Quantum Mechanics II.

Janos Polonyi
University of Strasbourg

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Video: `https://drive.google.com/drive/folders/11u3UNyrS6b-jXhRChaWBC3SXaZOz9yu_?usp=sharing`

Contents

Quantum mechanics is usually taught on four different levels:

1. Basic ideas, simple examples for a one dimensional particle, particle in spherical potential
2. More realistic, three dimensional cases with few particles ←
3. Several particles, relativistic effects (Quantum Field Theory)
4. Fundamental issues, challenges, paradoxes and interpretation of the quantum world

I. PERTURBATION EXPANSION

$$H = H_0 + gH_1$$

A. Stationary perturbations

- **Example:** Atom in an EM field
- **Goal:** solve stationary Schrödinger equation, $H|\psi_n\rangle = E_n|\psi_n\rangle$
- **Expansion in g:**

$$\begin{aligned}
|\psi_n\rangle &= |\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots , \\
E_n &= E_n^{(0)} + gE_n^{(1)} + g^2E_n^{(2)} + \cdots , \\
0 &= (H_0 + gH_1 - E_n^{(0)} - gE_n^{(1)} - g^2E_n^{(2)} - \cdots)(|\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots) \\
&= g^0\left(H_0|\psi_n^{(0)}\rangle - E_n^{(0)}|\psi_n^{(0)}\rangle\right) \\
&\quad + g\left(H_0|\psi_n^{(1)}\rangle + H_1|\psi_n^{(0)}\rangle - E_n^{(1)}|\psi_n^{(0)}\rangle - E_n^{(0)}|\psi_n^{(1)}\rangle\right) \\
&\quad + g^2\left(H_0|\psi_n^{(2)}\rangle + H_1|\psi_n^{(1)}\rangle - E_n^{(2)}|\psi_n^{(0)}\rangle - E_n^{(1)}|\psi_n^{(1)}\rangle - E_n^{(0)}|\psi_n^{(2)}\rangle\right) + \cdots
\end{aligned}$$

Orders one-by-one:

$$\begin{aligned}
\mathcal{O}\left(g^0\right): &\quad H_0|\psi_n^{(0)}\rangle &= E_n^{(0)}|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(1)}\rangle &= (E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g^2\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(2)}\rangle &= (E_n^{(1)} - H_1)|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle \\
\mathcal{O}\left(g^k\right): &\quad (H_0 - E_n^{(0)})|\psi_n^{(k)}\rangle &= (E_n^{(1)} - H_1)|\psi_n^{(k-1)}\rangle + E_n^{(2)}|\psi_n^{(k-2)}\rangle + \cdots + E_n^{(k)}|\psi_n^{(0)}\rangle
\end{aligned}$$

- **Zeroth order:** unperturbed stationary states, $|\psi_n^{(0)}\rangle$, $\langle\psi_m^{(0)}|\psi_n^{(0)}\rangle = \delta_{mn}$
- **Higher order:** no unique solution

1. $(H_0 - E_n^{(0)})^{-1}$ does not esists in the null space of $H_0 - E_n^{(0)}$

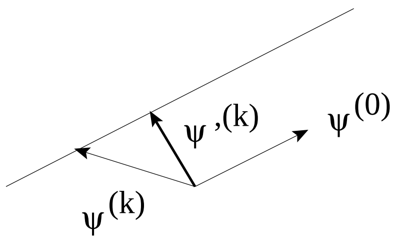
$$\nwarrow$$

$$\mathcal{N}_A = \{|\psi\rangle | A|\psi\rangle = 0\}$$

2. Another way to see: if $|\psi_n^{(k)}\rangle$ ($k > 0$ is a solution $\Longrightarrow |\psi_n'^{(k)}\rangle = |\psi_n^{(k)}\rangle + c|\psi_n^{(0)}\rangle$ is another solution

$$\begin{aligned}(H_0 - E_n^{(0)})(|\psi_n^{(k)}\rangle + c|\psi_n^{(0)}\rangle &= (H_0 - E_n^{(0)})|\psi_n^{(k)}\rangle \\ &= (E_n^{(1)} - H_1)|\psi_n^{(k-1)}\rangle + E_n^{(2)}|\psi_n^{(k-2)}\rangle + \cdots + E_n^{(k)}|\psi_n^{(0)}\rangle\end{aligned}$$

3. Unique solution: choose $c = -\langle\psi_n^{(0)}|\psi_n^{(k)}\rangle \Longrightarrow \langle\psi_n^{(0)}|\psi_n'^{(k)}\rangle = \langle\psi_n^{(0)}|(|\psi_n^{(k)}\rangle - |\psi_n^{(0)}\rangle\langle\psi_n^{(0)}|\psi_n^{(k)}\rangle) = 0$



- **First order:** One writes $|\psi_n^{(1)}\rangle = \sum_\ell c_{n,\ell}|\psi_\ell^{(0)}\rangle$

$$\begin{aligned}\langle\psi_k^{(0)}| \quad (H_0 - E_n^{(0)})|\psi_n^{(1)}\rangle &= (E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\ \sum_\ell c_{n,\ell}\langle\psi_k^{(0)}|(H_0 - E_n^{(0)})|\psi_\ell^{(0)}\rangle &= \langle\psi_k^{(0)}|(E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\ \sum_\ell c_{n,\ell}\langle\psi_k^{(0)}|(E_k^{(0)} - E_n^{(0)})|\psi_\ell^{(0)}\rangle &= \langle\psi_k^{(0)}|(E_n^{(1)} - H_1)|\psi_n^{(0)}\rangle \\ \langle\psi_k^{(0)}|\psi_\ell^{(0)}\rangle = \delta_{k,\ell} \quad \to \quad (E_k^{(0)} - E_n^{(0)})c_{n,k} &= E_n^{(1)}\delta_{k,n} - H_{1kn} \quad \leftarrow \quad \langle\psi_k^{(0)}|H_1|\psi_n^{(0)}\rangle\end{aligned}$$

Solution:

$$\boxed{c_{n,k} = \begin{cases}\frac{H_{1kn}}{E_n^{(0)} - E_k^{(0)}}, & k \neq n \\ 0 & k = n,\end{cases}}$$

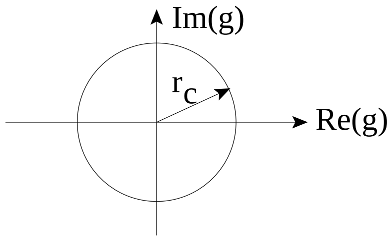
$$\boxed{E_n^{(1)} = H_{1nn}}$$

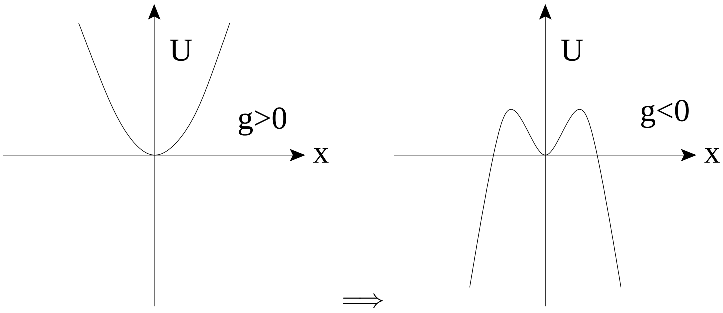
- **Necessary conditions:**

$$\begin{aligned}g\langle\psi_n^{(0)}|H_1|\psi_n^{(0)}\rangle &\ll E_n^{(0)} \\ g|\langle\psi_k^{(0)}|H_1|\psi_n^{(0)}\rangle| &\ll |E_n^{(0)} - E_k^{(0)}|.\end{aligned}$$

- **Convergence radius:** $r_c = 0$

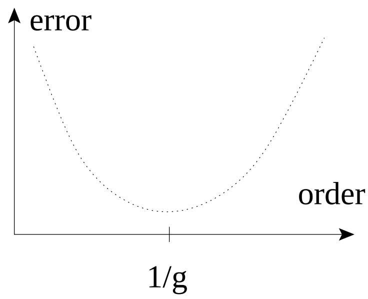
$$H = \frac{p^2}{2m} + U(x), \qquad U(x) = \frac{m\omega^2}{2}x^2 + \frac{g}{4!}x^4, \quad g \to -g \ ?$$





- **Asymptotic convergence:**

  1. Defintion: $f_N(g) = \sum_{n=0}^{N} f_n g^n \rightarrow_{as} f(g)$ if $\frac{f_N(g)-f(g)}{g^N} \rightarrow 0$ as $g \rightarrow 0$
  2. Quantum mechanical systems: $|f_N(g) - f(g)|$ starts to grow at $N = \mathcal{O}\left(\frac{1}{g}\right)$ (QED: $g = \frac{1}{137}$)



- **Degenerate perturbations:**

  1. *Problem:*
     (a) $H_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle \Longrightarrow |\psi_n^{(0)}\rangle$ is ill defined within the degenerate subspace
     (b) *The higher orders* in $|\psi_n\rangle = |\psi_n^{(0)}\rangle + g|\psi_n^{(1)}\rangle + g^2|\psi_n^{(2)}\rangle + \cdots$ are not small
     (c) *Singularity* at $g = 0$
     (d) $g|\langle\psi_k^{(0)}|H_1|\psi_n^{(0)}\rangle| \ll |E_n^{(0)} - E_k^{(0)}|$ is violated
  2. *Solution:* diagonalize H_1 within the degenerate subspace

3. *Degeneracy:* $E_k^{(0)} = E_\ell^{(0)}$ for $1 \le k, \ell \le N \ll \dim(H)$

$$H_1 = \begin{pmatrix} \begin{pmatrix} H_{1\ 1,1} & 0 & \cdots & 0 \\ 0 & H_{1\ 2,2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & H_{1\ N,N} \end{pmatrix} & B \\ B^\dagger & H_1' \end{pmatrix},$$

and suppose that $H_{1\ j,j} \neq H_{1\ k,k}$ for $j \neq k$.

4. *Secular equation:*
   - Eigenvalues: $A|\psi\rangle = a|\psi\rangle \quad \leftrightarrow \quad (A - a\mathbb{1})|\psi\rangle = 0 \quad \leftrightarrow \quad \det(A - a\mathbb{1}) = 0$
   - $\det[H_{1\ k\ell} - \delta_{k,\ell} E_k^{(1)}] = 0 \leftrightarrow E_k^{(1)} = H_{1\ kk}$

5. *Higher orders are regular:*

$$\begin{aligned} |\psi_k\rangle &= |\psi_k^{(0)}\rangle + \mathcal{O}(g) \\ E_k &= E_k^{(0)} + gH_{1kk} + \mathcal{O}\left(g^2\right) \end{aligned}$$

6. *Physical importance:*
   (a) The increased sensitivity of the eigenfunctions on the perturbations: large $\left|\frac{\langle\psi_n^{(0)}|H_1|\psi_k^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}\right|$ Weak interactions become more important for exact or approximate degeneracy

   (b) An atom interacting with an ideal gas in box L:
   - Typical level spacing of the gas: $\Delta E \sim \frac{\hbar^2}{mL^2}$
   - "Small" parameter of the perturbation expansion:

$$\frac{gH_{1kn}}{\frac{\hbar^2}{mL^2}} \sim 10^{54} mL^2 gH_{1kn} > 1$$

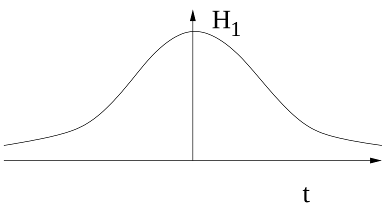
   (m, L expressed gram and centimeter)
   - Classical limit in quantum mechanics
   - Relaxation in statistical physics (starting point of Statistical Mechanics)

B. Time dependent perturbations

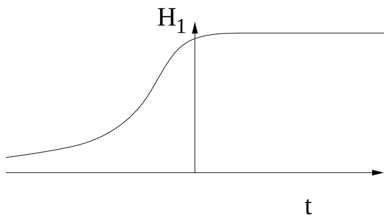
1. **Example:** Atom in time-dependent EM field

2. **Goal:**

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle, \qquad H = H_0 + gH_1(t),$$



or



3. **Typical problem:**

   Initial condition: $|\psi(t=t_i)\rangle = e^{-i\frac{t_i}{\hbar}E_k^{(0)}}|\psi_n^{(0)}\rangle$, $H_0|\psi_k^{(0)}\rangle = E_k^{(0)}|\psi_k^{(0)}\rangle$

   Transition probability:

$$\boxed{P_{n\to k}(t) = |\langle\psi_k^{(0)}|\psi(t)\rangle|^2}$$

4. **Time-dependence of the state:**

$$|\psi(t)\rangle = \sum_k c_k(t)|\psi_k^{(0)}(t)\rangle$$

$\nearrow$ interaction $\qquad$ $\nwarrow$ unperturbed dynamics

   Time dependent basis:

$$\begin{aligned} i\hbar\partial_t|\psi_k^{(0)}(t)\rangle &= H_0|\psi_k^{(0)}(t)\rangle \\ H_0|\psi_k^{(0)}(0)\rangle &= E_k^{(0)}|\psi_k^{(0)}(0)\rangle \\ |\psi_k^{(0)}(t)\rangle &= e^{-i\frac{t}{\hbar}E_k^{(0)}}|\psi_k^{(0)}(0)\rangle \end{aligned}$$

5. **Schrödinger equation:**

$$\begin{aligned} i\hbar\partial_t|\psi(t)\rangle &= [H_0 + gH_1(t)]|\psi(t)\rangle \\ i\hbar\sum_k(\partial_t c_k(t)|\psi_k^{(0)}(t)\rangle + c_k(t)\underbrace{\partial_t|\psi_k^{(0)}(t)\rangle}_{\frac{1}{i\hbar}H_0|\psi_k^{(0)}(t)\rangle}) &= [H_0 + gH_1(t)]|\sum_k c_k(t)|\psi_k^{(0)}(t)\rangle \\ \langle\psi_\ell^{(0)}| \qquad i\hbar\sum_k \partial_t c_k(t)|\psi_k^{(0)}(t)\rangle &= gH_1(t)|\sum_k c_k(t)|\psi_k^{(0)}(t)\rangle \\ i\hbar\partial_t c_\ell(t) &= g\sum_k \underbrace{\langle\psi_\ell^{(0)}(t)|H_1(t)|\psi_k^{(0)}(t)\rangle}_{H_{1\ell k}(t)} c_k(t) = g\sum_k H_{1\ell k}(t)c_k(t) \end{aligned}$$

   Order by order:

$$\begin{aligned} c_\ell(t) &= \sum_k g^k c_\ell^{(k)}(t) \\ \mathcal{O}\left(g^0\right): i\hbar\partial_t c_\ell^{(0)}(t) &= 0 \\ \mathcal{O}\left(g^m\right): i\hbar\partial_t c_\ell^{(m)}(t) &= \sum_k H_{1\ell k}(t)c_k^{(m-1)}(t) \end{aligned}$$

6. **Factorizable interaction:** $H_1(t) = f(t)H'$

$$\begin{aligned}
c_k^{(0)}(t) &= c_k^{(0)}(t_i) = \delta_{k,n} \quad \rightarrow \quad c_k^{(m)}(t_i) = \delta_{m,0}\delta_{k,n} \\
c_k^{(1)}(t) &= c_k^{(1)}(t_i) - \frac{i}{\hbar}\int_{-\infty}^{t} dt' H_{1kn}(t') = -\frac{i}{\hbar}\int_{-\infty}^{t} dt' H_{1kn}(t') \\
c_k(t) &= \delta_{k,n} - i\frac{g}{\hbar}\int_{-\infty}^{t} dt' H_{1kn}(t') + \mathcal{O}\left(g^2\right) \\
H_{1\ell k}(t) &= e^{i\frac{t}{\hbar}E_\ell^{(0)}}\langle\psi_\ell^{(0)}(0)|H'|\psi_k^{(0)}(0)\rangle e^{-i\frac{t}{\hbar}E_k^{(0)}} f(t) = H'_{\ell k}e^{i\omega_{\ell k}t}f(t) \\
\hbar\omega_{\ell k} &= E_\ell^{(0)} - E_k^{(0)}, \qquad H'_{\ell k} = \langle\psi_\ell^{(0)}(0)|H'|\psi_k^{(0)}(0)\rangle \\
c_k(t) &= \delta_{k,n} - i\frac{gH'_{kn}}{\hbar}\int_{-\infty}^{t} dt' f(t')e^{i\omega_{k,n}t'} + \mathcal{O}\left(g^2\right)
\end{aligned}$$

7. Transition probability:

$$\boxed{P_{n(\neq k)\to k}(t) = |c_k(t)|^2 = \left|\frac{gH'_{kn}}{\hbar}\right|^2 \left|\int_{-\infty}^{t} dt' f(t')e^{i\omega_{kn}t'}\right|^2 + \mathcal{O}\left(g^3\right)}$$

8. **Example:** Sinusoidal perturbation is turned on suddenly

- Transition amplitude:

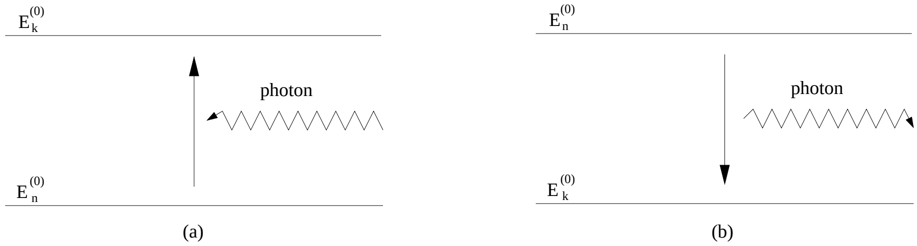
$$\begin{aligned}
f(t) &= \begin{cases} 2\cos\omega t, & \omega > 0 \quad t > 0 \\ 0 & t < 0, \end{cases} \\
c_{k\neq n} &= -\frac{igH'_{kn}}{\hbar}\int_0^t dt' e^{i\omega_{k,n}t'}\left(e^{i\omega t'} + e^{-i\omega t'}\right) \\
&= -\frac{gH'_{kn}}{\hbar}\left(\frac{e^{i(\omega_{k,n}-\omega)t} - 1}{\omega_{k,n} - \omega} + \frac{e^{i(\omega_{k,n}+\omega)t} - 1}{\omega_{k,n} + \omega}\right) \qquad \left[\int dte^{i\omega t} = \frac{e^{i\omega t}}{i\omega}\right] \\
e^{i\phi} - 1 &= e^{i\frac{\phi}{2}}\left(e^{i\frac{\phi}{2}} - e^{-i\frac{\phi}{2}}\right) = 2ie^{i\frac{\phi}{2}}\sin\frac{\phi}{2} \\
c_{k\neq n} &= -\frac{2igH'_{k,n}}{\hbar}\left(\frac{e^{\frac{i}{2}(\omega_{kn}-\omega)t}\sin\frac{\omega_{kn}-\omega}{2}t}{\omega_{kn} - \omega} + \frac{e^{\frac{i}{2}(\omega_{kn}+\omega)t}\sin\frac{\omega_{kn}+\omega}{2}t}{\omega_{kn} + \omega}\right)
\end{aligned}$$

- Transition probability for $\omega \approx |\omega_{kn}|$:

$$\begin{aligned}
P &\approx \begin{cases} P^+ & \omega_{kn} > 0 \text{ (absorption)}, \\ P^- & \omega_{kn} < 0 \text{ (emission)}, \end{cases} \\
P^\pm_{n\to k} &= \frac{4g^2|H'_{k,n}|^2}{\hbar^2(\omega_{kn} \mp \omega)^2}\sin^2\frac{1}{2}(\omega_{kn} \mp \omega)t.
\end{aligned}$$

- Small and large t asymptotics:

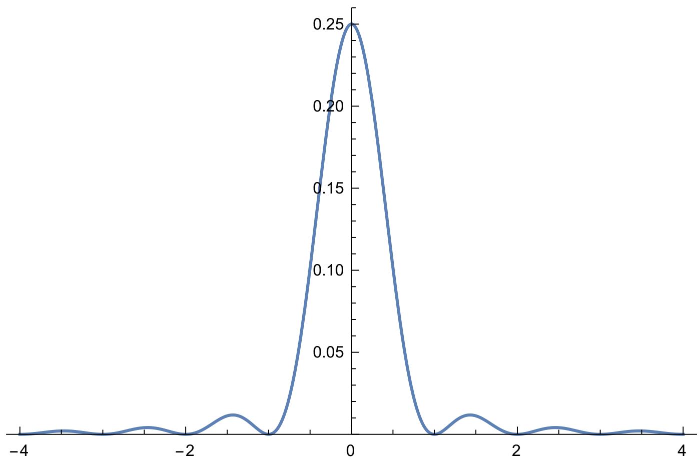
$$\begin{aligned}
t \approx 0: \ P^\pm_{n\to k} &= t^2\frac{|gH_{kn}|^2}{\hbar^2} \\
t \to \infty: \ w^\pm_{n\to k} &= \frac{P^\pm_{n\to k}}{t} = \frac{2\pi|gH'_{kn}|^2}{\hbar^2}\frac{2}{\pi}\frac{\sin^2\frac{t(\omega_{kn}\mp\omega)}{2}}{t(\omega_{kn} \mp \omega)^2} = \frac{2\pi|gH'_{kn}|^2}{\hbar^2}\delta_t(\omega_{kn} \mp \omega)
\end{aligned}$$



absorption: $P^+_{n\to k} = P(n+\gamma\to k)$ emission: $P^-_{n\to k} = P(n\to k+\gamma)$

$$\delta(x) = \frac{2}{\pi}\lim_{\eta\to\infty}\frac{\sin^2\frac{\eta x}{2}}{\eta x^2}$$

$x = \frac{t(\omega_{kn}\mp\omega)}{2\pi}$, $\frac{\sin^2\pi x}{(2\pi x)^2}$:



C. Non-exponential decay rate

1. **Time dependence:**

- Initial state: $|\psi_{in}\rangle$ at $t=0$, $H|\psi_{in}\rangle \neq E|\psi_{in}\rangle$
- Time evolution:

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle$$

- Probability to preserve the initial state:

$$P_0(t) = |A(t)|^2$$

- Persistence amplitude:

$$A(t) = \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle.$$

- The decay is usually not exponential and has short, intermediate and long time regimes.

