Special relativity

Janos Polonyi
University of Strasbourg
(Dated: March 21, 2023)

Video:

https://drive.google.com/drive/folders/1VpbWufW2YU_XwDJYz4Z7p_VxHpFREVHO?usp=sharing

Contents

I. PHYSICAL LAWS AS THE FUNCTIONS OF THE SCALE OF OBSERVATION

Newton's equation $F = ma$ is modified at $v \sim c = 2.9979 \cdot 10^8 m/s$ (speed of light)

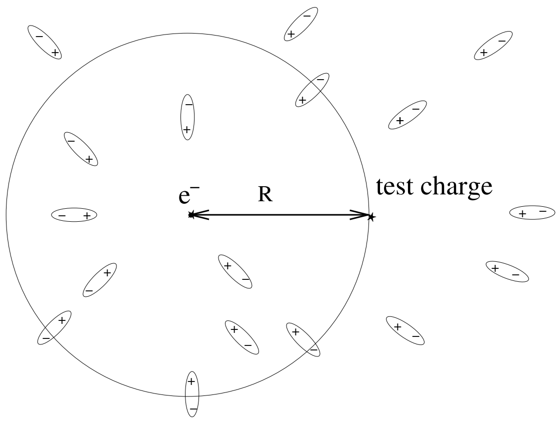
Problem: our intuition belongs to $\frac{v}{c} \ll 1$

$$\boxed{\text{The observed quantities and the physical laws depend on the scale of observation}}$$

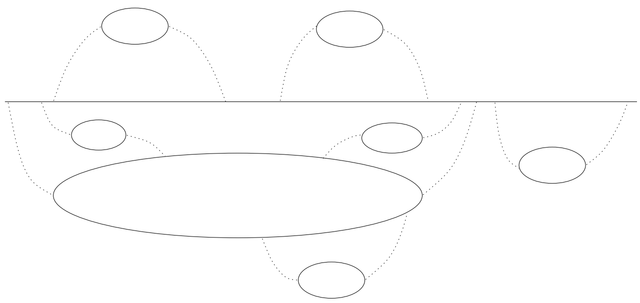
$\nearrow$

Scales: L, T, and M

Charge:



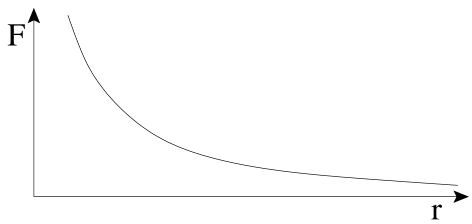
Polarizable medium



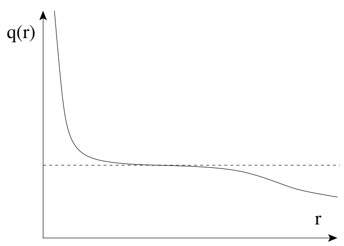
Relativistic vacuum

$$F_C(r) = \frac{qq'}{r^2} \neq F_{phys}(r) = \frac{q(r)q'}{r^2}$$

Vacuum polarization: $q \to q(r)$

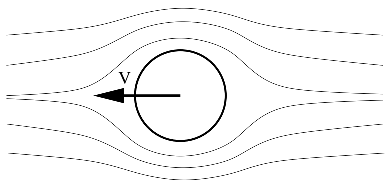


Physical constant $\equiv$ plateau



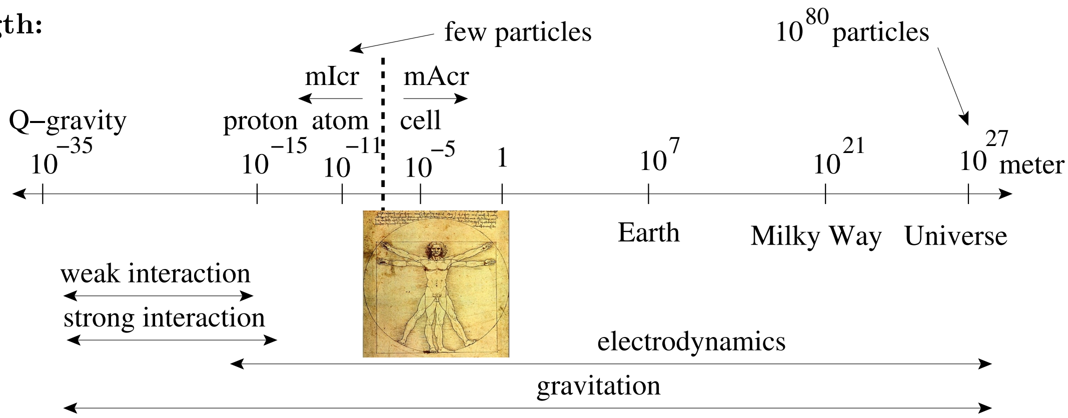
Mass:

