{r})$ and received at $p = (ct_o, \mathbf{0})$ by stationary source and receptor
- World line of the signal:

$$\begin{aligned} ds^2 &= d\tau^2 - a^2(\tau)\left[d\chi^2 + h^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ d\tau &= a(\tau)d\chi, \quad \int_{\tau_e}^{\tau_o} \frac{d\tau}{a(\tau)} = \chi \end{aligned}$$

- Emission of $dn_e = d\tau_e\omega_e$ light periods and reception of $dn_o = d\tau_o\omega_o$ periods

$$dn_e = dn_o \implies d\tau_e\omega_e = d\tau_o\omega_o$$

- Red shift: $\lambda = \frac{2\pi c}{\omega}$

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\omega_e}{\omega_o} - 1 = \frac{d\tau_o}{d\tau_e} - 1 = \frac{a(\tau_o)}{a(\tau_e)} - 1$$

$\nearrow$

stationarity:

$$\chi = \int_{\tau_e}^{\tau_o} \frac{d\tau}{a(\tau)} = \int_{\tau_e + d\tau_e}^{\tau_o + d\tau_o} \frac{d\tau}{a(\tau)} \implies \frac{d\tau_e}{a(\tau_e)} = \frac{d\tau_o}{a(\tau_o)}$$

- Slowly changing $a(\tau)$: $a_o = a(\tau_o)$

$$\begin{aligned} z &= \frac{a_o}{a_o + (\tau_e - \tau_o)\dot{a}_o + \frac{1}{2}(\tau_e - \tau_o)^2\ddot{a}_o + \cdots} - 1 \\ &= \frac{\dot{a}_o}{a_o}(\tau_o - \tau_e) + \left[\left(\frac{\dot{a}_o}{a_o}\right)^2 - \frac{\ddot{a}_o}{2a_o}\right](\tau_o - \tau_e)^2 + \cdots \\ &= \frac{\dot{a}_o}{a_o}(\tau_o - \tau_e) + \left(\frac{\dot{a}_o}{a_o}\right)^2\left(1 - \frac{a_o\ddot{a}_o}{2\dot{a}_o^2}\right)(\tau_o - \tau_e)^2 + \cdots \end{aligned}$$

- Deceleration parameter:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \sim 1$$

- Hubble's law:

$$z \approx \frac{\dot{a}_o}{a_o}(\tau_o - \tau_e) \approx \frac{\dot{a}_o}{a_o}\ell = H\ell.$$

D. Horizons

1. **Conformal flat Universe:**

$$ds^2 = a^2(\eta)\left[d\eta^2 - d\chi^2 - h^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad \eta = \int \frac{d\tau}{a(\tau)}$$

2. **Existence of horizons:**

$$\int_\epsilon^\infty \frac{d\tau'}{a(\tau')} < \infty \quad \Longrightarrow \quad \text{no signal from the whole universe}$$

   (a) $k = -1$ or 0: Dust dominated universe $a(\tau) = \mathcal{O}\left(\tau^{2/3}\right)$ for $\tau \approx 0 \quad \Longrightarrow \quad$ horizon

   (b) $k = 1$:

      i. dust: horizon disappears after a_{max}

      ii. radiation: remains present

   (c) Problem: no relaxation between acausal regions

   (d) Cosmic Microwave Background:

      i. Inhomogeities: quantum fluctuations at the Big Bang

      ii. Weak inhomogeneities:

      The Universe started in an unusually homogeneous state $\Longrightarrow$ inflationary model

E. Evolution of the universe

1. **Planck-era:** $t \approx 10^{-43}$s, $T \approx 10^{31}$K, $\rho \approx 10^{92}$ gm/cm^3

   (a) Quantum gravity (?)

   (b) Thermodynamical equilibrium (?)

   (c) Radiation dominated

   (d) Matter-anti matter domains?

2. **Strong interaction appears:** $t \approx 10^{-36}$s

3. **Electromagnetic and weak interactions separate:** $t \approx 10^{-12}$s

4. **Hadron formation:** $t \approx 10^{-6}$s, quark confinement

5. **Neutrinos decouple:** $t \approx 1$s

   (a) Until now: ν, γ, $e^{\pm}$, p, n

(b) $\nu\bar{\nu} \not\leftrightarrow e^- e^+ \implies \nu$ decouple and follow a passive red shift in the rest of the time.

(c) β-decay $n \to p + e^- + \nu$ becomes irreversible $\implies$ stable p

6. e^+ **disappear:** $t \approx 4$s, $k_B T \sim 2m_e$ $e^- e^+ \not\leftrightarrow \gamma$, photons heat up slightly

7. **Nucleosynthesis:** $10s < t < 10min$, thermal energy reaches the nuclear binding scale

   (a) He era: few minutes, ^{4}He nuclei are produced, no stable elements with $Z < 8$

   (b) later ^{2}H, ^{3}He and ^{7}Li

8. **Matter radiation equilibrium:** $t \approx 4 \cdot 10^5$year

9. **Neutral atoms:** $t \approx 10^6$year, $k_B T \approx 5 \cdot 10eV$

   (a) neutral universe

   (b) photons decouple, $T \sim 3000K$, today $2.7K$ Cosmic Microwave Background

10. **End of the quantum-driven evolution:** decoupling of matter and radiation

    (a) loss of radiation pressure

    (b) gravitational instabilities for $M > 10^5 M_{sun}$, $M_{sun} \approx 2 \cdot 10^{33}$g

    (c) galaxy formation

11. **Hadronic matter domination:** $t \approx 10^3 - 10^7$years

12. **Today:** $t \sim 14 \cdot 10^9$years

VIII. OPEN QUESTIONS

1. **Action:** best tested predictions in Schwarzschild geometry, several candidate theories are indistinguishable

   Einstein-Hilbert action:

   - *Galactical scale:* Galactical rotation, velocity distribution $\implies$ stronger gravitational force

     (a) dark matter?

     (b) corrections to the action? $\leftarrow$ renormalization group

   - *Cosmological scale:*

     (a) $\rho \sim \rho_{crit}$

     (b) deficit $\rho_{crit} - \rho_{matter} = \rho_\Lambda$

   i. dark energy?
   ii. corrections to the action? $\leftarrow$ renormalization group

- *Consistency (?):*
  (a) $H = 73.5 \pm 0.5$km/sec/Mpc (nearby galaxies)
  (b) $H = 67.4 \pm 1.4$km/sec/Mpc (inhomogeneity of CMB at galactical scales)

2. **Quantum gravity:**
   (a) No experimental indications
   (b) Theoretical difficulties:
      - Emergence of classical space-time coordinates, input for quantization of matter
      - Renormalizability? $\leftarrow$ renormalization group (may not be needed)

3. **Thermodynamics:** The thermodynamical laws are modified
   - Short long distances (information loss at horizon)
   - Long distances (long range gravitational force)

Oral exam questions:

1. Classical field theory
2. Gauge theories
3. Covariant derivative, curvature tensor, metric admissibility
4. Einstein-Hilbert action, equations of motion
5. Horizon in Schwarzschild geometry, regularity
6. Orbits in Schwarzschild geometry