2. **Short time regime:**

- Persistence amplitude:

$$
\begin{aligned}
A(t) &= 1 - \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle + \mathcal{O}\left(t^3\right) \\
P_0(t) &= \left(1 - \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle\right)\left(1 + \frac{it}{\hbar}\langle\psi_{in}|H|\psi_{in}\rangle - \frac{t^2}{2\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle\right) + \mathcal{O}\left(t^3\right) \\
&= 1 + \frac{t^2}{\hbar^2}\langle\psi_{in}|H|\psi_{in}\rangle^2 - \frac{t^2}{\hbar^2}\langle\psi_{in}|H^2|\psi_{in}\rangle + \mathcal{O}\left(t^3\right) \\
&= 1 - \frac{t^2}{t_Z^2} + \mathcal{O}\left(t^3\right), \qquad t_Z = \frac{\hbar}{\sqrt{\langle\psi_{in}|(H - \langle\psi_{in}|H|\psi_{in}\rangle)^2|\psi_{in}\rangle}} \quad \leftarrow \quad \text{Zeno time}
\end{aligned}
$$

3. **Intermediate time regime:**

- Projection operators: $P^\dagger = P$, $P^2 = P$ (spectrum$=\{0,1\}$)

  (a) Longitudinal:

$$
\begin{aligned}
L &= |\psi_{in}\rangle\langle\psi_{in}|, \qquad \langle\psi_{in}|\psi_{in}\rangle = 1 \\
L|\psi\rangle &= |\psi_{in}\rangle\langle\psi_{in}|\psi\rangle \\
L^2|\psi\rangle &= |\psi_{in}\rangle\langle\psi_{in}|\psi_{in}\rangle\langle\psi_{in}|\psi\rangle = |\psi_{in}\rangle\langle\psi_{in}|\psi\rangle = L = |\psi\rangle
\end{aligned}
$$

  (b) Transverse:

$$
\begin{aligned}
T &= \mathbb{1} - L \\
T^2 &= (\mathbb{1} - L)(\mathbb{1} - L)| = \mathbb{1} - 2L + L^2 = T \\
\langle\psi_{in}|T|\psi\rangle &= \langle\psi_{in}|(\mathbb{1} - |\psi_{in}\rangle\langle\psi_{in}|)|\psi\rangle = 0
\end{aligned}
$$

- Separation of the longitudinal and transverse parts of the state:

$$
\begin{aligned}
|\psi(t)\rangle &= (\underbrace{L + T}_{\mathbb{1}})e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle \\
&= |\psi_{in}\rangle\langle\psi_{in}|e^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle + Te^{-\frac{i}{\hbar}Ht}|\psi_{in}\rangle \\
&= |\psi_{in}\rangle A(t) + |\phi(t)\rangle
\end{aligned}
$$

$\nearrow$

decay product, $\langle\psi_{in}|\phi(t)\rangle = 0$

- Functional equation for the persistence ampitude:

$$
\begin{aligned}
\langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\psi(t)\rangle &= \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\psi_{in}\rangle A(t) + \langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\phi(t)\rangle \\
A(t + t') &= A(t)A(t') + \underbrace{\langle\psi_{in}|e^{-\frac{i}{\hbar}Ht'}|\phi(t)\rangle}_{\text{re}-\text{excitation}}
\end{aligned}
$$

- Without re-excitation: $A(t+t') = A(t)A(t') \Longrightarrow A(t) = A(0)e^{-\frac{t}{\tau}}$
- Evolution of the decay product back to the undecayed state: deviation from the exponential decay
- Irreversibility:
  (a) $H^\dagger = H \Longrightarrow$ there is always a regenerated undecayed state component:

$$\begin{aligned} P^{\pm}_{n\to k} &= \frac{4g^2|H'_{k,n}|^2}{\hbar^2(\omega_{kn}\mp\omega)^2}\sin^2\frac{1}{2}(\omega_{kn}\mp\omega)t \\ P^{+}_{n\to k} &= P^{-}_{k\to n} \end{aligned}$$

  (b) Irreversibility, non-unitary time evolution is needed to arrive at exponential decays
- Spectral representation:
  (a) Spectral function: $H|n\rangle = E_n|n\rangle$

$$\begin{aligned} |\psi_{in}\rangle &= \mathbb{1}|\psi_{in}\rangle = \sum_n |n\rangle\langle n|\psi_{in}\rangle \\ A(t) &= \sum_n |\langle n|\psi_{in}\rangle|^2 e^{-\frac{i}{\hbar}E_n t} \\ &= \sum_n |\langle n|\psi_{in}\rangle|^2 \int dE\delta(E-E_n)e^{-\frac{i}{\hbar}Et} \\ &= \int dE \underbrace{\sum_n |\langle n|\psi_{in}\rangle|^2\delta(E-E_n)}_{\rho(E)} e^{-\frac{i}{\hbar}Et} = \int dE\rho(E)e^{-\frac{i}{\hbar}Et} \end{aligned}$$

  (b) $A(t)$ and $\rho(E)$ are related by Fourier transformation
  (c) "Uncertainty relation": the width of $A(t)$ and $\rho(E)$ are inversely proportional
  (d) There is no universal decay law
  (e) Exponential decay: Lorentzian spectral weight,

$$\rho(E) = \frac{\Delta E}{\pi[(E-E_0)^2+\Delta E^2]} \quad\to\quad A(t) = e^{-i\frac{E_0}{\hbar}t}e^{-\frac{\Delta E}{\hbar}|t|}$$

  (f) Natural line width of atomic spectra:
    i. Partial resummation of the perturbation series of QED
    ii. Decay of excited state $\Longrightarrow$ finite life-time $\Longrightarrow E \to E - i\frac{\hbar}{\tau}$, $e^{-\frac{i}{\hbar}Et} \to e^{-\frac{i}{\hbar}(E-i\frac{\hbar}{\tau})t} = e^{-\frac{i}{\hbar}Et}e^{-\frac{t}{\tau}}$

4. **Long time regime:**
  (a) Bundedness of the Hamiltonian from below: $\rho(E) = 0$ for $E < E_0$
  (b) Shrunk of the support of a Lorentzian spectral function $\rho(E) = 0 \Longrightarrow$ spread of $A(t)$
  (c) Slower than exponential decay rate for long time

D. Quantum Zeno-effect

1. **Zeno:** (b. Elea, 488BC) Achilles can not pass a tortoise!

2. **Quantum Zeno effect:** (short time, the parabolic decay regime)

   - We observe the system at times $j\Delta t$, $\Delta t = t/n$, $j = 1, \ldots, n$
   - Schrödinger equation is local in time $\Longrightarrow$ the eventual decays are independent

$$\begin{aligned} i\hbar\partial_t|\psi(t)\rangle &= H|\psi(t)\rangle \\ |\psi((j+1)\Delta t)\rangle &= e^{-\frac{i}{\hbar}\Delta t H}|\psi(j\Delta t)\rangle \\ P_0(t+\Delta t) &= P_0(\Delta t)P_0(t) \end{aligned}$$

   - Probability of not having decay:

$$\begin{aligned} P_0(t) &= P_0^n(\Delta t) \\ &= \left[1 - \left(\frac{t}{nt_Z}\right)^2 + \mathcal{O}\left(n^{-3}\right)\right]^n \\ &= e^{n\ln[1-(\frac{t}{nt_Z})^2+\mathcal{O}\left(n^{-3}\right)]} \to 1 \end{aligned}$$

   - Continuously monitored radioactive atom does not decay:

     (a) Undecayed state is completely regenerated by the collapse of the wave function (observations)

     (b) Wave function has no time to spread, an $\mathcal{O}\left(\Delta t^2\right)$ effect

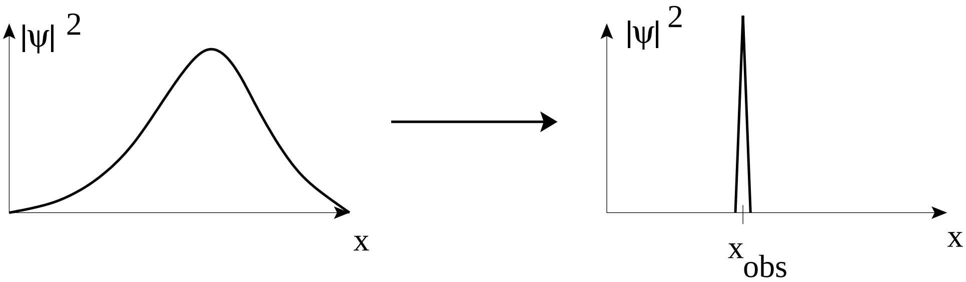
   - Watched pot paradox: (water does not boil in a continuously watched pot)

3. **Measurement process:**

   - Microscopic $\Longrightarrow$ macroscopic transition (e.g. tracks in Wilsons's could chamber)
   - Selection of a spectral element of the observable, a_n

     $A|n\rangle = a_n|n\rangle$, $|\psi\rangle = \sum_n c_n|n\rangle$, $\langle\psi|\psi\rangle = 1$, $\langle\psi|A|\psi\rangle = \sum_n |c_n|^2 a_n$

   - Collapse of the wave function



- Non-determinsitic choice of x_{obs}
  - QM: averages only.
  - No deterministic, causal theory for a single event
- Reality???
- Quantum Bar Kokhba game
- Hidden parameter theories:
  - Classical description of each microscopical quantity by the help of so far unobserved classical degrees of freedom
  - Non-local $\Longrightarrow$ acausality
  - Contextuel $\Longrightarrow$ no mathematical structure
    - Three observables, A, B and C, $[A, B] = [A, C] = 0$, $[B, C] \neq 0$
    - The value of A depends on whether we measure B or C simultaneously.

E. Time-energy uncertainty principle

1. **Heisenberg's uncertainty principle:**

(a) Algebraic derivation:

$$\begin{aligned}
[A, B] &= iC, \quad A = A^\dagger, \quad B = B^\dagger, \quad C = C^\dagger \\
A_0 &= A - \langle A \rangle, \quad B_0 = B - \langle B \rangle, \quad \langle A \rangle = \begin{cases} \langle \psi | A | \psi \rangle & \textit{pure state} \\ \mathrm{Tr}\rho A & \textit{mixed state} \end{cases}, \quad [A_0, B_0] = iC \\
\Delta A^2 &= \langle A_0^2 \rangle = \langle A^2 \rangle - \langle A \rangle^2, \quad \Delta B^2 = \langle B_0^2 \rangle = \langle B^2 \rangle - \langle B \rangle^2
\end{aligned}$$

Non-negative norm: $O = A_0 + ixB_0$, $x \in \mathcal{R}$

$$\begin{aligned}
\langle OO^\dagger \rangle &= \langle A_0^2 \rangle - ix\langle [A_0, B_0] \rangle + x^2 \langle B_0^2 \rangle \geq 0 \\
x_{min} &= -\frac{\langle C \rangle}{2\langle B_0^2 \rangle}
\end{aligned}$$

Uncertainty:

$$\boxed{\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|}$$

(b) Fourier transformatiun for x and p: Gaussian wave packet,

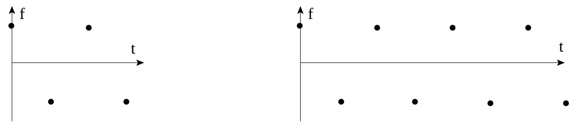
$$\psi(x) = \int \frac{dk}{2\pi} e^{ikx - \frac{k^2}{2\sigma^2}} = \frac{\sqrt{2\pi}}{\sigma} e^{-\frac{\sigma^2 x^2}{2}}$$

Uncertainty:

$$\psi(x) = e^{-\frac{x^2}{2\Delta x^2}}, \quad \tilde{\psi}(k) = e^{-\frac{k^2}{2\Delta k^2}} \quad \Longrightarrow \quad \Delta x \Delta k = 1, \quad \Delta x \Delta p = \hbar$$

2. **Frequency and observation time:**

   (a) Intuitive approach: $T\Delta\omega \approx 1$, $E = \hbar\omega$, $T\Delta E \approx \hbar$

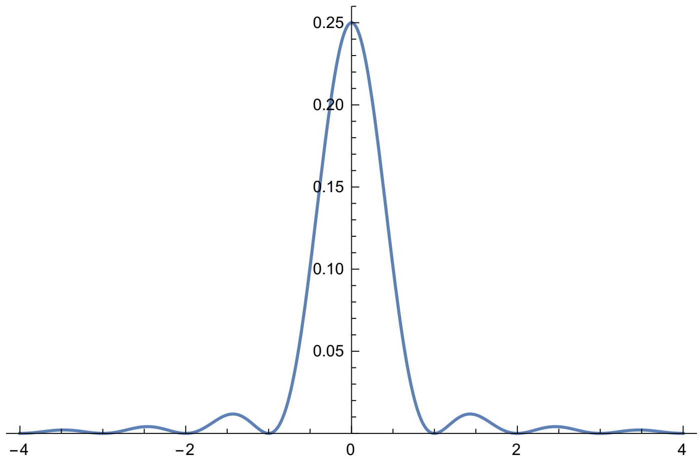


   (b) Fourier transformation

   (c) Width of the energy spread:

$$P^{\pm}_{n\to k\neq n} = \frac{4g^2|H'_{k,n}|^2}{\hbar^2(\omega_{kn}\pm\omega)^2}\sin^2\frac{1}{2}(\omega_{kn}\pm\omega)t$$
$$t\Delta|\omega\pm\omega_{k,n}| \approx 2\pi, \quad \Delta E t \approx 2\pi\hbar.$$

$\frac{\sin^2\pi x}{(2\pi x)^2}$:



F. Fermi's golden rule

- Transition from discrete to continuous spectrum
- Final states are assumed to be decohered (no interference)

$$P_{\text{cont.}\leftarrow\text{discr.}} = \int dE g(E)\frac{|gH_{\text{cont.,discr.}}|^2}{\hbar^2}\frac{4\sin^2\frac{1}{2}(\omega_{\text{cont.,discr.}}\pm\omega)t}{(\omega_{\text{cont.,discr.}}\pm\omega)^2},$$

- Spectral density: $g(E)$ the number of state in the energy interval $[E, E+\Delta E]$
- Change of variable: $E = \hbar\omega \to \beta = \frac{1}{2}(\omega_{\text{cont.,discr.}}\pm\omega)t$, $d\beta = dE\frac{t}{2\hbar}$

$$P_{\text{cont.}\leftarrow\text{discr.}} = \frac{2t}{\hbar}\int d\beta g(E)|gH_{\text{cont.,discr.}}|^2\frac{\sin^2\beta}{\beta^2}.$$

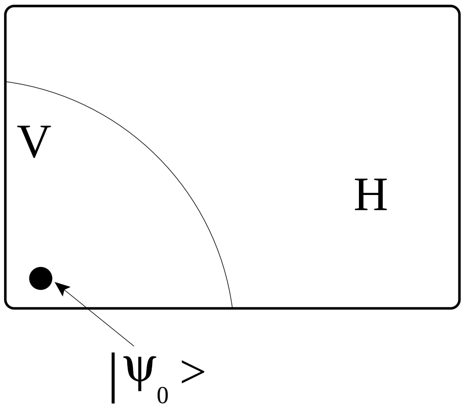
- Assuming that t is large enough to keep $g(E)$ approximately constant

$$\int_{-\infty}^{\infty} d\beta \frac{\sin^2\beta}{\beta^2} = \pi$$

$$\boxed{P_{\text{cont.}\leftarrow\text{discr.}} \approx t\frac{2\pi}{\hbar} g(E)|gH_{\text{cont.,discr.}}|^2}$$

G. Variational method

- **A non-perturbative and not completely systematic approximation**
- **An approach of the non-degenerate ground state:**
  - *Hilbert space of states: H*
  - *Variational subset: $V = \{|\psi(\alpha)\rangle\} \subset H$*



  - *Minimization of the energy:*

$$\begin{aligned} H|\psi_n\rangle &= E_n|\psi_n\rangle, \quad E_0 \le E_2 \le E_2 \le \cdots \\ |\psi(\alpha)\rangle &= \sum_n c_n(\alpha)|n\rangle \\ E(\alpha) &= \frac{\langle\psi(\alpha)|H|\psi(\alpha)\rangle}{\langle\psi(\alpha)|\psi(\alpha)\rangle} = \frac{\sum_n |c_n(\alpha)|^2 E_n}{\sum_n |c_n(\alpha)|^2} \ge E_0 \end{aligned}$$

  - *Lower is $E(\alpha)$, $|\psi(\alpha)\rangle$ is a better approximation of $|\psi_0\rangle$*

$$E(\alpha) = E_0 \quad \Longrightarrow \quad |\psi(\alpha)\rangle = |\psi_0\rangle$$

- **Problems with degenerate ground state or spectrum with small gap ($E_1 - E_0 \ll E_0$)**

II. ROTATIONS

A. Translations

1. **Classical physics:** coordinate space

$$\boldsymbol{r} \to T(\boldsymbol{a})\boldsymbol{r} = \boldsymbol{r} + \boldsymbol{a}.$$

2. **Functions in space:**

$$f(\boldsymbol{r}) \to f'(\boldsymbol{r}') = f(\boldsymbol{r}' - \boldsymbol{a}).$$

3. **Quantum mechanics:** Hilbert space

$$\psi(\boldsymbol{r}) \to U(T(\boldsymbol{a}))\psi(\boldsymbol{r}) = \psi(\boldsymbol{r} - \boldsymbol{a}).$$

4. **Representation:** $T(\boldsymbol{a}) \to U(T(\boldsymbol{a}))$ preserves the algebraic structure

$$U(T(\boldsymbol{a}))U(T(\boldsymbol{b}))\psi(\boldsymbol{r}) = \psi(\boldsymbol{r} - \boldsymbol{a} - \boldsymbol{b}) = U(T(\boldsymbol{a} + \boldsymbol{b}))\psi(\boldsymbol{r})$$

5. **Unitary representation:**

$$\begin{aligned} \langle\psi|\phi\rangle &= \langle U\psi|U\phi\rangle = \langle\psi| \underbrace{U^\dagger U}_{U^\dagger U = \mathbb{1}} |\phi\rangle \\ \int d\boldsymbol{x}\psi^*(\boldsymbol{x} - \boldsymbol{a})\phi(\boldsymbol{x} - \boldsymbol{a}) &= \int d\boldsymbol{x}\psi^*(\boldsymbol{x})\phi(\boldsymbol{x}) \end{aligned}$$

6. **Infinitesimal translations:**

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + \delta\boldsymbol{r} \\ \psi(\boldsymbol{r}) &\to \psi(\boldsymbol{r}) - \delta\boldsymbol{r}\boldsymbol{\nabla}\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \frac{i}{\hbar}\delta\boldsymbol{r}\vec{G}\psi(\boldsymbol{r}) \end{aligned}$$

Generator: $\vec{G} = \frac{\hbar}{i}\boldsymbol{\nabla} = \boldsymbol{p}$

7. **Finite translations:**

$$\psi(\boldsymbol{r}) \to \psi(\boldsymbol{r} - \boldsymbol{a}) = \sum_{n=0}^{\infty} \frac{(-\boldsymbol{a}\boldsymbol{\nabla})^n}{n!}\psi(\boldsymbol{r}) = e^{-\boldsymbol{a}\boldsymbol{\nabla}}\psi(\boldsymbol{r}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\boldsymbol{p}}\psi(\boldsymbol{r})$$

$$\boxed{U(\boldsymbol{a}) = e^{-\frac{i}{\hbar}\boldsymbol{a}\boldsymbol{p}}}$$

B. Rotations

1. **Classical physics:**

- *3x3 matrix:*

$$\boldsymbol{r} \to R_{\boldsymbol{n}}(\alpha)\boldsymbol{r}$$

$$\nearrow \qquad \nwarrow$$

$$\text{axis} \qquad\qquad \text{angle}$$

- *Orthogonality:*

$$\begin{aligned}(\boldsymbol{u},\boldsymbol{v}) &= \sum_j u_j v_j = (R\boldsymbol{u}, R\boldsymbol{v}) = \sum_j (R\boldsymbol{u})_j, (R\boldsymbol{v})_j = \sum_{jk\ell} R_{jk} u_k R_{j\ell} v_\ell = \sum_{jk\ell} u_k \underbrace{R^{\mathrm{tr}}_{kj} R_{j\ell}}_{R^{\mathrm{tr}}R=\mathbb{1}} v_\ell \\ R^{-1}_{\boldsymbol{n}}(\alpha) &= R_{\boldsymbol{n}}(-\alpha) = R^{\mathrm{tr}}_{\boldsymbol{n}}(\alpha)\end{aligned}$$

- *Rotation around the quantization axis* $\boldsymbol{z}$:

$$R_{\boldsymbol{z}}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *Rotation around an arbitrary axis* $\boldsymbol{v} = A\boldsymbol{u}$*:*

$$R_{\boldsymbol{v}}(\alpha) = AR_{\boldsymbol{u}}(\alpha)A^{-1}$$

Proof:

 – A rotation matrix has a single eigenvactor with zero eigenvalue, the axis, $R_{\boldsymbol{v}}(\alpha)\boldsymbol{v} = \boldsymbol{v}$

$$AR_{\boldsymbol{u}}(\alpha)A^{-1}\boldsymbol{v} = AR_{\boldsymbol{u}}(\alpha)\boldsymbol{u} = A\boldsymbol{u} = \boldsymbol{v}$$

 – The rotation angle remains the same during basis transformation

In particular: $\boldsymbol{n} = A\boldsymbol{z}$

$$R_{\boldsymbol{n}}(\alpha) = AR_z(\alpha)A^{-1}$$

2. **Functions in space:**

$$f(\boldsymbol{r}) \to U(R)f(\boldsymbol{r}) = f(R^{-1}\boldsymbol{r}).$$

3. **Quantum mechanics:** Hilbert space

$$\psi(\boldsymbol{r}) \to U(R)\psi(\boldsymbol{r}) = \psi(R^{-1}\boldsymbol{r}).$$

4. **Representation:**

$$\begin{aligned} U(R)U(R')\psi(\boldsymbol{r}) &= U(R)\psi(R'^{-1}\boldsymbol{r}) \\ &= \psi(R'^{-1}R^{-1}\boldsymbol{r}) \\ &= \psi((RR')^{-1}\boldsymbol{r}) \\ &= U(RR')\psi(\boldsymbol{r}) \\ U(R)U(R') &= U(RR') \end{aligned}$$

5. **Unitary representation:**

$$\int d\boldsymbol{x}\psi^*(R\boldsymbol{x})\phi(R\boldsymbol{x}) = \int d\boldsymbol{x}\psi^*(\boldsymbol{x})\phi(\boldsymbol{x}) \quad \Longrightarrow \quad U(R)U^\dagger(R) = \mathbb{1}$$

6. **Infinitesimal rotations:**

$$\begin{aligned} \boldsymbol{r} &\to \boldsymbol{r} + \epsilon\boldsymbol{n}\times\boldsymbol{r} \\ \psi(\boldsymbol{r}) &\to \psi(\boldsymbol{r}) - (\epsilon\boldsymbol{n}\times\boldsymbol{r})\boldsymbol{\nabla}\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \epsilon\boldsymbol{n}(\boldsymbol{r}\times\boldsymbol{\nabla})\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}) - \frac{i}{\hbar}\epsilon\boldsymbol{n}\boldsymbol{L}\psi(\boldsymbol{r}), \end{aligned}$$

$\nearrow$

Generator: angular momentum

7. **Finite rotations:**

   (a) *One dimensional subgroup of rotational around a fixed axis:* $\{R_{\boldsymbol{n}}(\alpha)\}$

   (b) *Generator:* $\boldsymbol{nL}$

   (c) *Representation:*

$$\boxed{U(R_{\boldsymbol{n}}(\alpha)) = e^{-\frac{i}{\hbar}\alpha\boldsymbol{n}\boldsymbol{L}}}$$

8. $\boldsymbol{L}$ **is a vector operator:**

   (a) *Definition:* transforms under rotations as a vector and as an operator and the two transformations agree.

   (b) $\boldsymbol{n}_j = A^{-1}\boldsymbol{e}_j$,

$$\begin{aligned} U(R_{\boldsymbol{n}_j}(\alpha)) &= U(A^{-1}R_{\boldsymbol{e}_j}(\alpha)A) \\ &= U(A^{-1})U(R_{\boldsymbol{e}_j}(\alpha))U(A) \\ &= U(A^{-1})e^{-\frac{i}{\hbar}\alpha\boldsymbol{e}_j\boldsymbol{L}}U(A) \\ &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}U(A^{-1})(\boldsymbol{e}_j\boldsymbol{L})^n U(A) \\ &= \sum_{n=0}^{\infty}\frac{(-\frac{i}{\hbar}\alpha)^n}{n!}[U(A^{-1})\boldsymbol{e}_j\boldsymbol{L}U(A)]^n \\ &= e^{-\frac{i}{\hbar}\alpha U(A^{-1})\boldsymbol{e}_j\boldsymbol{L}U(A)}. \end{aligned}$$

(c) *Another expression:* $\boldsymbol{n}_j = A^{-1}\boldsymbol{e}_j = A^{\rm tr}\boldsymbol{e}_j = \boldsymbol{e}_j A$

$$\begin{aligned} U(R_{\boldsymbol{n}_j}(\alpha)) &= e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}_j \boldsymbol{L}} \\ &= e^{-\frac{i}{\hbar}\alpha \boldsymbol{e}_j A \boldsymbol{L}} \end{aligned}$$

(d)

$$\boxed{A\boldsymbol{L} = U^\dagger(A)\boldsymbol{L}U(A)}$$

$\nearrow$ $\nwarrow$

vector operator

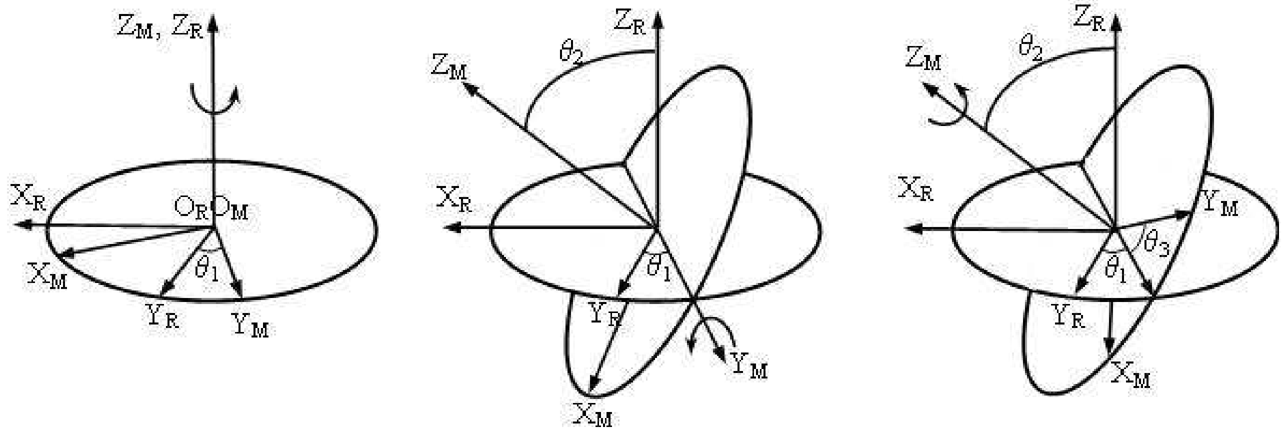
C. Euler angles

1. **Definition:**

$$R(\phi,\theta,\alpha) = R_{\boldsymbol{z}''}(\alpha)R_{\boldsymbol{y}'}(\theta)R_{\boldsymbol{z}}(\phi)$$

$\nearrow$ $\nwarrow$

$\boldsymbol{z}'' = R_{\boldsymbol{y}'}(\theta)R_{\boldsymbol{z}}(\phi)\boldsymbol{z} = R_{\boldsymbol{y}'}(\theta)\boldsymbol{z}$ $\boldsymbol{y}' = R_{\boldsymbol{z}}(\phi)\boldsymbol{y}$



$\theta_1 = \phi$ $\theta_2 = \theta$ $\theta_3 = \alpha$

2. **Another, equivalent expression:** $\boldsymbol{n} = A\boldsymbol{z}$, $R_{\boldsymbol{n}}(\alpha) = AR_z(\alpha)A^{-1}$

$$\begin{aligned} R_{\boldsymbol{z}''}(\alpha)R_{\boldsymbol{y}'}(\theta)R_{\boldsymbol{z}}(\phi) &= \underbrace{R_{\boldsymbol{y}'}(\theta)R_{\boldsymbol{z}}(\alpha)R^{-1}_{\boldsymbol{y}'}(\theta)}_{R_{\boldsymbol{z}''}(\alpha)} R_{\boldsymbol{y}'}(\theta)R_{\boldsymbol{z}}(\phi) \\ &= \underbrace{R_{\boldsymbol{z}}(\phi)R_{\boldsymbol{y}}(\theta)R^{-1}_{\boldsymbol{z}}(\phi)}_{R_{\boldsymbol{y}'}(\theta)} R_{\boldsymbol{z}}(\alpha)R_{\boldsymbol{z}}(\phi) \\ &= R_{\boldsymbol{z}}(\phi)R_{\boldsymbol{y}}(\theta)R_{\boldsymbol{z}}(\alpha). \end{aligned}$$

3. **Relation to the parameterization $R_{\boldsymbol{n}}(\alpha)$:**

$$\begin{aligned} \boldsymbol{n} &= R(\phi,\theta,\chi)\boldsymbol{z} = R_z(\phi)R_y(\theta)R_z(\chi)\boldsymbol{z} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\ R_{\boldsymbol{n}}(\alpha) &= R(\phi,\theta,\chi)R_z(\alpha)R^{-1}(\phi,\theta,\chi) \end{aligned}$$

Proof: $\boldsymbol{v} = A\boldsymbol{u}$, $R_v(\alpha) = AR_u(\alpha)A^{-1}$

$$\begin{aligned} R(\phi,\theta,\chi)R_z(\alpha)R^{-1}(\phi,\theta,\chi) &= R_z(\phi)R_y(\theta)R_z(\chi)R_z(\alpha)R_z(-\chi)R_y(-\theta)R_z(-\phi) \\ &= R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi), \quad R_y(\theta)\boldsymbol{z} = \boldsymbol{u} \\ &= R_z(\phi)R_u(\alpha)R_z(-\phi), \quad R_z(\phi)\boldsymbol{u} = \boldsymbol{v} \\ &= R_v(\alpha), \quad \boldsymbol{n} = R_z(\phi)R_y(\theta)\boldsymbol{z} \end{aligned}$$

D. Summary of the angular momentum algebra

1. **Orbital angular momentum:**

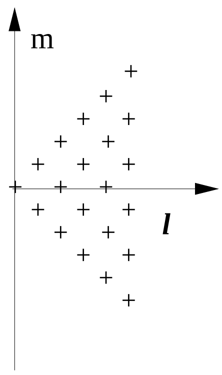
$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

2. **Commutation relations:**

$$[L_a, L_b] = i\hbar \sum_c \epsilon_{abc} L_c.$$

3. **Maximal set of commuting operators:** $\{L_z, \boldsymbol{L}^2\} \Longrightarrow$ eigenvalues to label the basis vectors,

$$\begin{aligned} L_z|\ell,m\rangle &= \hbar m|\ell,m\rangle, \quad \boldsymbol{L}^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle \\ \ell &= 0,1,\cdots, \quad m \in \{-\ell,-\ell+1,\cdots,\ell-1,\ell\} \end{aligned}$$



4. **Ladder operators:** $L_\pm = L_x \pm iL_y$

$$\begin{aligned} [L_z, L_\pm] &= \pm\hbar L_\pm, \quad [L_+, L_-] = 2\hbar L_z \\ L_\pm|\ell,m\rangle &= \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle \end{aligned}$$

$\nearrow$

to stop at the highers (lowest) state

5. ℓ **remains unvariant under $\boldsymbol{L}$:**

$$\langle \ell, m|L_a|\ell', m'\rangle = \delta_{\ell,\ell'} F_a(\ell, m, m')$$

block diagonal structure ℓ

E. Rotational multiplets

1. **Helicity basis:**

$$\begin{aligned} \boldsymbol{u} &= (u_x, u_y, u_z) \to (u_+, u_-, u_z), \quad u_\pm = u_x \pm iu_y \\ \boldsymbol{nL} &= n_x L_x + n_y L_y + n_z L_z \\ &= \frac{1}{2}(n_+ L_+ + n_- L_-) + n_z L_z = \frac{1}{2}[(n_x - in_y)(L_x + iL_y) + (n_x + in_y)(L_x - iL_y)] + n_z L_z \end{aligned}$$

2. **Rotation of $|\ell, m\rangle$:**

$$\begin{aligned} e^{-\frac{i}{\hbar}\alpha\boldsymbol{nL}}|\ell,m\rangle &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!}(\boldsymbol{nL})^n|\ell,m\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!}\left(n_3 L_3 + \frac{1}{2}n_+ L_- + \frac{1}{2}n_- L_+\right)^n |\ell,m\rangle \\ &= \sum_{-\ell\le m'}^{\ell} c_{m'}(\alpha, \boldsymbol{n})|\ell, m'\rangle \end{aligned}$$

and all coefficients are non-vanishing if $n_\pm \ne 0$

3. **Rotational multiplet:** $\mathcal{H}_\ell = \{\sum_{m=-\ell}^{\ell} x_m|\ell,m\rangle\}$

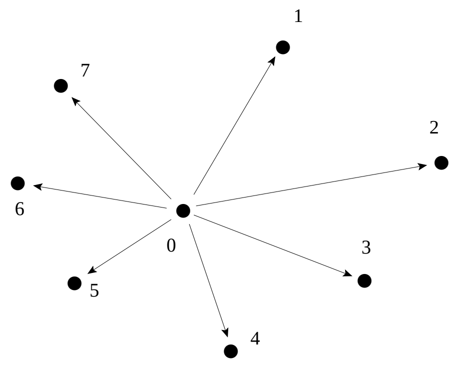
4. **Properties:**

   (a) *Basis:* $\{|\ell,m\rangle| - \ell \le m \le \ell\}$, $\mathrm{Dim}\mathcal{H}_\ell = 2\ell + 1$

   (b) $\mathcal{H}_\ell$ *is closed* with respect to rotations, $e^{-\frac{i}{\hbar}\alpha\boldsymbol{nL}}\mathcal{H}_\ell \subset \mathcal{H}_\ell$.

   (c) $\mathcal{H}_\ell$ *is irreducible* with respect to rotations.

      i. $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is reducible if each component is closed, $e^{-\frac{i}{\hbar}\alpha\boldsymbol{nL}}\mathcal{H}_j \subset \mathcal{H}_j$, $j = 1, 2$.



ii. Star condition of irreducibility of $\mathcal{H}$: $\exists|\psi_0\rangle$ such that $\forall|\psi\rangle \in \mathcal{H}$ $\exists R$ such that $\langle\psi|U(R)|\psi_0\rangle \neq 0$.

A suitable rotation of $|\psi_0\rangle$ has a projection onto any state.

$$\begin{aligned} e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}\boldsymbol{L}}|\ell, m\rangle &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!} \left(n_z L_z + \frac{1}{2}n_+ L_- + \frac{1}{2}n_- L_+\right)^n |\ell, m\rangle \\ \sum_{m'} c^*_{m'}\langle\ell, m'|e^{-\frac{i}{\hbar}\alpha \boldsymbol{n}\boldsymbol{L}}|\ell, m\rangle &= \sum_{n=0}^{\infty} \frac{(-\frac{i}{\hbar}\alpha)^n}{n!} \sum_{m'} c^*_{m'}\langle\ell, m'| \left(n_z L_z + \frac{1}{2}n_+ L_- + \frac{1}{2}n_- L_+\right)^n |\ell, m\rangle = 0 \end{aligned}$$

$\nearrow$

∞ equations for 3 variables (not a proof!)

F. Wigner's D matrix

1. D **matrix:** action of rotations within a rotational multiplet

2. **Definition:** $\sum_{\ell', m'} |\ell', m'\rangle\langle\ell', m'| = \mathbb{1}$

$$\begin{aligned} U(R)|\ell, m\rangle &= \mathbb{1}U(R)|\ell, m\rangle = \sum_{\ell', m'} |\ell', m\rangle\langle\ell', m'|U(R)|\ell, m\rangle \\ &= \sum_{m'} |\ell, m'\rangle \mathcal{D}^{(\ell)}_{m', m}(R) \end{aligned}$$

$$\boxed{\mathcal{D}^{(\ell)}_{m', m}(R) = \langle\ell, m'|U(R)|\ell, m\rangle}$$

3. **Euler angles:**

$$\begin{aligned} \mathcal{D}^{(\ell)}_{m', m}(R(\alpha, \beta, \gamma)) &= \mathcal{D}^{(\ell)}_{m', m}(R_{\boldsymbol{z}}(\alpha)R_{\boldsymbol{y}}(\beta))R_{\boldsymbol{z}}(\gamma)) \\ &= \sum_{m_1, m_2} \mathcal{D}^{(\ell)}_{m', m_1}(R_{\boldsymbol{z}}(\alpha))\mathcal{D}^{(\ell)}_{m_1, m_2}(R_{\boldsymbol{y}}(\beta))\mathcal{D}^{(\ell)}_{m_2, m}(R_{\boldsymbol{z}}(\gamma)) \\ \langle\ell, m'|e^{-i\frac{\alpha}{\hbar}L_z}|\ell, m\rangle &= \mathcal{D}^{(\ell)}_{m', m}(R_{\boldsymbol{z}}(\alpha)) = \delta_{m', m}e^{-i\alpha m} \\ \mathcal{D}^{(\ell)}_{m', m}(R_{\boldsymbol{y}}(\beta)) &= \langle\ell, m'|e^{-i\frac{\beta}{\hbar}L_y}|\ell, m\rangle = d^{(\ell)}_{m', m}(\beta) \\ \mathcal{D}^{(\ell)}_{m', m}(R(\alpha, \beta, \gamma)) &= e^{-i\alpha m' - i\gamma m} d^{(\ell)}_{m', m}(\beta) \end{aligned}$$

$$\nearrow$$

Reduced d-matrix

4. **Block diagonal structure:** Basis: $\{\underbrace{|0,0\rangle}_{\mathcal{H}_0}, \underbrace{|1,1\rangle, |1,0\rangle, |1,-1\rangle}_{\mathcal{H}_1}, \underbrace{|2,2\rangle, |2,1\rangle, |2,0\rangle, |2,-1\rangle, |2,-2\rangle}_{\mathcal{H}_2}, \cdots\}$

$$U = \begin{pmatrix} \mathcal{D}^{(0)} & 0 & 0 & \cdots \\ 0 & \mathcal{D}^{(1)} & 0 & \cdots \\ 0 & 0 & \mathcal{D}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \boldsymbol{L} = \begin{pmatrix} \boldsymbol{L}^{(0)} & 0 & 0 & \cdots \\ 0 & \boldsymbol{L}^{(1)} & 0 & \cdots \\ 0 & 0 & \boldsymbol{L}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$L_z^{(\ell)} = \hbar \begin{pmatrix} \ell & 0 & \cdots & 0 & 0 \\ 0 & \ell-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\ell+1 & 0 \\ 0 & 0 & \cdots & 0 & -\ell \end{pmatrix}, \; L_+^{(\ell)} = \hbar \begin{pmatrix} 0 & \sqrt{2\ell} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{2\ell} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; L_-^{(\ell)} = \hbar \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2\ell} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \sqrt{2\ell} & 0 \end{pmatrix}$$

5. $S = \frac{1}{2}$**:** Pauli matrices,

$$\langle \frac{1}{2}, m' | \boldsymbol{L} | \frac{1}{2}, m \rangle = \frac{\hbar}{2}\boldsymbol{\sigma} = \frac{\hbar}{2} \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

$$\left[\frac{\hbar}{2}\sigma_j, \frac{\hbar}{2}\sigma_k\right] = i\hbar \sum_\ell \epsilon_{jk\ell} \frac{\hbar}{2}\sigma_\ell \iff [\sigma_j, \sigma_k] = 2\sum_\ell \epsilon_{jk\ell}\sigma_\ell$$

6. **Two important relations:**

$$\sigma_a \sigma_b = \delta_{a,b} + i \sum_c \epsilon_{abc}\sigma_c \iff (\boldsymbol{u\sigma}) \cdot (\boldsymbol{v\sigma}) = \mathbb{1}\boldsymbol{uv} + i(\boldsymbol{u} \times \boldsymbol{v})\boldsymbol{\sigma}$$

$$\sigma_y \boldsymbol{\sigma} \sigma_y = -\boldsymbol{\sigma}^*$$

7. **Finite rotation:**

- *Euler's relation:*

$$\begin{aligned} e^{i\alpha} &= 1 + i\alpha + \frac{(i\alpha)^2}{2!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^4}{4!} + \cdots \\ &= 1 + i\alpha + \frac{(i\alpha)^2}{2!} + \frac{(i\alpha)^3}{3!} + \frac{(i\alpha)^4}{4!} + \cdots \\ &= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{1}{2}\left(e^{i\alpha} - e^{-i\alpha}\right) \\ &= \cos\alpha + i\sin\alpha \end{aligned}$$

- *Generalized Euler's relation:*

$$e^{i\alpha\boldsymbol{n\sigma}} = \mathbb{1} + i\alpha\boldsymbol{n\sigma} + \frac{(i\alpha)^2}{2!}(\boldsymbol{n\sigma})^2 + \frac{(i\alpha)^3}{3!}(\boldsymbol{n\sigma})^3 + \frac{(i\alpha)^4}{4!}(\boldsymbol{n\sigma})^4 + \cdots$$

$$
\begin{aligned}
&= \mathbb{1} + i\alpha \boldsymbol{n}\boldsymbol{\sigma} + \mathbb{1}\frac{(i\alpha)^2}{2!}\boldsymbol{n}^2 + \frac{(i\alpha)^3}{3!}\boldsymbol{n}^2\boldsymbol{n}\boldsymbol{\sigma} + \mathbb{1}\frac{(i\alpha)^4}{4!}\boldsymbol{n}^4 + \cdots \\
&= \mathbb{1}\frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{\boldsymbol{n}\boldsymbol{\sigma}}{2}\left(e^{i\alpha} - e^{-i\alpha}\right) \\
&= \mathbb{1}\cos\alpha + i\boldsymbol{n}\boldsymbol{\sigma}\sin\alpha.
\end{aligned}
$$

- *Reduced d matrix:*

$$
\begin{aligned}
d^{(\frac{1}{2})}_{m',m}(\beta) &= \langle\frac{1}{2}, m'|e^{-i\frac{\beta\sigma_y}{2}}|\frac{1}{2}, m\rangle = \left(\mathbb{1}\cos\frac{\beta}{2} - i\sigma_y \sin\frac{\beta}{2}\right)_{m',m}, \\
d^{(\frac{1}{2})}(\beta) &= \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}.
\end{aligned}
$$

G. Invariant integration

1. **Two sphere, S_2:**

- *Rotational invariance:*

$$
\boldsymbol{n}(\theta,\phi) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \qquad d\Sigma = d\theta\sin\theta d\phi = d\cos\theta d\phi
$$

$\uparrow$

elementary area on the unit sphere

- *Invariant integral:*

$$
\int_\Sigma d\phi d\cos\theta f(\boldsymbol{n}) = \int_{R\Sigma} d\phi d\cos\theta f(R^{-1}\boldsymbol{n})
$$

2. **Rotational group $SO(3)$:**

- *Invariant integral:*

$$
\boldsymbol{n} = R(\phi,\theta,\chi)\boldsymbol{z} = R_z(\phi)R_y(\theta)R_z(\chi)\boldsymbol{z} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}
$$

$$
\begin{aligned}
\int_V d\boldsymbol{n}d\chi f(R_{\boldsymbol{n}}(\chi)) &= \int_V d\phi d\cos\theta d\chi f(R_{\boldsymbol{n}}(\chi)) = \int_{R'V} d\phi d\cos\theta d\chi f(R'^{-1}R_{\boldsymbol{n}}(\chi)) \\
\int_V d\phi d\cos\theta d\chi f(R(\phi,\theta,\chi)) &= \int_{R'V} d\phi d\cos\theta d\chi f(R'^{-1}R(\phi,\theta,\chi))
\end{aligned}
$$

- *Equivalent form (Haar mesure):* $dR = d\phi d\cos\theta d\chi$

$$
\int dR f(R) = \int d(R'R) f(R) = \int dR f(R'^{-1}R)
$$

defined up to a normalization constant

- *Volumes:*

$$\int_{S_2} d\Sigma = \int_{-1}^{1} dc \int_{-\pi}^{\pi} d\phi = 4\pi,$$
$$\int_{SO(3)} dR = \int_{-1}^{1} dc \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\chi = 8\pi^2$$

H. Spherical harmonics

1. **Definition:** wave function of $|\ell, m\rangle$,

$$\langle \boldsymbol{n}|\ell, m\rangle = Y^\ell_m(\boldsymbol{n}) = Y^\ell_m(\theta, \phi), \quad \leftarrow \quad \boldsymbol{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

determined by the structure of the rotation group.

2. **Normalization:**

$$1 = \int_{S_2} d^2n |Y^\ell_m(\boldsymbol{n})|^2,$$

3. **Defining relations:**

- *Necessary:* and sufficient condition

$$\begin{aligned} L_z Y^\ell_m(\boldsymbol{n}) &= L_z\langle \boldsymbol{n}|\ell, m\rangle = \langle \boldsymbol{n}|L_z|\ell, m\rangle = \hbar m\langle \boldsymbol{n}|\ell, m\rangle = \hbar m Y^\ell_m(\boldsymbol{n}), \\ L_\pm Y^\ell_m(\boldsymbol{n}) &= \langle \boldsymbol{n}|L_\pm|\ell, m\rangle \\ &= \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\langle \boldsymbol{n}||\ell, m\pm 1\rangle \\ &= \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)} Y^\ell_{m\pm 1}(\boldsymbol{n}) \end{aligned}$$

- *Sufficient:*
  - Eigenvectors of hermitian operators $\Longrightarrow$ basis set on the unit sphere
  - Non-degeneracy in L_z: a set of functions on the unit sphere satisfying these eqs. are the spherical harmonics up to a constant

4. **Spherical harmonics in terms of $\mathcal{D}$ matrices:**

- *Relation between the Euler angles and the polar angles:*

$$\begin{aligned} \boldsymbol{n} &= \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = R(\phi, \theta, \chi)\boldsymbol{z} \\ |\boldsymbol{n}\rangle &= U(R(\phi, \theta, \chi))|\boldsymbol{z}\rangle \end{aligned}$$

with $\boldsymbol{z} = (0, 0, 1)$ and χ left arbitrary.

- *Resolution of unity:* $\sum_{\ell',m'} |\ell',m'\rangle\langle\ell',m'| = \mathbb{1}$

$$|\boldsymbol{n}\rangle = U(R(\phi,\theta,\chi))\mathbb{1}|\boldsymbol{z}\rangle = \sum_{\ell,m} U(R(\phi,\theta,\chi))|\ell,m\rangle\langle\ell,m|\boldsymbol{z}\rangle$$

- *Projection on* $\langle\ell,m'|$*:*

$$\langle\ell,m'|\boldsymbol{n}\rangle = Y^{\ell*}_{m'}(\boldsymbol{n}) = \sum_{m} \mathcal{D}^{(\ell)}_{m',m}(R(\phi,\theta,\chi))\langle\ell,m|\boldsymbol{z}\rangle$$

- *Last factor in three steps:*

  (a) Consider

$$\begin{aligned}\langle\ell,m|U(R_{\boldsymbol{z}}(\chi))|\boldsymbol{z}\rangle &= \sum_{\ell',m'}\langle\ell,m|U(R_{\boldsymbol{z}}(\chi))|\ell',m'\rangle\langle\ell',m'|\boldsymbol{z}\rangle \\ &= \sum_{m'}\mathcal{D}^{(\ell)}_{m,m'}(R_{\boldsymbol{z}}(\chi))\langle\ell,m'|\boldsymbol{z}\rangle \\ &= e^{-im\chi}\langle\ell,m|\boldsymbol{z}\rangle\end{aligned}$$

  (b) $\boldsymbol{z} = R_{\boldsymbol{z}}(\chi)\boldsymbol{z} \implies$ no χ-dependence,

$$\langle\ell,m|U(R_{\boldsymbol{z}}(\chi))|\boldsymbol{z}\rangle = \langle\ell,m|\boldsymbol{z}\rangle$$

  (c) Hence

$$e^{-im\chi}\langle\ell,m|\boldsymbol{z}\rangle = \langle\ell,m|\boldsymbol{z}\rangle$$

  acting on it by $\frac{\partial}{\partial\chi}$ and setting $\chi = 0$:

$$-im\langle\ell,m|\boldsymbol{z}\rangle = 0 \quad\implies\quad \langle\ell,m|\boldsymbol{z}\rangle = \delta_{m,0}c_\ell$$

- *Normalization:*

  – Resolution of unity: $\mathbb{1} = \int_{S_2} d\boldsymbol{n}|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$

  – Integration over the unit sphere:

$$\begin{aligned}\int_{S_2} d\Omega f(\boldsymbol{n}) &= \int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi f(\underbrace{\theta,\phi}_{\boldsymbol{n}}), \quad \boldsymbol{n} = \begin{pmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{pmatrix} = R(\phi,\theta,\chi)\boldsymbol{z} \\ &= \underbrace{\int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi}_{\int_{S_2} d\boldsymbol{n}}\, f(R(\phi,\theta,\chi)\boldsymbol{z}) \\ &= \frac{1}{2\pi}\underbrace{\int_{-1}^{1} d\cos\theta \int_{-\pi}^{\pi} d\phi \int_{-\pi}^{\pi} d\chi}_{\int_{SO(3)} dR}\, f(R(\phi,\theta,\chi)\boldsymbol{z})\end{aligned}$$

– Normalization:

$$\begin{aligned} 1 &= \langle \ell, 0 | \ell, 0 \rangle \\ &= \langle \ell, 0 | \mathbb{1} | \ell, 0 \rangle \\ &= \int_{S_2} d\boldsymbol{n} \langle \ell, 0 | \boldsymbol{n} \rangle \langle \boldsymbol{n} | \ell, 0 \rangle \\ &= \frac{1}{2\pi} \int_{SO(3)} dR \langle \ell, 0 | U(R) | \boldsymbol{z} \rangle \langle \boldsymbol{z} | U^\dagger(R) | \ell, 0 \rangle \end{aligned}$$

– Resolution of identity: $\mathbb{1} = \sum_{\ell,m} |\ell, m\rangle\langle \ell, m|$

$$\begin{aligned} 1 &= \frac{1}{2\pi} \int_{SO(3)} dR \langle \ell, 0 | U(R) \mathbb{1} | \boldsymbol{z} \rangle \langle \boldsymbol{z} | \mathbb{1} U^\dagger(R) | \ell, 0 \rangle \\ &= \frac{1}{2\pi} \sum_{\ell,\ell',m,m'} \int_{SO(3)} dR \langle \ell, 0 | U(R) | \ell', m' \rangle \underbrace{\langle \ell', m' | \boldsymbol{z} \rangle}_{\delta_{m',0} c_{\ell'}} \underbrace{\langle \boldsymbol{z} | \ell, m \rangle}_{\delta_{m,0} c_\ell} \langle \ell, m | U^\dagger(R) | \ell, 0 \rangle \\ &= \frac{|\langle \ell, 0 | \boldsymbol{z} \rangle|^2}{2\pi} \underbrace{\int_{SO(3)} dR |\mathcal{D}^{(\ell)}_{0,0}(R)|^2}_{\frac{8\pi^2}{2\ell+1}} \quad \Longrightarrow \quad c_\ell = \sqrt{\frac{2\ell+1}{4\pi}} \end{aligned}$$

(assuming that c_ℓ is real and positive)

- *Finally:*

$$\begin{aligned} Y^\ell_m(\boldsymbol{n}) &= \sum_{m'} \mathcal{D}^{(\ell)*}_{m,m'}(R(\phi,\theta,\chi)) \langle \ell, m' | \boldsymbol{z} \rangle^* \\ &= \sqrt{\frac{2\ell+1}{4\pi}} \mathcal{D}^{(\ell)*}_{m,0}(R(\phi,\theta,\chi)) \end{aligned}$$

$$\boxed{Y^\ell_m(\boldsymbol{n}) = \sqrt{\frac{2\ell+1}{4\pi}} e^{im\phi} d^{(\ell)*}_{m,0}(\theta)}$$

5. **Example:** Y^1_m :

- Three functions on the unit sphere, transforming under rotations in an irreducible manner
- $\boldsymbol{n} = (\frac{x}{r}, \frac{y}{r}, \frac{z}{r})$ do the same
- Two different bases for $\mathcal{H}_1$: Y^1_m and $\boldsymbol{n}$
  - Y^1_0: $L_z Y^1_0 = 0$, $L_z z = 0$, normalization: $\int_{S_2} d\boldsymbol{n} |Y(\boldsymbol{n}|^2 = 1$, $Y^1_0 = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$
  - $Y^1_{\pm 1}$:

$$\begin{aligned} Y^1_{\pm 1}(\boldsymbol{n}) &= \frac{1}{\sqrt{2}\hbar} L_\pm Y^1_0(\boldsymbol{n}) \\ &= \frac{1}{\sqrt{2}\hbar} (L_z \pm i L_y) Y^1_0(\boldsymbol{n}) \\ &= \frac{1}{\sqrt{2}\hbar r} \sqrt{\frac{3}{4\pi}} [y p_z - z p_y \pm i (z p_x - x p_z)] z, \end{aligned}$$

$\Longrightarrow$

$$
\begin{aligned}
Y_1^1(\boldsymbol{n}) &= -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \\
Y_0^1(\boldsymbol{n}) &= \sqrt{\frac{3}{4\pi}}\frac{z}{r} = \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{-1}^1(\boldsymbol{n}) &= \sqrt{\frac{3}{8\pi}}\frac{x-iy}{r} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}.
\end{aligned}
$$

III. ADDITION OF ANGULAR MOMENTUM

A. Composite systems

1. **Two independent systems:** linear spaces $\mathcal{H}_1$ and $\mathcal{H}_2$

   The two systems together: linear space consisting the pairs $(|\psi_1\rangle, |\psi_2\rangle)$, $|\psi_j\rangle \in \mathcal{H}_j$

   Two widely used algebraic structures:

2. **Direct sum:** $|\psi_1\rangle \oplus |\psi_2\rangle = |\psi_1\rangle + |\psi_2\rangle \in \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_1 + \mathcal{H}_2$ treated as an orthogonal sum $\mathcal{H}_1 \perp \mathcal{H}_2$

   (a) Multiplication:

   $$(c|\psi_1\rangle) \oplus |\psi_2\rangle, |\psi_1\rangle \oplus (c|\psi_2\rangle) \in \mathcal{H}_1 \oplus \mathcal{H}_2$$

   (b) Addition:

   $$(|\psi_1\rangle \oplus |\psi_2\rangle) + (|\psi_1'\rangle \oplus |\psi_2'\rangle) = (|\psi_1\rangle + |\psi_1'\rangle) \oplus (|\psi_2\rangle + |\psi_2'\rangle)$$

   (c) Scalar product:

   $$(\langle\psi_1| \oplus \langle\psi_2|)(|\psi_1'\rangle \oplus |\psi_2'\rangle) = \langle\psi_1|\psi_1'\rangle + \langle\psi_2|\psi_2'\rangle \qquad \leftarrow \quad \textbf{sum}$$

   (d) Operators: $A_j : \mathcal{H}_j \to \mathcal{H}_j \Longrightarrow A_1 \oplus A_2 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$

   $$(\langle\psi_1| \oplus \langle\psi_2|)(A_1 \oplus A_2)(|\psi_1'\rangle \oplus |\psi_2'\rangle) = \langle\psi_1|A_1|\psi_1'\rangle + \langle\psi_2|A_2|\psi_2'\rangle \qquad \leftarrow \quad \textbf{sum}$$

   (e) Basis: $\{|n_j\rangle\}$ a basis for $\mathcal{H}_j$

      i. $\Longrightarrow \{|n_1\rangle \oplus |n_2\rangle\}$ a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$

      ii. $\dim \mathcal{H}_1 \oplus \mathcal{H}_2 = \dim \mathcal{H}_1 + \dim \mathcal{H}_2$ ← **sum**

      iii. Components: $\langle j|(|\psi_1\rangle \otimes |\psi_2\rangle) = \langle j|\psi_1\rangle + \langle j|\psi_2\rangle$ ← **sum**

      iv. Wave function: $(\psi_1 \oplus \psi_2)(x) = \langle x|(|\psi_1 \oplus \psi_2\rangle) = \psi_1(x_1) + \psi_2(x_2)$ ← **sum**

3. **Direct product:** $|\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ treated as a linear space generated by the pairs $(\{\mathcal{H}_1\}, \{\mathcal{H}_2\})$

(a) Multiplication:

$$(c|\psi_1\rangle) \otimes |\psi_2\rangle = |\psi_1\rangle \otimes (c|\psi_2\rangle) = c(|\psi_1\rangle \otimes |\psi_2\rangle).$$

(b) Addition:

$$(|\psi_1\rangle \otimes |\psi_2\rangle) + (|\psi_1'\rangle \otimes |\psi_2'\rangle) \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

(c) Scalar product:

$$(\langle\psi_1| \otimes \langle\psi_2|)(|\psi_1'\rangle \otimes |\psi_2'\rangle) = \langle\psi_1|\psi_1'\rangle\langle\psi_2|\psi_2'\rangle \qquad \leftarrow \quad \textbf{product}$$

(d) Operators: $A_j : \mathcal{H}_j \to \mathcal{H}_j \Longrightarrow A_1 \otimes A_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$

$$(\langle\psi_1| \otimes \langle\psi_2|)(A_1 \otimes A_2)(|\psi_1'\rangle \otimes |\psi_2'\rangle) = \langle\psi_1|A_1|\psi_1'\rangle\langle\psi_2|A_2|\psi_2'\rangle \qquad \leftarrow \quad \textbf{product}$$

(e) Basis: $\{|n_j\rangle\}$ a basis for $\mathcal{H}_j$

i. $\Longrightarrow \{|n_1\rangle \otimes |n_2\rangle|$ a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$

ii. $\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \dim \mathcal{H}_1 \dim \mathcal{H}_2$ $\leftarrow$ **product**

iii. Components: $\langle j_1, j_2|\psi_1\rangle \otimes |\psi_2\rangle = \langle j_1|\psi_1\rangle\langle j_2|\psi_2\rangle$ $\leftarrow$ **product**

iv. Wave function: $(\psi_1 \otimes \psi_2)(x_1, x_2) = \langle x_1, x_2|(|\psi_1 \otimes \psi_2\rangle) = \psi_1(x_1)\psi_2(x_2)$ $\leftarrow$ **product**

4. **Usage:**

(a) Direct sum: exclusively existing components

example: s, p, d, etc, atomic shells, $\psi(\boldsymbol{x}) = \sum_{nml} c_{nml}\psi_{nml}(\boldsymbol{x})$

(b) Direct product: simultaneously existing components

example: two-particle state, $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \psi_1(\boldsymbol{x}_1)\psi_2(\boldsymbol{x}_2)$

B. Additive observables and quantum numbers

1. **Momentum:** Generator of translations, $\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{\epsilon}$

$$\begin{aligned}\delta\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \psi(\boldsymbol{r}_1 - \boldsymbol{\epsilon}, \boldsymbol{r}_2 - \boldsymbol{\epsilon}) - \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar}\boldsymbol{\epsilon}(\boldsymbol{p}_1 + \boldsymbol{p}_2)\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ &= -\frac{i}{\hbar}\boldsymbol{\epsilon}\boldsymbol{P}\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \quad \Longrightarrow \quad \boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2\end{aligned}$$

2. **Angular momentum:** Generator of rotations, $\boldsymbol{r} \to \boldsymbol{r} - \frac{i}{\hbar}\epsilon\boldsymbol{n}\boldsymbol{L}$

- An infinitesimal rotation around the z axis:

$$
\begin{aligned}
\boldsymbol{r} &= \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix} \to \begin{pmatrix} r\sin\theta\cos(\phi+\epsilon) \\ r\sin\theta\sin(\phi+\epsilon) \\ r\cos\theta \end{pmatrix} \\
\delta\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) &= -\epsilon(\partial_{\phi_1}+\partial_{\phi_2})\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) \\
&= -\frac{i}{\hbar}\epsilon(L_{1z}+L_{2z})\psi(\boldsymbol{r}_1,\boldsymbol{r}_2) \\
&= -\frac{i}{\hbar}\epsilon L_z\psi(\boldsymbol{r}_1,\boldsymbol{r}_2)
\end{aligned}
$$

- General case: $R_{\boldsymbol{n}}(\epsilon)$ is generated by $\boldsymbol{n}(\boldsymbol{L}_1+\boldsymbol{L}_2)$, $\Longrightarrow \boldsymbol{L}=\boldsymbol{L}_1+\boldsymbol{L}_2$
- Commutation relations:

$$
\begin{aligned}
[L_a, L_b] &= [L_{1a}+L_{2a}, L_{1b}+L_{2b}] \\
&= i\hbar\sum_c \epsilon_{a,b,c}(L_{1c}+L_{2c}) \\
&= i\hbar\sum_c \epsilon_{a,b,c}L_c.
\end{aligned}
$$

- But$\boldsymbol{L}^2=\boldsymbol{L}_1^2+\boldsymbol{L}_2^2+2\boldsymbol{L}_1\boldsymbol{L}_1$ is not additive $\Longrightarrow \ell$ is not additive neither
- Allowed values of ℓ?
  - Classical mechanics

$$
\left(\sqrt{\boldsymbol{L}_1^2}-\sqrt{\boldsymbol{L}_2^2}\right)^2 \le \boldsymbol{L}^2 \le \left(\sqrt{\boldsymbol{L}_1^2}+\sqrt{\boldsymbol{L}_2^2}\right)^2.
$$

  - Quantum mechanics?

C. System of two particles

1. **System of two particles:**
   - States $|\phi_1\rangle \in \mathcal{H}_{\ell_1}$, $|\phi_2\rangle \in \mathcal{H}_{\ell_2}$
   - Representation of rotations: $e^{-\frac{i}{\hbar}\alpha\boldsymbol{n}\boldsymbol{L}}|\phi_1\rangle\otimes|\phi_2\rangle$ in $\mathcal{H}=\mathcal{H}_{\ell_1}\otimes\mathcal{H}_{\ell_2}$.
   - Spectrum of $\boldsymbol{L}^2=(\boldsymbol{L}_1+\boldsymbol{L}_2)^2$: $\{\ell_1,\ell_2,\ldots,\ell_n\} \Longleftrightarrow \mathcal{H}=\mathcal{H}_{\ell_1}\oplus\mathcal{H}_{\ell_2}\oplus\cdots\oplus\mathcal{H}_{\ell_n}$
   - A reducible unitary representation can always be broken up into the direct sum of irreducible representations

2. **Two different bases:**

   (a) Decoupled basis:

$$
|\ell_1,\ell_2,m_1,m_2\rangle = |\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle, \quad -\ell_j \le m_j \le \ell_j, \quad dim\mathcal{H}=(2\ell_1+1)(2\ell_2+1)
$$

(b) Coupled basis: $\{|L, M\rangle\}$:

$$\begin{aligned} \boldsymbol{L}^2|L, M\rangle &= \hbar^2 L(L+1)|L, M\rangle, \\ L_3|L, M\rangle &= \hbar M|L, M\rangle, \end{aligned}$$

3. **Reduction (construction of the coupled basis)**:

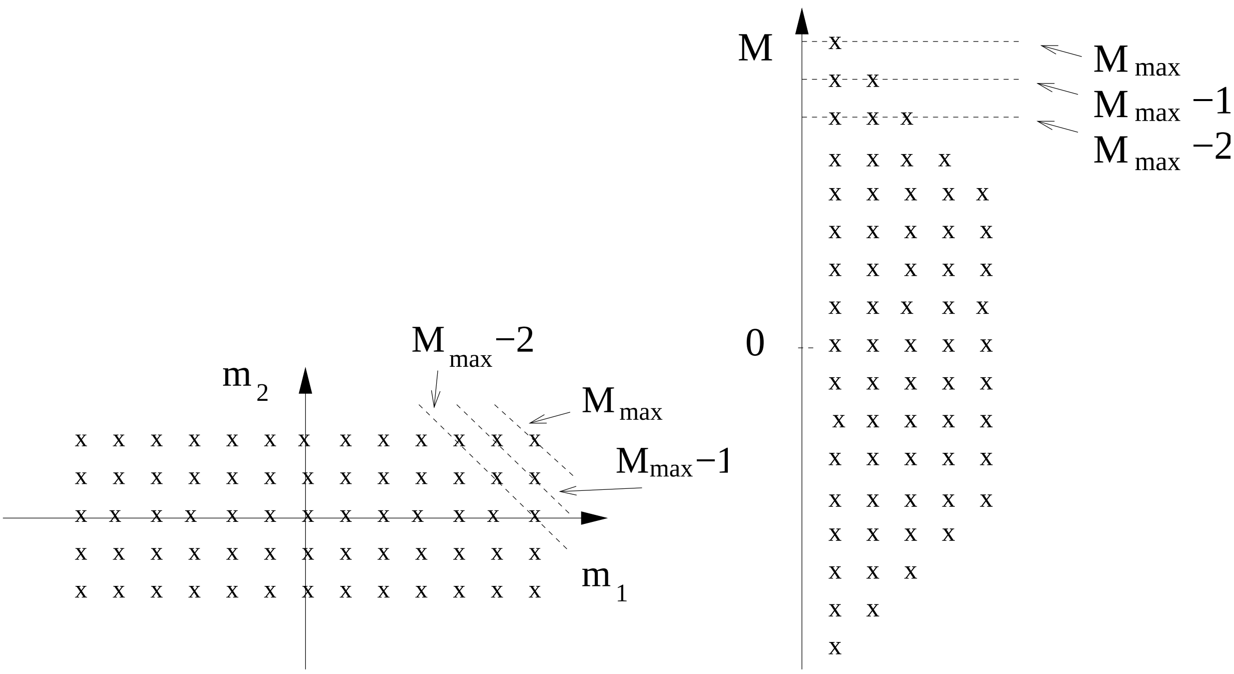
- $M = M_{max} = m_1 + m_2$:

  (a) $|\ell_1, \ell_2, \ell_1, \ell_2\rangle = |M_{max}, M_{max}\rangle \in \mathcal{H}_{M_{max}} \subset \mathcal{H}$

  (b) $|\ell_1, \ell_2, \ell_1, \ell_2\rangle$ is unique $\Longrightarrow$ no other $\mathcal{H}_{M_{max}} \subset \mathcal{H}$

  (c) No $\mathcal{H}_\ell \subset \mathcal{H}$ with $\ell > M_{max}$

  (d) $U(R)\mathcal{H} \subset \mathcal{H} \Longrightarrow \mathcal{H} = \mathcal{H}_{M_{max}} \oplus \cdots$



- $M = M_{max} - 1 = m_1 + m_2 - 1$:

  (a) Application of $L_- = L_{1-} + L_{2-}$:

$$|M_{max}, M_{max} - 1\rangle = \frac{1}{\hbar\sqrt{2M_{max}}} L_-|M_{max}, M_{max}\rangle.$$

  (b) Two decoupled basis elements with $M = M_{max} - 1$: $|\ell_1, \ell_2, \ell_1 - 1, \ell_2\rangle$ and $|\ell_1, \ell_2, \ell_1, \ell_2 - 1\rangle$

  (c) $S_{M_{max}-1} = \{c_1|\ell_1, \ell_2, \ell_1 - 1, \ell_2\rangle + c_2|\ell_1, \ell_2, \ell_1, \ell_2 - 1\rangle\}$ ← dashed lines

    – $\dim(S_{M_{max}-1}) = 2$

    – $S_{M_{max}-1} \subset \mathcal{H}$

  (d) Choose a basis vector $|M_{max}, M_{max} - 1\rangle \in S_{M_{max}-1}$ such that $|M_{max}, M_{max} - 1 \in\rangle \mathcal{H}_{M_{max}}$

(e) The other, orthogonal basis vector belongs to a new multiplet, $|M_{max}-1, M_{max}-1\rangle \in S_{M_{max}-1}$, $|M_{max}-1, M_{max}-1\rangle \in \mathcal{H}_{M_{max}-1}$

(f) $\in \mathcal{H}_{M_{max}-1} \subset \mathcal{H}$ comes with multiplicity one in $\mathcal{H}$.

(g) $U(R)\mathcal{H} \subset \mathcal{H} \Longrightarrow \mathcal{H} = \mathcal{H}_{M_{max}} \oplus \mathcal{H}_{M_{max}-1} \oplus \cdots$

- Iteration:

$$\mathcal{H} = \mathcal{H}_{|\ell_1-\ell_2|} \oplus \cdots \oplus \mathcal{H}_{\ell_1+\ell_2}$$

or

$$\boxed{\ell_1 \otimes \ell_2 = |\ell_1-\ell_2| \oplus |\ell_1-\ell_2|+1 \oplus \cdots \oplus \ell_1+\ell_2-1 \oplus \ell_1+\ell_2}$$

- Sum rule:

$$\mathrm{dim}\mathcal{H} = (2\ell_1+1)(2\ell_2+1) = \sum_{|\ell_1-\ell_2|\le\ell\le\ell_1+\ell_2} (2\ell+1),$$

- Resolution of the identity in $\mathcal{H}$:

$$\mathbb{1} = \sum_{m_1,m_2} |\ell_1,\ell_2,m_1,m_2\rangle\langle\ell_1,\ell_2,m_1,m_2| = \sum_{L,M} |L,M\rangle\langle L,M|$$

- Appears reasonable in the semiclassical limit, $l_1, l_2 \to \infty$

4. **Clebsch-Gordan coefficients:**

- Definition:

$$\boxed{(\ell_1,\ell_2,m_1,m_2|L,M) = \langle\ell_1,\ell_2,m_1,m_2|L,M\rangle}$$

- Decoupled $\to$ coupled:

$$\begin{aligned} |L,M\rangle &= \sum_{m_1,m_2} |\ell_1,\ell_2,m_1,m_2\rangle\langle\ell_1,\ell_2,m_1,m_2|L,M\rangle \\ &= \sum_{m_1,m_2} |\ell_1,\ell_2,m_1,m_2\rangle(\ell_1,\ell_2,m_1,m_2|L,M) \end{aligned}$$

- Additivity of L_z:

$$(\ell_1,\ell_2,m_1,m_2|L,M) = \delta_{m_1+m_2,M}(\ell_1,\ell_2,m_1,M-m_1|L,M)$$

- Theorem: One can choose the phase of $|\ell,m\rangle$ in such a manner that Clebsch-Gordan coefficients become real.

- Coupled → decoupled:

$$\begin{aligned} |\ell_1,\ell_2,m_1,m_2\rangle &= \sum_{L,M} |L,M\rangle\langle L,M|\ell_1,\ell_2,m_1,m_2\rangle \\ &= \sum_{L,M} |L,M\rangle\langle \ell_1,\ell_2,m_1,m_2|L,M\rangle^* \\ &= \sum_{L,M} |L,M\rangle(\ell_1,\ell_2,m_1,m_2|L,M)^* \\ &= \sum_{L,M} |L,M\rangle(\ell_1,\ell_2,m_1,m_2|L,M) \end{aligned}$$

5. **Simplest non-trivial example:** $\frac{1}{2}\otimes\frac{1}{2} = 0 \oplus 1$:

- $M = \pm 1$:

$$\begin{aligned} |1,\pm 1\rangle &= |\pm\frac{1}{2},\pm\frac{1}{2}\rangle \\ (\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}|1,\pm 1) &= 1 \end{aligned}$$

- $M = 0$:

(a) $L = 1$:

$$\begin{aligned} |1,0\rangle &= \frac{1}{\sqrt{2}\hbar}L_-|1,1\rangle \\ &= \frac{1}{2\sqrt{2}}[\sigma_{1x}+\sigma_{2x}-i(\sigma_{1y}+\sigma_{2y})]|\frac{1}{2},\frac{1}{2}\rangle \\ &= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0&0\\1&0\end{pmatrix}_1+\begin{pmatrix}0&0\\1&0\end{pmatrix}_2\right]|\frac{1}{2},\frac{1}{2}\rangle \\ &= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle+|-\frac{1}{2},\frac{1}{2}\rangle\right) \\ (\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2}|1,0) &= \frac{1}{\sqrt{2}} \end{aligned}$$

(b) $L = 0$:

$$\begin{aligned} |0,0\rangle &= \frac{1}{\sqrt{2}}\left(|\frac{1}{2},-\frac{1}{2}\rangle-|-\frac{1}{2},\frac{1}{2}\rangle\right) \\ (\frac{1}{2},\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2}|1,0) &= \pm\frac{1}{\sqrt{2}} \end{aligned}$$

(c) $\mathcal{H}_1$: symmetric with respect to the exchange of the two particles

$\mathcal{H}_0$: antisymmetric with respect to the exchange of the two particles

IV. SELECTION RULES

A. Tensor operators

1. **Definition:** $\{T^{(\ell)}_m\}$, $-\ell \le m \le \ell$, transform according to two equivalent ways

- operators acting in the Hilbert space and
- as tensors, basis vectors of an irreducible multiplet in the linear space of operators,

$$\boxed{U^\dagger(R)T^{(\ell)}_m U(R) = \sum_{m'} T^{(\ell)}_{m'} \mathcal{D}^\ell_{m',m}(R^{-1})} = \sum_{m'} \mathcal{D}^{\ell*}_{m,m'}(R) T^{(\ell)}_{m'}$$

- $\ell = 1$:

$$A\boldsymbol{L} = \boldsymbol{L}A^{-1} = U^\dagger(A)\boldsymbol{L}U(A),$$

$\nearrow$ vector $\qquad\qquad$ $\nwarrow$ operator

2. **Invariance:**

$$\sum_{m'} U^\dagger(R)T^{(\ell)}_{m'} U(R)\mathcal{D}^\ell_{m',m}(R) = T^{(\ell)}_m$$

3. **Spherical harmonics:**

(a) Transformation of kets and wave functions:

$$\begin{aligned} Y^\ell_m(R\boldsymbol{n}) &= U(R^{-1})Y^\ell_m(\boldsymbol{n}) = \langle \boldsymbol{n}|U(R^{-1})|\ell,m\rangle = \langle \boldsymbol{n}|\mathbb{1}U(R^{-1})|\ell,m\rangle \\ &= \sum_{m'} \langle \boldsymbol{n}|\ell,m'\rangle \underbrace{\langle \ell,m'|U(R^{-1})|\ell,m\rangle}_{D^{(\ell)}_{m',m}(R^{-1})} = \sum_{m'} Y^\ell_{m'}(\boldsymbol{n}) D^{(\ell)}_{m',m}(R^{-1}) \end{aligned}$$

(b) Transformation of the spherical function of operators: $Y^\ell_m(\hat{\boldsymbol{n}})$, $R\hat{\boldsymbol{n}} = U^\dagger(R)\hat{\boldsymbol{n}}U(R)$

$\nearrow$ vector operator

$$\begin{aligned} Y^\ell_m(R\hat{\boldsymbol{n}}) &= \sum_{m'} Y^\ell_{m'}(\hat{\boldsymbol{n}}) D^{(\ell)}_{m',m}(R^{-1}) \\ &= Y^\ell_m(U^\dagger(R)\hat{\boldsymbol{n}}U(R)) = U^\dagger(R)Y^\ell_m(\hat{\boldsymbol{n}})U(R) \end{aligned}$$

B. Orthogonality relations

1. **Orthogonality theorem:** The set of matrix elements of all irreducible representations of a group form a full, orthogonal basis for functions on the group.

2. $SO(3)$**:**

$$\begin{aligned} \mathcal{D}^{(\ell)}_{m',m}(R(\phi,\theta,\chi) &= \langle \ell,m'|U(R_{\boldsymbol{z}}(\phi)U(R_{\boldsymbol{y}}(\theta)U(R_{\boldsymbol{z}}(\chi)|\ell,m\rangle \\ &= e^{-im'\phi-im\chi} d^{(\ell)}_{m',m}(\theta), \end{aligned}$$

is a basis for $SO(3) = \{R(\phi,\theta,\chi)\}$ with the integral measure $d\phi d(\cos\theta)d\chi$,

- *Orthogonality:*

$$\boxed{\int dR \mathcal{D}^{(\ell_1)*}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) = \frac{8\pi^2}{2\ell_1+1}\delta_{\ell_1,\ell_2}\delta_{m_1',m_2'}\delta_{m_1,m_2}}$$

- *Completeness:*

$$f(\phi,\theta,\chi) = \sum_{\ell,m,m'} f_{\ell,m,m'}\mathcal{D}^{(\ell)}_{m,m'}(R(\phi,\theta,\chi))$$

where

$$f_{\ell,m,m'} = \frac{2\ell_1+1}{8\pi^2}\int_{-\pi}^{\pi} d\phi \int_{-1}^{1} d(\cos\theta)\int_{-\pi}^{\pi} d\chi \mathcal{D}^{(\ell)*}_{m,m'}((\phi,\theta,\chi))f(\phi,\theta,\chi)$$

for square integrable functions over $SO(3)$.

- *Hand waving argument:* $\mathcal{D}^{\ell}_{m,m'}(\phi,\theta,\chi) = d^{(\ell)}_{m,m'}(\theta)e^{-im\phi-im'\chi}$
  - Set of spherical harmonics,

$$Y^{\ell}_m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}d^{(\ell)*}_{m,0}(\theta)e^{im\phi}$$

    is a basis over θ,ϕ (S_2) with the integral measure $d\phi d(\cos\theta)$.
  - χ-depenence: $\{e^{-im'\chi}\}$ is a basis for $S_1 = U(1)$ with the integral measure $d\chi$
  - Normalization:

$$\int_{SO(3)} dR|\mathcal{D}^{(\ell)}_{0,0}(R)|^2 = \frac{8\pi^2}{2\ell+1}$$

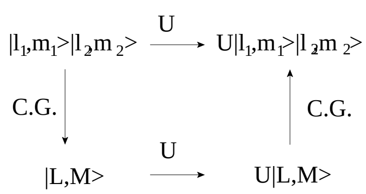
3. **Applied for the addition of angular momentum:**

- *Clebsch-Gordan coefficient are real* $\Longrightarrow$ the basis transformation from the decoupled to the coupled basis is not only unitary but orthogonal,

$$\begin{aligned} |L,M\rangle &= \sum_{m_1,m_2}|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle(\ell_1,\ell_2,m_1,m_2|L,M) \\ |\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle &= \sum_{L,M}|L,M\rangle(\ell_1,\ell_2,m_1,m_2|L,M) \end{aligned}$$

- *Two ways of calculating the result of a rotation:*

(a) Commutative diagram:



$$
\begin{aligned}
U(R)|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle &= \sum_{m_1',m_2'} |\ell_1,m_1'\rangle\otimes|\ell_2,m_2'\rangle \mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) \\
&= \sum_{L,M,M'} |L,M'\rangle \mathcal{D}^{(L)}_{M',M}(R)\underbrace{(\ell_1,\ell_2,m_1,m_2|L,M)}_{\langle L,M|\ell_1,\ell_2,m_1,m_2\rangle} \\
&= \sum_{L,M,M',m_1',m_2'} \underbrace{|\ell_1,m_1'\rangle\otimes|\ell_2,m_2'\rangle(\ell_1,\ell_2,m_1',m_2'|L,M')}_{|L,M'\rangle} \\
&\quad \mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)
\end{aligned}
$$

Projection on $\langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|$:

$$
\begin{aligned}
\langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|U(R)|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle &= \mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) \\
&= \sum_{L,M,M'} (\ell_1,\ell_2,m_1',m_2'|L,M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)
\end{aligned}
$$

(b) Resolution of the identity:

i. Trivial (single basis):

$$
\begin{aligned}
\mathbb{1} &= \sum_n |n\rangle\langle n| \\
\langle n|A|n'\rangle &= \langle n| \underbrace{\mathbb{1}}_{\sum_m |m\rangle\langle m|} A \underbrace{\mathbb{1}}_{\sum_{m'} |m'\rangle\langle m'|} |n'\rangle = \langle n|A|n'\rangle
\end{aligned}
$$

ii. Less trivial (several bases):

$$
\begin{aligned}
\mathbb{1}_d &= \sum_{m_1,m_2} |\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle\langle\ell_2,m_1|\otimes\langle\ell_2,m_2| \\
\mathbb{1}_c &= \sum_{L,M} |L,M\rangle\langle L,M| \\
\mathbb{1}_d U(R)\mathbb{1}_d &= \mathbb{1}_d\mathbb{1}_c U(R)\mathbb{1}_c\mathbb{1}_d \\
&= \mathbb{1}_d \sum_{L,M,M'} \underbrace{|L,M\rangle\langle L,M|U(R)|L,M'\rangle\langle L,M'|}_{\mathbb{1}_c U(R)\mathbb{1}_c}\mathbb{1}_d
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|U(R)|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle = \\
&= \sum_{L,M,M'} \langle\ell_1,m_1'|\otimes\langle\ell_2,m_2'|L,M\rangle\langle L,M|U(R)|L,M'\rangle\langle L,M'|\ell_1,m_1\rangle\otimes|\ell_2,m_2\rangle \\
&= \sum_{L,M,M'} \langle\ell_1,\ell_2,m_1',m_2'|L,M'\rangle\mathcal{D}^{(L)}_{M',M}(R)\langle L,M|\ell_1,\ell_2,m_1,m_2\rangle \\
&= \sum_{L,M,M'} (\ell_1,\ell_2,m_1',m_2'|L,M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)
\end{aligned}
$$

- *Multiplication by $\mathcal{D}^{(L)*}_{M',M}(R)$ and integration over R:*

$$
\begin{aligned}
\mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) &= \sum_{L,M,M'} (\ell_1,\ell_2,m_1',m_2'|L,M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M) \\
\int dR\mathcal{D}^{(L)*}_{M',M}\mathcal{D}^{\ell_1}_{m_1',m_1}(R)\mathcal{D}^{\ell_2}_{m_2',m_2}(R) &= \\
&\int dR\mathcal{D}^{(L)*}_{M',M}(R)(\ell_1,\ell_2,m_1',m_2'|L,M')\mathcal{D}^{(L)}_{M',M}(R)(\ell_1,\ell_2,m_1,m_2|L,M)
\end{aligned}
$$

- *Orthogonality relation for Clebsch-Gordan coefficients:*

$$\int dR\mathcal{D}^{(L)*}_{M',M}(R)\mathcal{D}^{(\ell_1)}_{m_1',m_1}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R) = \frac{8\pi^2}{2L+1}(\ell_1,\ell_2,m_1',m_2'|L,M')(\ell_1,\ell_2,m_1,m_2|L,M)$$

$$\nearrow \qquad\qquad\qquad \nwarrow$$

$$\langle \ell_1,\ell_2,m_1',m_2'|L,M'\rangle \qquad\qquad \langle L,M|\ell_1,\ell_2,m_1,m_2\rangle$$

C. Wigner-Eckart theorem

1. **Selection rules for a tensor operator:** Rotational quantum numbers $\{\ell,m\}$, remaining quantum numbers n

$$\mathcal{M} = \langle n_1,\ell_1,m_1|T^{(\ell)}_m|n_2,\ell_2,m_2\rangle$$

$$\nearrow \qquad\qquad \nwarrow$$

Rotational quantum numbers $\{\ell,m\}$ remaining quantum numbers n

2. **Derivation:**

- *Tensor operator invariance:* $\sum_{m'} U^\dagger(R)T^{(\ell)}_{m'}U(R)\mathcal{D}^\ell_{m',m}(R) = T^{(\ell)}_m$

$$\begin{aligned}
\mathcal{M} &= \langle n_1,\ell_1,m_1|T^{(\ell)}_m|n_2,\ell_2,m_2\rangle \\
&= \sum_{m'}\langle n_1,\ell_1,m_1|U^\dagger(R)T^{(\ell)}_{m'}U(R)|n_2,\ell_2,m_2\rangle\mathcal{D}^\ell_{m',m}(R) \\
&= \sum_{m'}\langle n_1,\ell_1,m_1|U^\dagger(R)\mathbb{1}T^{(\ell)}_{m'}\mathbb{1}U(R)|n_2,\ell_2,m_2\rangle\mathcal{D}^\ell_{m',m}(R) \quad \leftarrow \quad \mathbb{1} = \sum_m |\ell,m\rangle\langle\ell,m| \\
&= \sum_{m_1'm_2'm'} \underbrace{\langle n_1,\ell_1,m_1|U^\dagger(R)|n_1,\ell_1,m_1'\rangle}_{\langle n_1,\ell_1,m_1'|U(R)|n_1,\ell_1,m_1\rangle^*}\langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle\underbrace{\langle n_2,\ell_2,m_2'|U(R)|n_2,\ell_2,m_2\rangle}_{\langle n_2,\ell_2,m_2'|U(R)|n_2,\ell_2,m_2\rangle} \\
&\quad \times\mathcal{D}^\ell_{m',m}(R)
\end{aligned}$$

Integration over R:

$$\mathcal{M}\int dR = \sum_{m_1',m_2',m'} \langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle \int dR\mathcal{D}^{(\ell_1)*}_{m_1',m_1}(R)\mathcal{D}^{(\ell)}_{m',m}(R)\mathcal{D}^{(\ell_2)}_{m_2',m_2}(R)$$

- *Orthogonality relation for Clebsch-Gordan coefficients:*

$$\mathcal{M}\underbrace{\int dR}_{8\pi^2} = \frac{8\pi^2}{2\ell_1+1}(\ell,\ell_2,m,m_2|\ell_1,m_1)\sum_{m_1',m_2',m'}(\ell,\ell_2,m',m_2'|\ell_1,m_1')\langle n_1,\ell_1,m_1'|T^{(\ell)}_{m'}|n_2,\ell_2,m_2'\rangle.$$

- **Wigner-Eckart theorem:**

$$\boxed{\mathcal{M} = (\ell,\ell_2,m,m_2|\ell_1,m_1) \ll n_1,\ell_1|T^{(\ell)}|n_2,\ell_2 \gg}$$

Factorization of the rotational kinematics from the rest of the dynamics

$\nearrow$ $\nwarrow$

$(\ell, \ell_2, m, m_2|\ell_1, m_1)$ reduced matrix element:

$$\ll n_1, \ell_1|T^{(\ell)}|n_2, \ell_2 \gg = \frac{1}{2\ell_1+1}\sum_{m_1', m_2', m'}(\ell, \ell_2, m', m_2'|\ell_1, m_1')\langle n_1, \ell_1, m_1'|T^{(\ell)}_{m'}|n_2, \ell_2, m_2'\rangle,$$

3. **Selection rule:** $\langle n_1, \ell_1, m_1|T^{(\ell)}_m|n_2, \ell_2, m_2\rangle$ is vanishing if $(\ell, \ell_2, m, m_2|\ell_1, m_1) = 0$

4. **Examples:**

   (a) $\ell = 0$:

$$\begin{aligned}(\ell_2, 0, m_2, 0|\ell_1, m_1) &= \delta_{\ell_1,\ell_2}\delta_{m_1,m_2}\\ \langle n_1, \ell_1, m_1|T^{(0)}_m|n_2, \ell_2, m_2\rangle &= \delta_{\ell_1,\ell_2}\delta_{m_1,m_2} \ll n_1, \ell_1|T^{(0)}|n_2, \ell_2 \gg\end{aligned}$$

   Rotation invariant potential $U(r) = r^p$

$$\langle n_1, \ell_1, m_1|r^p|n_2, \ell_2, m_2\rangle = \underbrace{\int d\phi\int d(\cos\theta)Y^{\ell_1 *}_{m_1}(\theta,\phi)Y^{\ell_2}_{m_2}(\theta,\phi)}_{(\ell_2,0,m_2,0|\ell_1,m_1)}\underbrace{\int dr r^{2+p}\eta^*_{n_1,\ell_1}(r)\eta_{n_2,\ell_2}(r)}_{\ll n_1,\ell_1|r^p|n_2,\ell_2\gg}$$

   (b) $\ell = 1$: angular momentum

$$\langle n_1, \ell_1, m_1|T^{(1)}_m|n_2, \ell_2, m_2\rangle = (1, \ell_2, m, m_2|\ell_1, m_1) \ll n_1, \ell_1|T|n_2, \ell_2 \gg$$

   To find the reduced matrix elements for the angular momentum $T^{(1)}_0 = L_z$, $T^{(1)}_{\pm 1} = \mp\frac{1}{\sqrt{2}}L_\pm$:

$$\begin{aligned}\langle n_1, \ell_1, \ell_1|L_0|n_2, \ell_2, \ell_2\rangle &= (1, \ell_2, 0, \ell_2|\ell_1, \ell_1) \ll n_1, \ell_1|L|n_2, \ell_2 \gg\\ \langle n_1, \ell_1, m|L_0|n_2, \ell_2, m\rangle &= \hbar m\delta_{n_1,n_2}\delta_{\ell_1,\ell_2}\\ (1, \ell_1, 0, \ell_1|\ell_1, \ell_1) &= \sqrt{\frac{\ell_1}{\ell_1+1}}\\ \Longrightarrow \quad \ll n_1, \ell_1|L|n_2, \ell_2 \gg &= \frac{\langle n_1, \ell_1, \ell_1|L_0|n_2, \ell_2, \ell_2\rangle}{(1, \ell_2, 0, \ell_2|\ell_1, \ell_1)} = \delta_{n_1,n_2}\delta_{\ell_1,\ell_2}\hbar\sqrt{\ell(\ell+1)}.\end{aligned}$$

V. RELATIVISTIC CORRECTIONS TO THE HYDROGEN ATOM

A. Scale dependence of physical laws

1. **No "constants" in physics:** no truly isolate system

   Measured results depend on the scale of observation (environment)

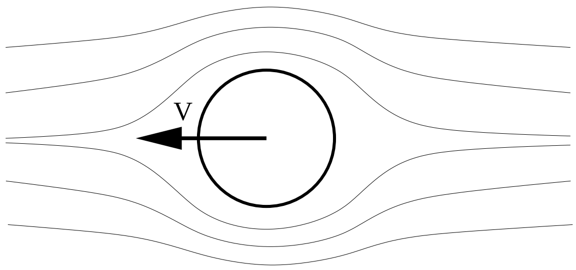
   Scales in physics: dimensional quantities, M, L, T, and ...

   $\nwarrow$

to establish relation between physical quantities

Bureau of Standard: to assure unchanged environment

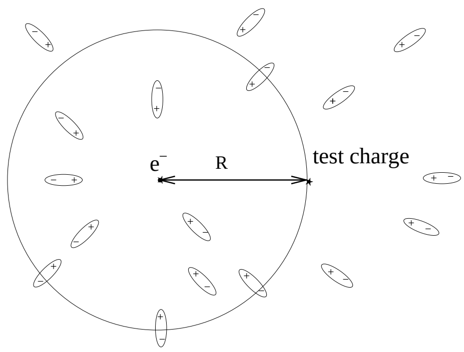
(a) *Mass:* a ball moving with velocity v in a viscous fluid

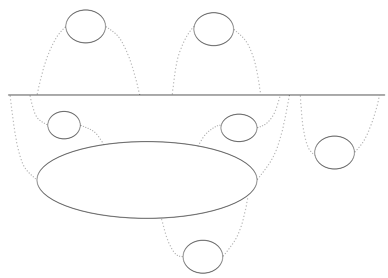


$$E_{tot}(v) = E_{ball}(v) + E_{fl}, \quad E_{ball} = \frac{m(v)}{2}v^2 \quad \Longrightarrow \quad m(v) = \frac{d^2 E_{tot}(v)}{dv^2}$$

(b) *Charge:*

- Polarization:



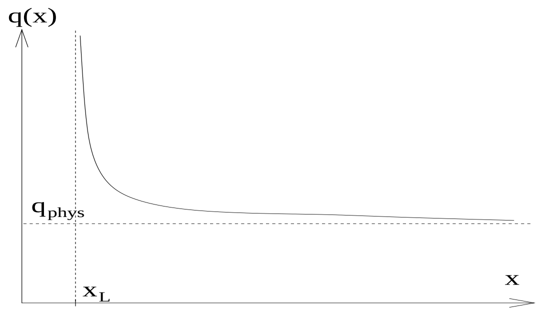


Classical polarizable medium

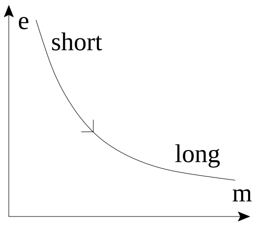
Vacuum polarization around a charge in QED

- Running electric charge:

$$F(R) \neq \frac{q_t q}{R^2} \quad \Longrightarrow \quad F(R) = \frac{q_t q(R)}{R^2}$$



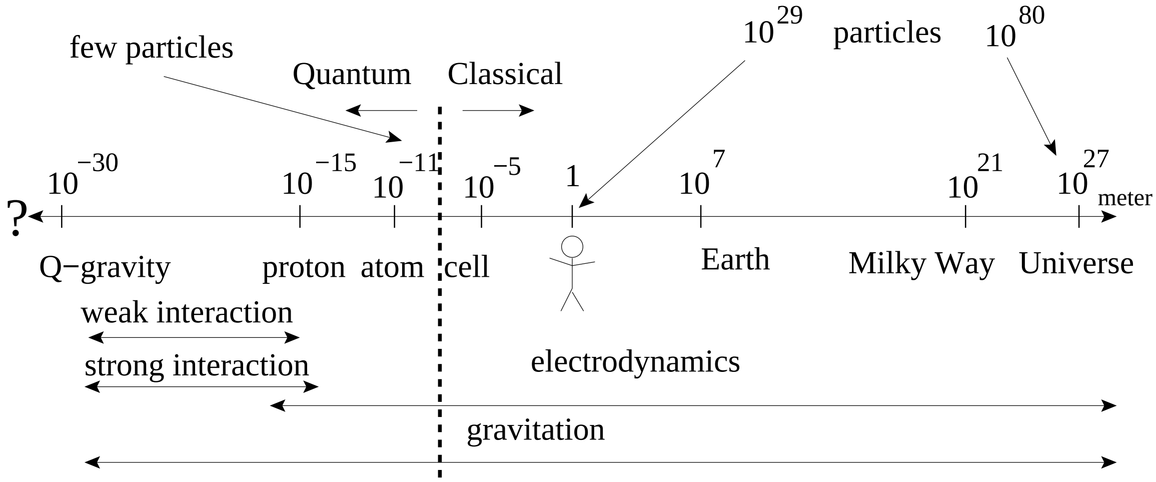
- Renormalized trajectory: Identical physics, changing resolution



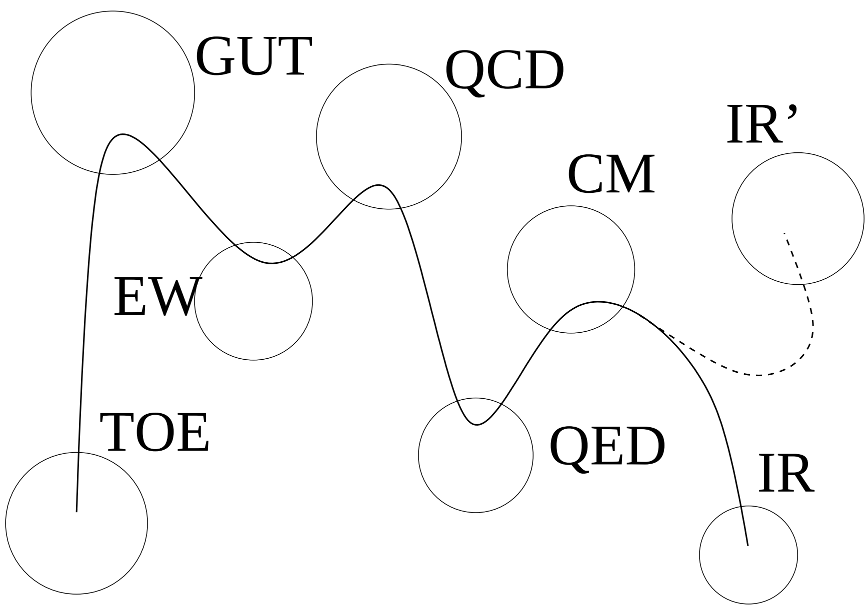
(c) *Speed of light:*

$$v = \frac{c}{\sqrt{\epsilon\mu}}$$

(d) *Relevant length scales:*



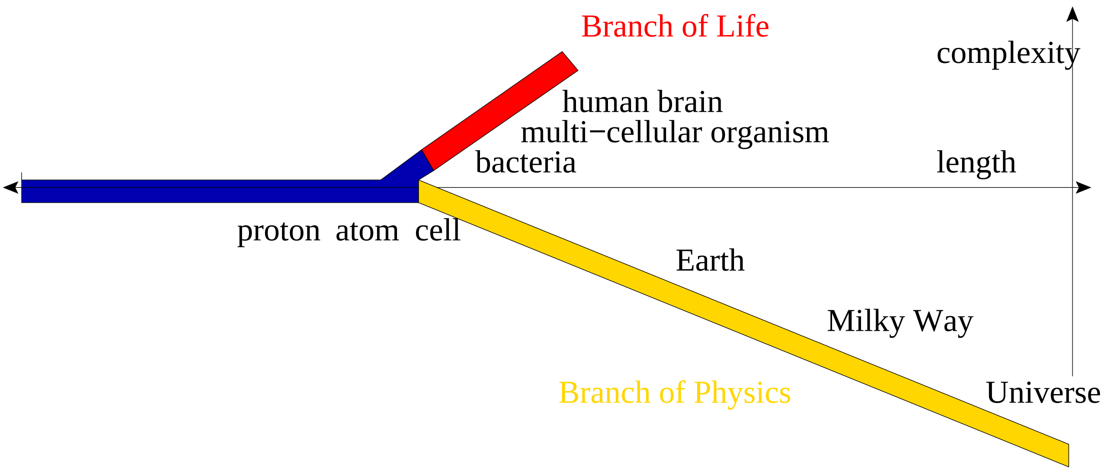
(e) *Theory Of Everything:* parameter space of all "constants", a guided tour of physics



2. **Why can not we understand Quantum Mechanics?**

   - The brain is a problem solver organ for the problems presented by the senses
   - We learn about the classical world in childhood by playing with macroscopic objects
   - Intuition, logics are based on macroscopic, classical physics,
   - We have no clue to the quantum world
   - What is left is the universal language of mathematics without "understanding"

3. **Quantum Biology:** Life = microscopic order enfolding on macroscopic level



   - Average of micr. events
     - Photosynthesis (molecular antennas)
     - Electron transfer in proteins (transport at the mIcr-mAcr edge)
   - A single micr. event
     - Rhodopsin in the retina (photon detector)
     - Olfaction (spectrum analyser)
     - Bird navigation by the Earth's magnetic field (quantum measuring device)
     - Neuron dynamics (brain as an amplifier)
   - Evolution:
     - 10^{60} possible proteins, 6225 appear in living organisms
     - How were they selected?
       * At least 165 nucleic acid bases in RNA for reproductibility
       * $4^{165} \sim 10^{99}$ possibilities
       * One from each in a primordial soup: $10^{25} \times M_{Univ.}$

    - "Survival of the fittest" is not enough

  - Quantum criticality

    - 500 randomly chosen proteins function in between the micr. and the macr. domain
    - Life exploits the more efficient quantum transport processes on the macr. scale

B. Hierarchy of scales in QED

1. **Fine-structure constant:**

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

   (a) *Relativistic effects in the hydrogen atom:* $a_0 = \frac{\hbar^2}{me^2}$

$$\frac{v^2}{c^2} \approx \frac{\frac{\hbar^2}{m^2 a_0^2}}{c^2} = \frac{e^4}{\hbar^2 c^2} = \alpha^2.$$

   (b) *Hierarchy of length scales:*

Bohr radius, $a_0 \Longrightarrow$ perturbation expansion $\Longrightarrow$ length scales $r_n = a_0\alpha^n$, $n = 1, 2, ...$
$\Longrightarrow$ semiclassical expansion $\Longrightarrow$ length scales $r_n = a_0\alpha^{-n}$, $n = 1, 2, ...$

2. **Bohr radius:** $n = 0$,

   - $a_0 = \frac{\hbar^2}{me^2} \approx 0.053nm$
   - size of a hydrogen atom
   - $\mathcal{O}\left(c^0\right) \Longrightarrow$ non-relativistic physics

3. **Compton wavelength:** $n = 1$

   - $\lambda_C = \frac{\hbar}{mc} \approx 3.86 \cdot 10^{-11} cm = 386 fm$
   - $\mathcal{O}\left(e^0\right) \Longrightarrow$ relativistic dynamics of a neutral particle
   - particle localized in a region of length $\ell \lessapprox \lambda_C \Longrightarrow$ pair creation

$$E = c\sqrt{m^2c^2 + \boldsymbol{p}^2} \approx c\sqrt{m^2c^2 + \frac{\hbar^2}{\ell^2}}$$

4. **Classical electron radius:** $n = 2$

   - electron-proton Coulomb energy creates electron-positron pairs
   - r_c :

$$\frac{e^2}{r_c} = mc^2 \quad \Longrightarrow \quad r_c = \frac{e^2}{mc^2} \approx 2.8 fm$$

- $\mathcal{O}\left(\hbar^0\right) \Longrightarrow$ classical physics, embedded deeply into the quantum domain
- Abraham-Lorentz force, the last more or less open chapter of classical electrodynamics

5. **Lamb shift:** $n = 3$

   - $\ell_L = \frac{e^4}{mc^3\hbar} \approx 0.02fm$
   - accidental degeneracy of the hydrogene atom spectrum

6. **Beyond** $n = 0, 1, 2, 3$:

$$\underbrace{\cdots\ -2,\ -1}_{\substack{\uparrow \\ \text{Overwritten by classical physics}}},\ \overbrace{0,\ 1,\ 2,\ 3}^{\text{visible}},\ \underbrace{4,\ 5,\ 6,\ \cdots}_{\substack{\uparrow \\ \text{by the electro-weak interaction}}}$$

C. Unperturbed, non-relativistic dynamics

1. **Hamiltonian:** $\boldsymbol{P} = \boldsymbol{p}_e + \boldsymbol{p}_p$, $\boldsymbol{p} = \boldsymbol{p}_e - \boldsymbol{p}_p$, $\boldsymbol{r} = \boldsymbol{r}_e - \boldsymbol{r}_p$

$$H = \frac{\boldsymbol{P}^2}{2M} + \frac{\boldsymbol{p}^2}{2m} - \frac{e^2}{r}, \quad M = m_e + m_p, \quad \frac{1}{m} = \frac{1}{m_e} + \frac{1}{m_p} \approx \frac{1}{m_e}$$

2. **Eigenstates:** $\boldsymbol{P}$ free motion

$$\psi_{n,\ell,m,s_e,s_p}(r,\theta,\phi,\sigma,\Sigma) = \eta_{n,\ell}(r) Y^\ell_m(\theta,\phi)\chi_{s_e}(\sigma)\chi_{s_p}(\Sigma),$$

3. **Eigenvalues:** Rydberg constant: $R = \frac{\hbar^2}{2ma_0^2} \approx 13.6eV$

$$E_{n,\ell,m,s} = -\frac{R}{n^2}, \quad \ell = 0,\dots,n-1, \quad -\ell \le m \le \ell$$

(accidental) Degeneracy: $(2S_p + 1)(2S_e + 1)n^2 = 4n^2$-fold

D. Fine structure

1. **Relativistic effects:**

   - *Kinetic energy:* relativistic free particle
   - *Interactions:* dynamical degrees of freedom of the E.M. field are resolved

2. **Kinetic energy**

(a) *Origin:*

$$E = c\sqrt{m^2c^2 + \boldsymbol{p}^2} = mc^2 + \frac{\boldsymbol{p}^2}{2m} - \frac{\boldsymbol{p}^4}{8m^3c^2} + \mathcal{O}\left(\left(\frac{v}{c}\right)^6\right)$$

(b) *Form:*

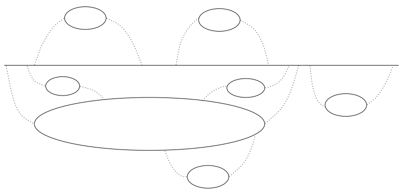
$$H_0 = \frac{\boldsymbol{p}^2}{2m} - \frac{\boldsymbol{p}^4}{8m^3c^2} = \frac{\boldsymbol{p}^2}{2m} + H_m.$$

(c) *Magnitude:*

$$\frac{|H_m|}{\frac{\boldsymbol{p}^2}{2m}} \approx \frac{\frac{\boldsymbol{p}^4}{m^3c^2}}{\frac{\boldsymbol{p}^2}{m}} = \frac{v^2}{c^2} = \alpha^2$$

3. **Darwin term:**

(a) *Origin:* cloud of virtual electron-positron pairs, the vacuum polarization of the Dirac-see



(b) *Form:* Smearing, $\rho(\boldsymbol{r}) = \delta(\boldsymbol{r}) \ \rightarrow \ \rho(\boldsymbol{r})$, $\int d\boldsymbol{r}\rho(\boldsymbol{r}) = 1$, $U_C(r) = \frac{e^2}{r} \ \rightarrow \ U(\boldsymbol{r}) = U_C(\boldsymbol{r}) + U_D(\boldsymbol{r})$

Multipole expansion:

$$\begin{aligned}
U(\boldsymbol{r}) &= \int d\boldsymbol{r}'\rho(\boldsymbol{r}')U_C(\boldsymbol{r}+\boldsymbol{r}') \\
&= \int d\boldsymbol{r}'\rho(\boldsymbol{r}')\left[U_C(\boldsymbol{r}) + \boldsymbol{r}'\boldsymbol{\nabla}U_C(\boldsymbol{r}) + \frac{1}{2}r'_jr'_k\partial_j\partial_kU_C(\boldsymbol{r}) + \cdots\right] \\
\int d\boldsymbol{r}'\boldsymbol{r}' &= 0, \qquad \int d\boldsymbol{r}'\rho(\boldsymbol{r}')r'_jr'_k = \frac{1}{3}\delta_{jk}\underbrace{\int d\boldsymbol{r}'\boldsymbol{r}'^2\rho(\boldsymbol{r}')}_{\frac{3}{4}\lambda_C^2} \\
&= U_C(\boldsymbol{r}) + \frac{\lambda_C^2}{4}\boldsymbol{\nabla}^2U_C(\boldsymbol{r}) \\
H_D &= -U_D = \frac{1}{8}\lambda_C^2\boldsymbol{\nabla}^2U_C(\boldsymbol{r}) = -\frac{1}{2}\pi e^2\lambda_C^2\delta(\boldsymbol{r}) = -\frac{\pi\hbar^2e^2}{2m^2c^2}\delta(\boldsymbol{r}) \\
&\uparrow \\
\boldsymbol{\nabla}_x^2\tfrac{1}{|\boldsymbol{x}-\boldsymbol{y}|} &= -4\pi\delta(\boldsymbol{x}-\boldsymbol{y})
\end{aligned}$$

(c) *Magnitude:*

$$\begin{aligned}
-\langle U_D\rangle &= \frac{\pi\hbar^2e^2}{2m^2c^2}\underbrace{|\psi(0)|^2}_{\approx\frac{1}{a_0^3}} \approx \frac{e^2\hbar^2}{m^2c^2}\frac{m^3e^6}{\hbar^6} = mc^2\frac{e^8}{\hbar^4c^4} = mc^2\alpha^4 \\
\langle H_0\rangle &\approx R = \frac{\hbar^2}{2ma_0^2} = \frac{\hbar^2}{2m}\frac{m^2e^4}{\hbar^4} = \frac{me^4}{2\hbar^2} = \frac{1}{2}mc^2\alpha^2 \\
\frac{|\langle H_D\rangle|}{\langle H_0\rangle} &\approx \alpha^2
\end{aligned}$$

4. **Spin-orbit coupling:**

(a) *Origin:* spin $\Longrightarrow$ magnetic moment & moving charge (proton!) $\Longrightarrow$ current $\Longrightarrow$ magnetic field

- *Form:*

$$\boxed{H_i = -\boldsymbol{m}\boldsymbol{B}}$$

- *E.M. field in quantum mechanics:* canonical quantization

$$\begin{aligned}
L &= \frac{m}{2}\dot{\boldsymbol{x}}^2 - e\phi(t,\boldsymbol{x}) + \frac{e}{c}\dot{\boldsymbol{x}}\boldsymbol{A}(t,\boldsymbol{x}) \\
\boldsymbol{p} &= \frac{\partial L(\dot{\boldsymbol{x}},\boldsymbol{x})}{\partial \dot{\boldsymbol{x}}} = m\dot{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}(t,\boldsymbol{x}) \quad \Longrightarrow \quad \dot{\boldsymbol{x}} = \frac{1}{m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) \\
[x_j, p_k] &= i\hbar\delta_{j,k} \quad \Longrightarrow \quad \boldsymbol{p} = \frac{i}{\hbar}\boldsymbol{\nabla} \\
H &= \boldsymbol{p}\dot{\boldsymbol{x}} - L = \boldsymbol{p}\dot{\boldsymbol{x}} - \frac{m}{2}\dot{\boldsymbol{x}}^2 + e\phi(t,\boldsymbol{x}) - \frac{e}{c}\dot{\boldsymbol{x}}\boldsymbol{A}(t,\boldsymbol{x}) \\
&= \boldsymbol{p}\frac{1}{m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) - \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + e\phi - \frac{e}{cm}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\boldsymbol{A} = \frac{(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2}{2m} + e\phi,
\end{aligned}$$

- *Magnetic moment of the orbital angular momentum:*
  - Homogeneous magnetic field

$$\begin{aligned}
A^\mu &= (0,\boldsymbol{A}), \quad A_\mu = (0,-\boldsymbol{A}), \quad \boldsymbol{A} = -\frac{1}{2}\boldsymbol{r}\times\boldsymbol{B} \\
B_i &= (\boldsymbol{\nabla}\times\boldsymbol{A})_i = -\frac{1}{2}\epsilon_{ijk}\nabla_j\epsilon_{k\ell m}x_\ell B_m = -\frac{1}{2}\epsilon_{ijk}\epsilon_{kjm}B_m = \frac{1}{2}\underbrace{\epsilon_{ijk}\epsilon_{jkm}}_{2\delta_{im}}B_m = B_i
\end{aligned}$$

  - Hamiltonian:

$$\begin{aligned}
H &= \frac{\boldsymbol{p}^2}{2m} - \frac{e}{2mc}(\boldsymbol{p}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{p}) + \frac{e^2}{2mc^2}\boldsymbol{A}^2 \\
&= \frac{\boldsymbol{p}^2}{2m} - \frac{e}{mc}\boldsymbol{A}\boldsymbol{p} + i\frac{e\hbar}{2mc}\boldsymbol{\nabla}\boldsymbol{A} + \frac{e^2}{2mc^2}\boldsymbol{A}^2 \\
&= \frac{\boldsymbol{p}^2}{2m} + \frac{e}{2mc}(\boldsymbol{r}\times\boldsymbol{B})\boldsymbol{p} + \frac{e^2}{2mc^2}\boldsymbol{A}^2 \\
&= \frac{\boldsymbol{p}^2}{2m} - \frac{e}{2mc}\boldsymbol{L}\boldsymbol{B} + \frac{e^2}{2mc^2}\boldsymbol{A}^2
\end{aligned}$$

  - Magnetic moment:

$$\boxed{\boldsymbol{m} = \frac{e}{2mc}\boldsymbol{L} = \mu_B\frac{\boldsymbol{L}}{\hbar}, \quad \mu_B = \frac{e\hbar}{2mc}}$$

- *Magnetic moment of the spin:* Pauli 1927, Dirac 1928
  - Pauli: $\boldsymbol{P} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} \to \boldsymbol{\sigma}\boldsymbol{P}$,

$$\begin{aligned}
H &= \frac{\boldsymbol{P}^2}{2m} \to \frac{(\boldsymbol{\sigma}\boldsymbol{P})^2}{2m} = \frac{\{\sigma_j,\sigma_k\}\{P_j,P_k\} + [\sigma_j,\sigma_k][P_j,P_k]}{8m} \\
\sigma_a\sigma_b &= \delta_{a,b} + i\sum_c \epsilon_{abc}\sigma_c \quad \Longrightarrow \quad \{\sigma_j,\sigma_k\} = 2\delta_{jk}, \quad [\sigma_j,\sigma_k] = 2i\epsilon_{jk\ell}\sigma_\ell \\
H &= \frac{2\delta_{jk}\{P_j,P_k\} + 2i\epsilon_{jk\ell}\sigma_\ell[P_j,P_k]}{8m} = \frac{\boldsymbol{P}^2}{2m} + i\frac{\epsilon_{jk\ell}\sigma_\ell[P_j,P_k]}{4m}
\end{aligned}$$

$$
\begin{aligned}
[\nabla_j + f_j, \nabla_k + f_k]\, h &= [\nabla_j, \nabla_k]h + [f_j, f_k]h + \underbrace{[\nabla_j, f_k]h}_{\nabla_j(f_k h) - f_k \nabla_j h} + [f_j, \nabla_k]h = (\nabla_j f_k - \nabla_k f_j)h \\
\frac{(\boldsymbol{\sigma P})^2}{2m} &= \frac{\boldsymbol{P}^2}{2m} - \frac{i}{4m}\epsilon_{jk\ell}\sigma_\ell \frac{\hbar}{i}\frac{e}{c}(\nabla_j A_k - \nabla_k A_j) = \frac{\boldsymbol{P}^2}{2m} - \frac{\hbar e}{2mc}\epsilon_{jk\ell}\nabla_j A_k \sigma_\ell \\
&= \frac{\boldsymbol{P}^2}{2m} - \frac{\hbar e}{2mc} B_\ell \sigma_\ell = \frac{\boldsymbol{P}^2}{2m} - \boldsymbol{mB} \quad \Longrightarrow \quad \boldsymbol{m} = \frac{\hbar e}{2mc}\boldsymbol{\sigma} = \frac{\hbar e}{mc}\frac{\boldsymbol{s}}{\hbar}
\end{aligned}
$$

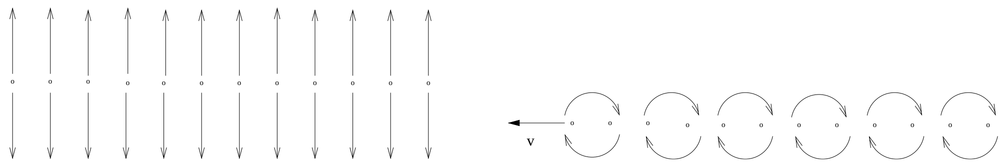
$$
\boxed{\mu = g_S \mu_B, \quad g_S = 2}
$$

– Magnetic moment of a composite particle: $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$

$$
\boldsymbol{m} = \mu_B \frac{\boldsymbol{L} + g\boldsymbol{S}}{\hbar} = \mu_B \frac{\boldsymbol{J} + (g-1)\boldsymbol{S}}{\hbar} \approx g_P \mu_B \frac{\boldsymbol{J}}{\hbar}, \quad g_P \neq 1
$$

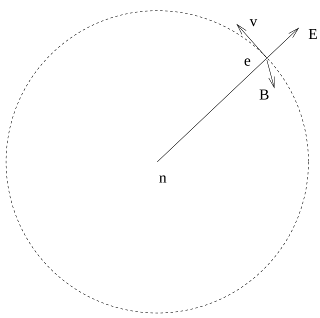
- *Magnetic field in the hydrogen atom:*
  - The best frame: rest frame of the electron
  - Homogeneous electric field $\boldsymbol{E}$ seen by velocity $\boldsymbol{v}$:

$$
\boldsymbol{B} = -\frac{1}{c}\boldsymbol{v} \times \boldsymbol{E}
$$



  - Magnetic field of the moving nucleus

$$
\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{e}_r \partial_r \frac{e}{r} = \frac{e}{r^2}\boldsymbol{e}_r \quad \Longrightarrow \quad \boldsymbol{B} = \frac{1}{c}\partial_r \frac{e}{r}\boldsymbol{v} \times \boldsymbol{e}_r = \frac{1}{rc}\partial_r \frac{e}{r}\boldsymbol{v} \times \boldsymbol{r}
$$



(b) *Form:*

$$
\begin{aligned}
H_{so} &= -\boldsymbol{m}_s \boldsymbol{B} = -2\mu_B \frac{\boldsymbol{s}}{\hbar}\frac{1}{rc}\partial_r \frac{e}{r}\boldsymbol{v} \times \boldsymbol{r} = -2\frac{e\hbar}{2mc}\frac{\boldsymbol{s}}{\hbar}\frac{1}{rc}\partial_r \frac{e}{r}\boldsymbol{v} \times \boldsymbol{r} = -\frac{e^2}{m^2c^2}\frac{1}{r}\partial_r \frac{1}{r}\boldsymbol{s}(\boldsymbol{p} \times \boldsymbol{r}) \\
&= \frac{e^2}{m^2c^2}\frac{1}{r}\partial_r \frac{1}{r}\boldsymbol{sL}
\end{aligned}
$$

(c) *Magnitude:*

$$\frac{\langle H_{so}\rangle}{\langle U_C\rangle} \approx \frac{\frac{e^2\hbar^2}{m^2c^2a_0^3}}{\frac{e^2}{a_0}} = \frac{\hbar^2}{m^2c^2a_0^2} = \frac{\hbar^2}{m^2c^2(\frac{\hbar^2}{me^2})^2} = \frac{e^4}{\hbar^2c^2} = \alpha^2$$

E. Hyperfine structure

1. **Origin:**

(a) $\boldsymbol{S}_e$, $\boldsymbol{L}_e \Longrightarrow$ magnetic field for $\boldsymbol{S}_p$

(b) suppressed by $\frac{m_e}{m_p} \sim \frac{0.51MeV}{938MeV} \sim \frac{1}{2000}$ compared to the fine structure

2. **Form:**

$$\begin{aligned} H_{hf} &= -\frac{1}{c^2}\left\{\frac{e}{m_eR^3}\boldsymbol{L}\boldsymbol{m}_p + \frac{1}{R^3}[3(\boldsymbol{m}_e\boldsymbol{n})(\boldsymbol{m}_p\boldsymbol{n}) - \boldsymbol{m}_e\boldsymbol{m}_p] + \frac{8\pi}{3}\boldsymbol{m}_e\boldsymbol{m}_p\delta^{(3)}(R)\right\} \\ \boldsymbol{m}_e &= 2\frac{e\hbar}{2m_e}\frac{\boldsymbol{s}_n}{\hbar} \\ \boldsymbol{m}_p &= g_p\frac{e\hbar}{2m_n}\frac{\boldsymbol{s}_n}{\hbar}, \quad g_p \approx 5.585\ (\neq 2) \end{aligned}$$

3. **Magnitude:**

$$\langle H_{hf}\rangle \approx \frac{e^2\hbar^2}{m_em_pc^2a_0^3} \approx \langle H_{so}\rangle\frac{m_e}{m_p}.$$

F. Splitting of the fine structure degeneracy

1. **Hamiltonian:**

$$H_f = H_m + H_D + H_{so}.$$

$\boldsymbol{L}\boldsymbol{s}$ in H_{so}:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{s} \quad \Longrightarrow \quad \boldsymbol{L}\boldsymbol{s} = \frac{1}{2}(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{s}^2)$$

Coupled basis:

$$|n, J, M, \ell\rangle = \sum_{s_e}|n, \ell, M - s_e, s_e\rangle(\ell, \frac{1}{2}, M - s_e, s_e|J, M).$$

Spectroscopic quantum numbers: $n\ell_J$, $\ell = 0, 1, 2, 3, \ldots = s, p, d, f, g, \ldots$.

2. $n = 1$**:** $1s$ level, 2 dimensional degeneracy (s_e)

- *We seek* $E^{(1)} = \langle\psi^{(0)}|H_f|\psi^{(0)}\rangle$

$$\begin{aligned} E^{(1)} &= \langle n,\ell,m,s_s|H_f|n,\ell,m,s_s\rangle \\ \langle r,\theta,\phi,s_s|n,\ell,m,s_s\rangle &= R_{n,\ell}(r)Y_m^\ell(\theta,\phi)u(s_e), \quad R_{1,0}(r) = \frac{2}{a_0^{\frac{3}{2}}}e^{-\frac{r}{a_0}}, \quad Y_0^0 = \frac{1}{\sqrt{4\pi}} \end{aligned}$$

- H_m*:*

$$\begin{aligned} \boldsymbol{p}^4 &= 4m^2\left(H_0 + \frac{e^2}{r}\right)^2, \quad H_0 = \frac{\boldsymbol{p}^2}{2m} - \frac{e^2}{r}, \quad E_n = -\frac{\alpha^2mc^2}{2n^2} \\ H_m &= -\frac{\boldsymbol{p}^4}{8m^3c^2} = -\frac{(H_0+\frac{e^2}{r})^2}{2mc^2} \\ \langle H_m\rangle &= -\frac{1}{2mc^2}\left(E_n^2 + 2E_n\langle\frac{e^2}{r}\rangle + \langle\frac{e^4}{r^2}\rangle\right) \end{aligned}$$

Generator functional:

$$\begin{aligned} I(\kappa) &= \int_0^\infty dre^{-\kappa r} = \frac{1}{\kappa} \\ \langle\frac{1}{r^n}\rangle &= \frac{4\pi}{4\pi}\int_0^\infty drr^{2-n}\frac{4}{a_0^3}e^{-\frac{2r}{a_0}} = \frac{4}{a_0^3}(-1)^{2-n}\frac{d^{2-n}I(\kappa)}{d\kappa^{2-n}}_{|\kappa=\frac{2}{a_0}} \\ \langle\frac{1}{r}\rangle &= -\frac{4}{a_0^3}\frac{dI(\kappa)}{d\kappa}_{|\kappa=\frac{2}{a_0}} = \frac{4}{a_0^3}\frac{a_0^2}{4} = \frac{1}{a_0}, \qquad \langle\frac{1}{r^2}\rangle = \frac{4}{a_0^3}(-1)^{2-n}\frac{d^{2-n}I(\kappa)}{d\kappa^{2-n}}_{|\kappa=\frac{2}{a_0}} = \frac{4}{a_0^3}\frac{a_0}{2} = \frac{2}{a_0^2} \\ \langle H_m\rangle &= -\frac{1}{2mc^2}\left(\frac{\alpha^4m^2c^4}{4} - \frac{\alpha^2mc^2e^2}{a_0} + \frac{2e^4}{a_0^2}\right) = -\frac{5}{8}\alpha^4mc^2 \end{aligned}$$

- H_D*:*

$$\langle H_D\rangle = -\frac{\pi\hbar^2e^2}{2m^2c^2}\langle\delta(\boldsymbol{r})\rangle = \frac{e^2\hbar^2\pi}{2m^2c^2}|\psi_{n,\ell,m}(0)|^2 = \frac{e^2\hbar^2}{8m^2c^2}\underbrace{|R_{1,0}(0)|^2}_{4a_0^{-3}=4(\frac{me^2}{\hbar^2})^3} = \frac{1}{2}\alpha^4mc^2.$$

- H_{so} : $\langle H_{so}\rangle \sim \langle\boldsymbol{sL}\rangle = 0$
- *Finally:* $\Delta E = -\frac{1}{8}\alpha^4mc^2$, the spin degeneracy prevails in $1s_{\frac{1}{2}}$

3. $n = 2$**:**

- *Degeneracy:*

$$\underbrace{2}_{2s_{\frac{1}{2}}} + \underbrace{2}_{2p_{\frac{1}{2}}} + \underbrace{4}_{2p_{\frac{3}{2}}} = 8$$

- *Absence of mixing of* $2s$ *and* $2p$*:*

$$H_f = \begin{pmatrix} H_{2s} & 0 \\ 0 & H_{2p} \end{pmatrix}.$$

- $2s$:

$$
\begin{aligned}
R_{2,0} &= \frac{2}{(2a_0)^{\frac{3}{2}}}\left(1-\frac{r}{2a_0}\right)e^{-\frac{r}{2a_0}}, \qquad R_{2,1} = \frac{1}{\sqrt{2}(2a_0)^{\frac{3}{2}}}\frac{r}{a_0}e^{-\frac{r}{2a_0}} \\
\langle 2s|\frac{1}{r}|2s\rangle &= \frac{1}{4a_0}, \qquad \langle 2s|\frac{1}{r^2}|2s\rangle = \frac{1}{12a_0^2}, \qquad \langle 2s|\frac{1}{r^3}|2s\rangle = \frac{1}{24a_0^3} \\
\langle H_m\rangle &= -\frac{13}{128}mc^2\alpha^4 \\
\langle H_D\rangle &= -\frac{1}{16}mc^2\alpha^4 \\
\langle H_{so}\rangle &= 0
\end{aligned}
$$

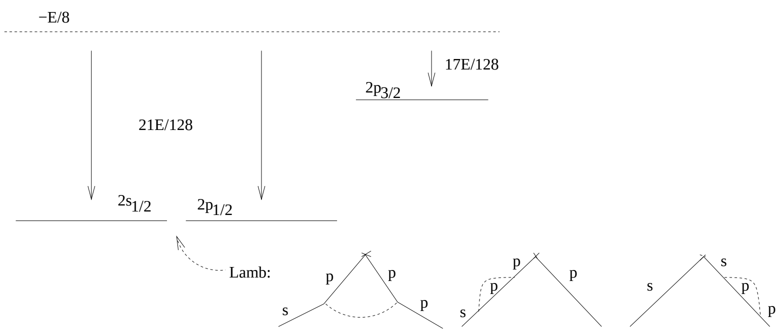
Energy shift: degeneracy remains

$$\Delta E_{2s_{\frac{1}{2}}} = -\frac{21}{128}\alpha^4 mc^2$$

- $2p$:

$$
\begin{aligned}
\langle H_m\rangle &= -\frac{55}{384}mc^2\alpha^4 \\
\langle H_D\rangle &= 0 \quad \leftarrow \quad R_\ell = \mathcal{O}\left(r^\ell\right) \\
\langle H_{so}\rangle &= \langle 2p, m, s_e|\frac{e^2}{2m^2c^2}\frac{1}{r^3}\boldsymbol{SL}|2p, m', s_e'\rangle = \underbrace{\frac{e^2}{2m^2c^2}\int_0^\infty dr r^2\frac{1}{r^3}|R_{2,1}(r)|^2}_{\frac{mc^2\alpha^4}{48\hbar^2}}\langle 1, m, s_e|\boldsymbol{SL}|1, m', s_e'\rangle
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{SL}|\ell, m, s\rangle &= \frac{1}{2}(\vec{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2)|\ell, m, s\rangle \\
&= \frac{\hbar^2}{2}\left[J(J+1) - \ell(\ell+1) - \frac{1}{2}\frac{3}{2}\right]|\ell, m, s\rangle \\
&= \frac{\hbar^2}{2}\left[J(J+1) - \frac{11}{4}\right]|\ell, m, s\rangle \\
&= \begin{cases} -\hbar^2|1, m, s_e\rangle & J = \frac{1}{2} \\ \frac{\hbar^2}{2}|1, m, s_e\rangle & J = \frac{3}{2} \end{cases} \\
\langle H_{so}\rangle &= \begin{cases} -\frac{1}{48}mc^2\alpha^4 & J = \frac{1}{2}, \\ \frac{1}{96}mc^2\alpha^4 & J = \frac{3}{2}. \end{cases} \\
\Longrightarrow \Delta E_{2p_{\frac{1}{2}}} &= -\frac{21}{128}\alpha^4 mc^2, \quad \Delta E_{2p_{\frac{3}{2}}} = -\frac{17}{128}\alpha^4 mc^2
\end{aligned}
$$



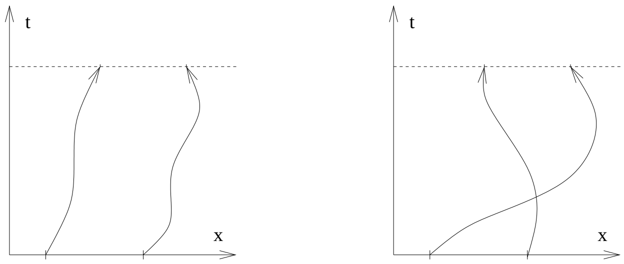
(a) Degeneracy in J is split

(b) Subspaces $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ remain degenerate, they split up in $\mathcal{O}\left(\alpha^2\right)$ by photon emission and absorption processes (Lamb shift)

VI. IDENTICAL PARTICLES

1. **Macroscopic quantum effect**

- *Classical physics:* trajectories distinguish the particles



- *Quantum physics:*
  - Heisenberg's uncertaintuy principle $\Longrightarrow \nexists$ trajectories
  - The difficulty of distinguishability is generalised to the principle of undistinguishability
  - $\hbar$-independent quantum effect
  - Realization:
    - ∗ π: exchange of two particles, $|x_1, x_2\rangle \neq \pi|x_1, x_2\rangle = |x_2, x_1\rangle$
    - ∗ Hilbert space: ray representation of physical states, $|\psi\rangle_{phys} = \{e^{i\alpha}|\psi\rangle\}$

$$\begin{aligned} |x_2, x_1\rangle &= e^{i\theta_e}|x_1, x_2\rangle \\ \psi(x_2, x_1) &= e^{i\theta_e}\psi(x_1, x_2). \end{aligned}$$

  - Gibbs paradox: entropy of the ideal gas is non-extensive

    Solution: N identical particles has $N!$ identical rearrangements

2. **Fermions and bosons:**

- *Naive expectation:*

$$\begin{aligned} \pi^2 &= 1 \quad \Longrightarrow \quad e^{2i\theta_e} = 1 \quad \Longrightarrow \quad e^{i\theta_e} = \pm 1 \\ \text{However} \quad \pi^2|x_1, x_2\rangle &= e^{2i\theta_e}|x_1, x_2\rangle = e^{i\alpha}|x_1, x_2\rangle \quad \Longrightarrow \quad 2\theta_e = \alpha \neq 2n\pi \end{aligned}$$

- *Spin-statistic theorem:*

(a) Rotationial phase:

$$U_j(R_{\boldsymbol{n}}(2\pi))|x_1, x_2\rangle = e^{i\theta_r}|x_1, x_2\rangle,$$

(b) Exchange phase:

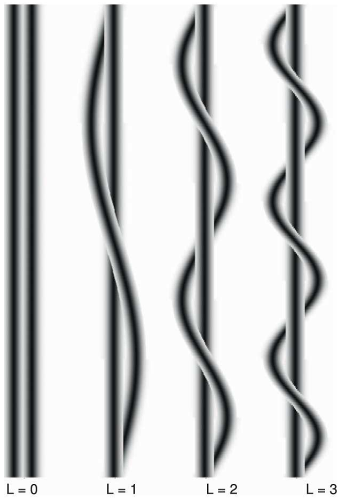
$$\psi(x_2, x_1) = e^{i\theta_e}\psi(x_1, x_2).$$

(c) Theorem:

$$\boxed{\theta_r = \theta_e}$$

(d) Topological proof:

- Twist number of a closed ribbon: number of rotation by 2π



$$\nu = \frac{1}{2\pi}\int_0^L dx \frac{d\alpha(x)}{dx}$$

- Ribbon, attached to each particle and to the wall
- Exchange of the ends of a ribbon generates 2π rotation

(e) Fermions and bosons in three dimensions:

$$U(R_{\boldsymbol{n}}(2\pi)) = \xi = \pm 1$$

(f) Anyons in two dimensions: phase (irreducible) representations of rotationangourp $SO(2)$

$$U_\theta(2\pi)|x_1, x_2\rangle = e^{i\theta}|x_1, x_2\rangle, \qquad -\pi < \theta \leq \pi$$

3. **Superselection rule:**

$$\boxed{\text{No mixing between fermions and bosons} \Longleftrightarrow \text{classical physics has no fermionic coordinate}}$$

(a) Matrix element of a tensor operator of integer angular momentum:

$$\begin{aligned}
\langle\psi_{\xi'}|T^{(\ell)}_m|\phi_\xi\rangle &= \langle\psi_{\xi'}|U^\dagger(R_{\boldsymbol{n}}(2\pi))U(R_{\boldsymbol{n}}(2\pi))T^{(\ell)}_m U^\dagger(R_{\boldsymbol{n}}(2\pi))U(R_{\boldsymbol{n}}(2\pi))|\phi_\xi\rangle \\
T^{(\ell)}_m &= \sum_{m'} U^\dagger(R)T^{(\ell)}_{m'}U(R)\mathcal{D}^\ell_{m',m}(R) \\
\langle\psi_{\xi'}|T^{(\ell)}_m|\phi_\xi\rangle &= \sum_{m'} \mathcal{D}^\ell_{m',m}(R_{\boldsymbol{n}}(2\pi))\langle\psi_{\xi'}|U^\dagger(R_{\boldsymbol{n}}(2\pi))T^{(\ell)}_{m'}U(R_{\boldsymbol{n}}(2\pi))|\phi_\xi\rangle \\
&= \xi'\xi\sum_{m'} \underbrace{\mathcal{D}^\ell_{m',m}(R_{\boldsymbol{n}}(2\pi))}_{\delta_{m,m'}}\langle\psi_{\xi'}|T^{(\ell)}_{m'}|\phi_\xi\rangle \\
&= \xi'\xi\langle\psi_{\xi'}|T^{(\ell)}_{m'}|\phi_\xi\rangle \quad\Longrightarrow\quad \langle\psi_{\xi'}|T^{(\ell)}_{m'}|\phi_\xi\rangle = 0 \ \text{ for } \ \xi' \neq \xi
\end{aligned}$$

(b) Interactions do not mix fermionic and bosonic states (Hamiltonian is an $\ell = 0$ tensor operator)

$$\begin{aligned}
\underbrace{\psi(1,2)}_{\mathcal{H}_{12}} &= \underbrace{\frac{1}{2}(\psi(1,2)+\psi(2,1))}_{\mathcal{H}_s}\underbrace{+}_{\oplus}\underbrace{\frac{1}{2}(\psi(1,2)-\psi(2,1))}_{\mathcal{H}_a} \\
\mathcal{H}_{12} &= \mathcal{H}_s \oplus \mathcal{H}_a
\end{aligned}$$

4. **Several particles:**

(a) *Exchange of two neighbouring particles:*

$$\psi(x_1,\ldots,x_j,x_{j+1},\ldots,x_n) = \xi\psi(x_1,\ldots,x_{j+1},x_j,\ldots,x_n)$$

(b) *Exchange a pair:*

$$\psi(x_1,\ldots,x_j,\ldots,x_k,\ldots,x_n) = \xi\psi(x_1,\ldots,x_k,\ldots,x_j,\ldots,x_n)$$

beacuse each particle $j+1,\ldots,k-1$ are skipped twice by x_j and x_k producing $\xi^{2|j-k-1|}$

(c) *Parity of a permutation:*

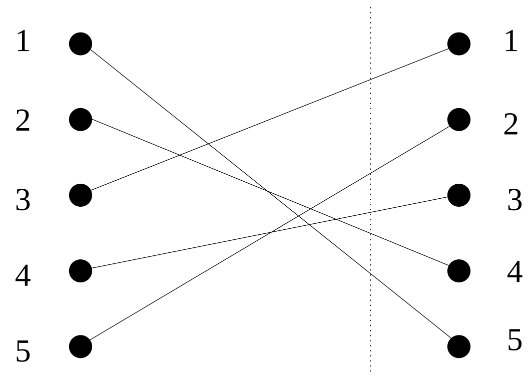
- Each permutation

$$\pi = \binom{1\ ,\ldots,\ \ N}{\pi(1),\ldots,\pi(N)}$$

is the product of the exchange of neighbours

Example:

$$\binom{1,2,3,4,5}{3,5,4,2,1} = (1,4)(1,5)(4,5)(2,3)(3,5)(2,4)(3,4)$$



- Number of exchanged neigbhours, $n(\pi)$, is not unique but its parity

$$\sigma(\pi) = n(\pi) \mod (2)$$

is unique and well defined

Proof: continuous deformation of the lines

- can reproduce any factorization
- changes the number of crossing in units of 2

- Each crossing generates a multiplicative factor $\xi \Longrightarrow$ total exchange factor is $\xi^{\sigma(\pi)}$

Example: $N = 3$

$$\begin{aligned} 1 &= \sigma\left(\binom{1,2,3}{1,2,3}\right) = \sigma\left(\binom{1,2,3}{3,1,2}\right) = \sigma\left(\binom{1,2,3}{2,3,1}\right) \\ -1 &= \sigma\left(\binom{1,2,3}{1,3,2}\right) = \sigma\left(\binom{1,2,3}{3,2,1}\right) = \sigma\left(\binom{1,2,3}{2,1,3}\right). \end{aligned}$$

(d) *N particle ket state from N one-particle states $\{|k\rangle\}$:*

$$\boxed{|k_1, \dots, k_N\rangle = \mathcal{N} \sum_{\pi \in S_N} \xi^{\sigma(\pi)} |k_{\pi(1)}\rangle \otimes \cdots \otimes |k_{\pi(N)}\rangle}$$

Proof:

$$\sum_{\pi \in S_N} F(\pi) = \sum_{\pi \in S_N} F(\pi\pi') = \sum_{\pi \in S_N} F(\pi'\pi)$$

the maps $\pi \to \pi\pi'$, $\pi \to \pi'\pi$ are onto and one-to-one $\Longrightarrow$ same sums in different order

$$\begin{aligned} |k_{\pi'(1)}, \dots, k_{\pi'(N)}\rangle &= \mathcal{N} \sum_{\pi \in S_N} \xi^{\sigma(\pi)} |k_{\pi\pi'(1)}\rangle \otimes \cdots \otimes |k_{\pi\pi'(N)}\rangle \\ \sigma(\pi\pi') &= \sigma(\pi) \pm \sigma(\pi') \quad \leftarrow \quad \xi^{2\sigma(\pi')} = 1 \\ |k_{\pi'(1)}, \dots, k_{\pi'(N)}\rangle &= \xi^{\sigma(\pi')} \mathcal{N} \sum_{\pi \in S_N} \xi^{\sigma(\pi\pi')} |k_{\pi\pi'(1)}\rangle \otimes \cdots \otimes |k_{\pi\pi'(N)}\rangle = \xi^{\sigma(\pi')} |k_1, \dots, k_N\rangle \end{aligned}$$

(e) *N particle function:*

$$
\begin{aligned}
\psi^{(+)}_{k_1,\ldots,k_n}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &= \mathcal{N}\sum_{\pi\in S_N}\psi_{k_1}(\boldsymbol{x}_{k_{\pi(1)}})\cdots\psi_{k_N}(\boldsymbol{x}_{k_{\pi(N)}})\\
\psi^{(-)}_{k_1,\ldots,k_n}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) &= \mathcal{N}\sum_{\pi\in S_N}(-1)^{\sigma(\pi)}\psi_{k_1}(\boldsymbol{x}_{k_{\pi(1)}})\cdots\psi_{k_N}(\boldsymbol{x}_{k_{\pi(N)}})\\
&= \mathcal{N}\det\begin{vmatrix}\psi_{k_1}(\boldsymbol{x}_1) & \psi_{k_1}(\boldsymbol{x}_2) & \cdots & \psi_{k_1}(\boldsymbol{x}_N)\\ \psi_{k_2}(\boldsymbol{x}_1) & \psi_{k_2}(\boldsymbol{x}_2) & \cdots & \psi_{k_2}(\boldsymbol{x}_N)\\ \vdots & \vdots & \cdots & \vdots\\ \psi_{k_N}(\boldsymbol{x}_1) & \psi_{k_N}(\boldsymbol{x}_2) & \cdots & \psi_{k_N}(\boldsymbol{x}_N)\end{vmatrix}
\end{aligned}
$$

$\nearrow$

Slater determinant

(f) *Pauli's exclusion principle:*

Two fermions can not occupy the same quantum state

5. **Occupation number representation:** Counting the number of particles of different types

$$
\begin{aligned}
|k_1,k_2,\ldots,k_N\rangle &\implies |n_k\rangle\\
N[n] &= \sum_k n_k, \quad \boldsymbol{P}=\sum_k n_k\boldsymbol{p}_k, \quad E[n]=\sum_k n_kE_k
\end{aligned}
$$

Does not contain unphysical information $\Longrightarrow$ no need of (anti)symmetrization

6. **Exchange interaction:** (wrong) historical name

(a) *Two particle state:*

$$
\psi_{12}(\boldsymbol{x}_1,\sigma_1,\boldsymbol{x}_2,\sigma_2)=\frac{1}{\sqrt{2}}[\psi_1(\boldsymbol{x}_1,\sigma_1)\psi_2(\boldsymbol{x}_2,\sigma_2)+\xi\psi_2(\boldsymbol{x}_1,\sigma_1)\psi_1(\boldsymbol{x}_2,\sigma_2)].
$$

(b) *Factorizable one- and two-particle wave functions:*

$$
\begin{aligned}
\psi_j(\boldsymbol{x},\sigma) &= \chi_j(\boldsymbol{x})\phi_j(\sigma)\\
\psi_{12}(\boldsymbol{x}_1,\sigma_1,\boldsymbol{x}_2,\sigma_2) &= \chi_{12}(\boldsymbol{x}_1,\boldsymbol{x}_2)\phi_{12}(\sigma_1,\sigma_2),
\end{aligned}
$$

(c) *Exchange statistics:*

$$
\chi_{12}(\boldsymbol{x}_2,\boldsymbol{x}_1)=\xi_c\chi_{12}(\boldsymbol{x}_1,\boldsymbol{x}_2), \quad \chi_{12}(\boldsymbol{x}_2,\boldsymbol{x}_1)=\xi_s\phi_{12}(\sigma_1,\sigma_2) \quad \Longrightarrow \quad \xi_c\xi_s=\xi
$$

(anti)symmetrization of states may introduce correlations among quantum numbers

(d) *Bound states of two identical fermions:*

- Hamiltonian:

$$H = \frac{\boldsymbol{p}_1^2}{2m} + \frac{\boldsymbol{p}_2^2}{2m} + U(r_{12}).$$

$$\nearrow$$

strongly attractive spherical symmetric potential at short distances

- New variables:

$$\begin{aligned} \boldsymbol{X} &= \frac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2), \quad \boldsymbol{P} = \frac{1}{2}(\boldsymbol{p}_1 + \boldsymbol{p}_2), \quad \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2, \quad \boldsymbol{p} = \boldsymbol{p}_1 - \boldsymbol{p}_2 \\ \chi(\boldsymbol{x}_1, \boldsymbol{x}_2) &= e^{-\frac{i}{\hbar}\boldsymbol{P}\vec{X}} \eta_{n,\ell}(r) Y_m^\ell(\theta, \phi) \end{aligned}$$

- Spatial inversion:

$$\boldsymbol{x} \to -\boldsymbol{x} \quad \Longrightarrow \quad \xi_x = (-1)^\ell$$

- Correlation among quantum numbers for $\xi = -1$:
  - $S = 0$: $\xi_s = -1 \Longrightarrow \xi_c = 1 \Longrightarrow \ell = 0$ in the bound state
  - $S = 1$: $\xi_s = 1 \Longrightarrow \xi_c = -1 \Longrightarrow \ell = 1$ in the bound state
  - $\eta_{n,\ell}(r) = \mathcal{O}\left(r^\ell\right)$
  - Spin-dependen ground state energy: $\int drr^2 U(r) < \int drr^4 U(r)$

VII. DENSITY MATRIX

A. Gleason theorem

1. **Classical probability:**

- *Elementary events:* $\mathcal{H}$
- *σ-algebra:* $\mathcal{M}$, the measurable subsets of $\mathcal{H}$
  (a) $a_n \in \mathcal{M} \Longrightarrow \cup_n a_n \in \mathcal{M}$
  (b) $a \in \mathcal{M} \Longrightarrow \mathcal{H} \backslash a \in \mathcal{M}$
- *Probability measure:* $\mu : \mathcal{M} \to R$
  (a) $0 \le p(a) < \infty$ ($p(a) < 1$ for discrete values of a)
  (b) $p(\emptyset) = 0$
  (c) $a_n \in \mathcal{M}$ and $a_m \cap a_n = \emptyset \Longrightarrow p(\cup_n a_n) = \sum_n p(a_n)$

2. **Quantum probability:**

- *Elementary events:* $\mathcal{H}$
- *σ-algebra:* $\mathcal{M}$, the measurable linear subspaces of $\mathcal{H}$
  (a) $a_n \in \mathcal{M} \Longrightarrow \sum_n c_n a_n \in \mathcal{M}$
  (b) $\forall a \in \mathcal{M}$, $\{v \in \mathcal{H} | \langle v|w\rangle = 0, \forall w \in a\} \in \mathcal{M}$.
- *Probability measure:* $\mu : \mathcal{M} \to R$
  (a) $0 \leq p(a) < \infty$
  (b) $p(\emptyset) = 0$
  (c) $a_n \in \mathcal{M}$ and $a_m \perp a_n = 0 \Longrightarrow p(\{\sum_n c_n a_n\}) = \sum_n p(a_n)$

3. **Gleason's theorem:** Any measure p in a separable Hilbert space of at least 3 dimensions is of the form

$$\boxed{p(a) = \mathrm{Tr}[\rho\Lambda(a)]}$$

- *Projector onto the subspace a:* $\{|n\rangle\}$ is a basis fora linear subspace $a \subset \mathcal{H}$

$$\Lambda(a) = \sum_n |n\rangle\langle n|,$$

- *Quantum state:*
  - collection of information abut the system,
  - probability distribuiton for all subspaces
  - density matrix ρ

4. **Expectation value of an observable:**

$$\begin{aligned}
A &= \sum_n |\psi_n\rangle\lambda_n\langle\psi_n| = \sum_n \lambda_n|\psi_n\rangle\langle\psi_n| = \sum_n \lambda_n\Lambda(|n\rangle) \qquad |\psi_n\rangle \leftrightarrow \lambda_n \\
\langle A\rangle &= \sum_n p_n\lambda_n \\
&= \sum_n \mathrm{Tr}[\rho\Lambda(|n\rangle)]\lambda_n \\
&= \sum_n \mathrm{Tr}[\rho\lambda_n\Lambda(|n\rangle)] \\
&= \mathrm{Tr}\rho A
\end{aligned}$$

$$\boxed{\langle A\rangle = \mathrm{Tr}[\rho A]}$$

B. Properties

1. **Hermiticity:**

$$\rho = \rho_h + \rho_{ah}, \quad \rho_h = \frac{1}{2}(\rho + \rho^\dagger), \quad \rho_{ah} = \frac{1}{2}(\rho - \rho^\dagger)$$
$$\mathrm{Tr} P_\psi \rho = \langle\psi|\rho|\psi\rangle \in \mathbb{R} \Longrightarrow \langle\psi|\rho|\psi\rangle = \langle\psi|\rho^\dagger|\psi\rangle \Longrightarrow \rho_{ah} = 0$$

2. *Positive operator:*

$$\langle\psi|\rho|\psi\rangle = \mathrm{Tr}[\Lambda(\psi)\rho] \geq 0$$

3. *Unit trace:*

$$\mathrm{Tr}\rho = \mathrm{Tr}[\rho \mathbb{1}] = 1$$

4. *Diagonalizable:* $\{|\psi_n\rangle\}$ is an orthonormal base

$$\rho = \sum_n |\psi_n\rangle p_n \langle\psi_n|, \quad 0 \leq p_n, \quad \sum_n p_n = 1$$

Interpretation: p_n is the probability of finding the system in $|\psi_n\rangle$

5. *Pure states:* (factorizable density matrix)

$$\rho = |\psi\rangle\langle\psi| \quad \leftrightarrow \quad \mathrm{Tr}[\rho^2] = \mathrm{Tr}[\rho] = 1$$

6. *Mixed states:* (non-factorizable density matrix)

$$\rho = \sum_{n=1}^N |\psi_n\rangle p_n \langle\psi_n|, \quad (N \geq 2) \quad \leftrightarrow \quad \mathrm{Tr} p^2 = \sum_n p_n^2 < \sum_n p_n = \mathrm{Tr}\rho = 1$$

7. *Degeneracy:* non-unique decomposition

8. *Example:* Two-state system:

$$\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{p}\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$
$$\langle\boldsymbol{\sigma}\rangle = \mathrm{tr}[\rho\boldsymbol{\sigma}] = \boldsymbol{p}$$

C. Physical origin

Two different(?) physical origins of the same mathematical structure:

1. **Loss of classical information:** $|\psi_n\rangle \leftrightarrow p_n$

$$\begin{aligned}\rho &= \sum_n |\psi_n\rangle p_n \langle\psi_n| \\ \langle A\rangle &= \mathrm{Tr}\rho A = \sum_n p_n \mathrm{Tr}[|\psi_n\rangle\langle\psi_n|A] = \sum_n p_n \langle\psi_n|A|\psi_n\rangle\end{aligned}$$

- *Expectation value:*
  - quadratic in the wave function

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \int dxdy\psi^*(x)\langle x|A|y\rangle\psi(y)$$

  - linear in the density matrix

$$\begin{aligned}\rho(x,y) &= \langle x|\rho|y\rangle \\ \mathrm{Tr}[\rho A] &= \int dxdy\langle x|\rho|y\rangle\langle y|A|x\rangle = \int dxdy\rho(x,y)\langle y|A|x\rangle\end{aligned}$$

  - no interference in $\mathrm{Tr}[\rho A] = \sum_n p_n\langle\psi_n|A|\psi_n\rangle$ between the eigenstate of ρ

- *Coherent average in a pure state:*

$$\begin{aligned}|\psi\rangle &= \sum_n \sqrt{p_n}|\psi_n\rangle \\ \langle\psi|A|\psi\rangle &= \sum_{m,n}\sqrt{p_m p_n}\langle\psi_m|A|\psi_n\rangle \neq \sum_n p_n\langle\psi_n|A|\psi_n\rangle \\ \rho &= \sum_{m,n}\sqrt{p_m p_n}|\psi_m\rangle\langle\psi_n| \neq \sum_n p_n|\psi_n\rangle\langle\psi_n|.\end{aligned}$$

- *Decoherence:* reduced coherence in the expectation values

$$\langle\psi|A|\psi\rangle = \mathrm{Tr}[\rho A] = \sum_n p_n\langle\psi_n|A|\psi_n\rangle$$

2. **Entangled states:**

- *Bipartite system:* $\mathcal{H} = \mathcal{H}_\phi \otimes \mathcal{H}_\chi$, with bases $\{|\phi_m\rangle\}$ and $\{|\chi_n\rangle\}$

  ↗ observed system ↖ environment

- *Complete system:* pure state

$$|\psi\rangle = \sum_{m,n} c_{m,n}|\phi_m\rangle \otimes |\chi_n\rangle$$

- *Schmidt decomposition:*

$$|\psi\rangle = \sum_{n=1}^N c_n|u_n\rangle \otimes |v_n\rangle, \quad \langle u_m|u_{m'}\rangle = \delta_{m,m'}, \quad \langle v_n|v_{n'}\rangle = \delta_{n,n'}$$

- *Classification of pure states:*

  (a) $N = 1$: factorisable state $|\psi\rangle = |u\rangle| \otimes |v\rangle$

  (b) $N \geq 2$: entangled state

- *Properties of a sub-system are well defined in a factorisable state*

$$\langle\psi|A_1|\psi\rangle = \langle u| \otimes \langle v|A_1 \otimes 1\!\!1_2|u\rangle \otimes |v\rangle = \langle u|A_1|u\rangle$$

- *Properties of a sub-system depend on the entangled environment*

$$\begin{aligned}|\psi\rangle &= |u_1\rangle| \otimes |v_1\rangle + |u_1\rangle| \otimes |v_1\rangle \\ \langle\psi|A_1|\psi\rangle &= \langle u_1|A_1|u_1\rangle\langle v_1|v_1\rangle + \langle u_2|A_1|u_2\rangle\langle v_2|v_2\rangle \\ &\quad + \langle u_1|A_1|u_2\rangle\langle v_1|v_2\rangle + \langle u_1|A_1|u_2\rangle\langle v_1|v_2\rangle\end{aligned}$$

- *No state vectors for an entangled subsystem:*

  (a) Suppose the contrary, $\exists\ |\phi_{obs}\rangle$, $N \geq 2$

$$|\psi\rangle = \sum_{n=1}^{N} c_n |u_n\rangle \otimes |v_n\rangle$$

  (b) $P(|\phi_{obs}\rangle) =?$ 1. way:

$$p(|\phi_{obs}\rangle\langle\phi_{obs}|) = \langle\phi_{obs}|\phi_{obs}\rangle\langle\phi_{obs}|\phi_{obs}\rangle = 1$$

  (c) $P(|\phi_{obs}\rangle) =?$ 2. way:

$$\begin{aligned}\Lambda(|\phi_{obs}\rangle) &= |\phi_{obs}\rangle\langle\phi_{obs}| \otimes 1\!\!1_\chi \\ p(|\phi_{obs}\rangle\langle\phi_{obs}|) &= \langle\psi|\Lambda(|\phi_{obs}|\psi\rangle = \langle\psi||\phi_{obs}\rangle\langle\phi_{obs}| \otimes 1\!\!1_2|\psi\rangle \\ &= \sum_{n,n'} c^*_{n'} c_n \langle u_{n'}| \otimes \langle v_{n'}|(|\phi_{obs}\rangle\langle\phi_{obs}| \otimes 1\!\!1)|u_n\rangle \otimes |v_n\rangle \\ &= \sum_{n=1}^{N} \underbrace{|c_n|^2}_{<1} \underbrace{|\langle u_n|\phi\rangle|^2}_{\leq 1} < 1\end{aligned}$$

$$\nearrow$$

$\sum_{n=1}^{N} |c_n|^2 = 1$, $N \geq 2$

  (d) The state of an entangled subsystem is given by the (reduced) density matrix

- *Reduced density matrix:*

$$\langle A_\phi\rangle = \sum_{n,n'} c^*_n c_{n'} \langle u_n| \otimes \langle v_n|A_\phi \otimes 1\!\!1_\chi|u_{n'}\rangle \otimes |v_{n'}\rangle$$

$$= \sum_n |c_n|^2 \langle u_n|A_\phi|u_n\rangle$$

$$= \mathrm{Tr}[\rho_\phi A_\phi], \quad \rho_\phi = \sum_{n=1}^{N} |u_n\rangle |c_n|^2 \langle u_n|$$

$\nearrow$

non-factorizable

General form:

$$\begin{aligned}
\rho_{tot} &= \sum_{n,n'} c_n c_{n'}^* |u_n\rangle \otimes |v_n\rangle\langle u_{n'}| \otimes \langle v_{n'}| \\
\rho_\phi &= \mathrm{Tr}_\chi[\rho_{tot}] \\
&= \sum_{\bar{n}} \langle\chi_{\bar{n}}|\rho_{tot}|\chi_{\bar{n}}\rangle \\
&= \sum_{\bar{n},n,n'} c_n c_{n'}^* \langle\chi_{\bar{n}}|(|u_n\rangle \otimes |v_n\rangle\langle u_{n'}| \otimes \langle v_{n'}|)|\chi_{\bar{n}}\rangle \\
&= \sum_{\bar{n},n,n'} c_n c_{n'}^* |u_n\rangle\langle u_{n'}|\langle v_n|\chi_{\bar{n}}\rangle\langle\chi_{\bar{n}}|v_{n'}\rangle \\
&= \sum_{n,n'} c_n c_{n'}^* |u_n\rangle\langle u_{n'}\langle v_n| \underbrace{\sum_{\bar{n}} |\chi_{\bar{n}}\rangle\langle\chi_{\bar{n}}}_{\mathbb{1}}|v_{n'}\rangle \\
&= \sum_{n,n'} c_n c_{n'}^* |u_n\rangle\langle u_{n'}|\langle v_n|v_{n'}\rangle \\
&= \sum_n |c_n|^2 |u_n\rangle\langle u_n|.
\end{aligned}$$

- *Lessons:*

  (a) An entangled sub-system has no indivual properties

  (b) Entanglement and mixed states arise from interactions

  (c) Entanglement is more general than interactions (no interaction Hamiltonian needed)

  (d) Mathematical equivalence of points 1. and 2.:

Loss of classical information $\longleftrightarrow$ observed system is entanglened

VIII. MEASUREMENT THEORY

1. **Tripartide system:**

- *Hamiltonian:* $H_{s,a}(t) \neq 0$ for $t_m - \tau_m < t < t_m + \tau_m$

$$H = \underbrace{H_s}_{\text{system}} + \underbrace{H_a + H_{s,a}(t)}_{\text{apparatus}} + \underbrace{H_e + H_{s,a,e}}_{\text{environment}}$$

- *Non-demolishing measurement:*

$$[H_s, H_{s,a}] = 0$$

- *Initial state:*

$$|\Psi(t_i)\rangle = \sum_n c_n |s_n\rangle \otimes |a_0\rangle \otimes |e_0\rangle$$

2. **Measurement process:**

   (a) *Pre-measurement:* $t_m - \tau_m < t < t_m + \tau_m$

   - Environment ignored: $2\tau_m H_{a,e} \ll \int dt H_{s,a}(t)$
   - System-apparatus correlations:

$$|\Psi(t)\rangle = \sum_n c_n |s_n\rangle \otimes |a_0\rangle \otimes |e_0\rangle \to \sum_n c_n |s_n\rangle \otimes |a_n\rangle \otimes |e_0\rangle$$

   - Interaction generates entanglement
   - Microscopic information spreads over macroscopic size
   - Understood

   (b) *Decoherence* $t_m + \tau_m < t < t_m + \tau_m + \tau_d$*:*

   - Apparatus-environment interaction

$$|\Psi(t)\rangle = \sum_n c_n |s_n\rangle \otimes |a_n\rangle \otimes |e_0\rangle \to \sum_n c_n |s_n\rangle \otimes |a_n\rangle \otimes |e_n\rangle$$

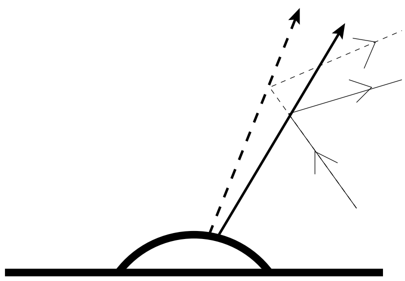
   - Reduced density matrix:

$$\rho_{s,a} = \sum_{n,n'} c_n^* c_{n'} \langle e_n | e_{n'} \rangle |s_n\rangle \otimes |a_n\rangle \langle s_{n'}| \otimes \langle s_{n'}|$$

   - Decoherence of the pointer of an ampermeter in an ideal gas environment:
     - Independent particles $\Longrightarrow$ factorizable pure state

$$|\chi\rangle = |\chi^{(1)}\rangle \otimes |\chi^{(2)}\rangle \otimes \cdots$$

     - Macroscopically different pointer states $|a_n\rangle$ and $|a_{n'}\rangle$



     - The overlap of gas particle states after bouncing back from the pointer

$$\langle e_n^{(j)} | e_{n'}^{(j)} \rangle < 1 - \epsilon, \quad j = 1, \ldots, N_p$$

- Macroscopical limit of the pointer

$$\lim_{N_p\to\infty}\langle e_n|e_{n'}\rangle = \lim_{N_p\to\infty}\prod_{j=1}^{N_s}\langle e_n^{(j)}|e_{n'}^{(j)}\rangle < \lim_{N_p\to\infty}(1-\epsilon)^{N_p} = 0$$

- Macroscopically different apparatus states $|a_n\rangle$, $|a_{n'}\rangle$
  $\Longrightarrow$ orthogonal environment states $\langle e_n|e_{n'}\rangle = 0$
- Macroscopically off diagonal elements of the density matrix in the pointer basis are suppressed
- Requires non-unitary time evolution
- Loss of phase differences $\Longrightarrow$ irreversibility
- Understood

(c) *Collapse:*

- The result of the measurement is the apparatus state $|\phi_{n_m}\rangle$

$$\begin{aligned}\rho_{s,a} &= \sum_{n,n'} c_n^* c_{n'}\langle\chi_n|_e\chi_{n'}\rangle_e|\psi_n\rangle_s\otimes|\phi_n\rangle_a\langle\psi_{n'}|_s\otimes\langle\phi_{n'}| \\ &\to \frac{\Lambda(|\phi_{n_m}\rangle)\rho_{s,a}\Lambda(|\phi_{n_m}\rangle)}{\mathrm{Tr}_{s,a}[\Lambda(|\phi_{n_m}\rangle)\rho_{s,a}\Lambda(|\phi_{n_m}\rangle)]} \\ &= |\psi_{n_m}\rangle_s\otimes|\phi_{n_m}\rangle_a\langle\psi_{n_m}|_s\otimes\langle\phi_{n_m}|_a\end{aligned}$$

- Collapse of a structure: $c_n\to\delta_{n,n_m}$
- A complicated, fast many-body effect
- Nondeterministic, far from being understood
- Determinism emerges in macroscopic physics as thermodynamics appears in statistical physics

3. **Escape route:** Hidden variable theory

- Deterministic theory containing yet unresolved degrees of freedom
- The statistical treatment of the hidden variables repoduces the predictions of quantum theory
- It must be

(a) *Non-local*

- Entanglement is distance independent
- Einstein-Podolski-Rosen effect
  * $|S=0\rangle = \frac{1}{\sqrt{2}}(|(\boldsymbol{x},\uparrow),(\boldsymbol{y},\downarrow)\rangle - |(\boldsymbol{x},\downarrow),(\boldsymbol{y},\uparrow)\rangle)$
  * An interaction at $\boldsymbol{x}$ influences the state at $\boldsymbol{y}$
  * Such a correlation spreads with $c+\epsilon$ (and infinite) speed (!!!)

(b) *Contextual*

  - Three observable A, B and C
  - $[A, B] = [A, C] = 0 \neq [B, C]$
  - The result of a *single measurement* of A depends on whether B or C has been measured
- Is this an acceptable price to save microscopic determinism?
- Why do we think that microscopic physics is deterministic?

Oral exam topics:

1. Time independent perturbation expansion
2. Time dependent perturbation expansion
3. Rotations
4. Addition of angular momentum
5. Relativistic corrections
6. Wigner's D-matrix, Indistinguishability of particles