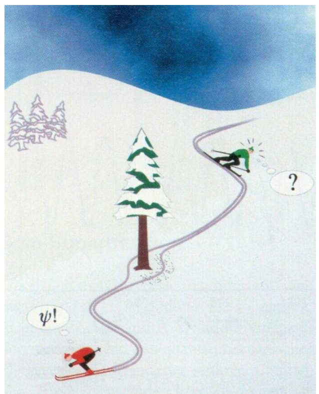
$$
\begin{aligned}
E(v) &= E(v_0) + (v - v_0)\frac{dE(v_0)}{dv} \\
&\quad + \frac{(v-v_0)^2}{2}\frac{d^2E(v_0)}{dv^2} + \cdots \\
M(v_0) &= \frac{d^2E(v_0)}{dv^2}
\end{aligned}
$$



Interactions with the environment $\implies$ effective parameters: $q \to q(r)$, $M \to M(v)$, etc.

Length:





Quantum mechanics:

- The Reality can not fully be known
- Quantum mechanics:

  Optimized and consistent treatment of partial information
- Quantum state: a list of virtual realities

  like a phone book: name $\leftrightarrow$virtual reality

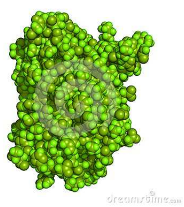
  tel. number$\leftrightarrow$probability

Life:

- elementary unit: protein
- living forms consist of 4500 proteins
- Origin: primordial soup?

  4^{165} possible RNA chains

  $M = 2 \times 10^{77} kg \sim M_{Universe} \times 10^{25}$

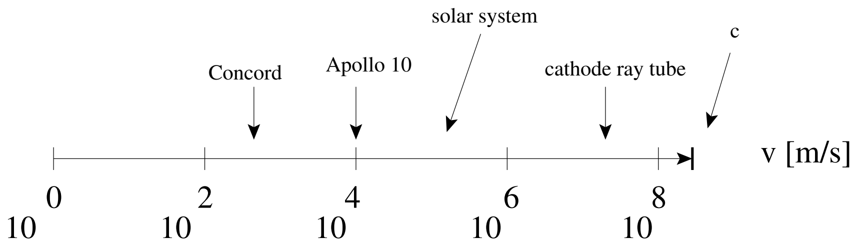


450 randomly chosen proteins

at the quantum-classical border

Quantum origin of life?

Velocity:



Orders of magnitudes in free fall: $v = 9.8t[MKS]$,

after 1 year $v = 9.8 \times 365 \times 24 \times 3600 \approx 3 \times 10^8 m/s$

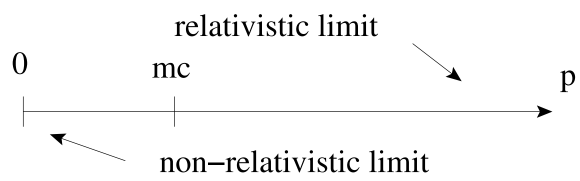
Momentum:

measure of "relativisticness":

one needs dimensionless ratios

$0 < \frac{v}{c} < 1$

$0 < \frac{p}{mc} < \infty$ with crossover at $\frac{p}{mc} \sim 1$



II. A CONFLICT AND ITS SOLUTION

A. Reference frame and Galilean symmetry

Inertial reference frame: free motion = constant velocity

$\boldsymbol{x} = t\boldsymbol{v} + \boldsymbol{x}_0$

E. O. M.: $\ddot{\boldsymbol{x}} = 0$

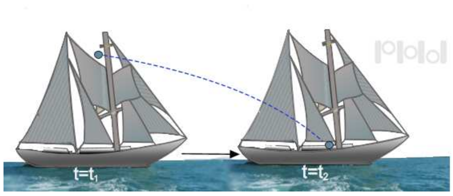
identical in each inertial reference frame

Galilean relativity: a generalization of the dynamics of a free particle to the interactive case

$\boxed{\text{The mechanical laws are identical in each inertial reference frame}}$

Example:

An object, falling freely from the the mast, ends up at the bottom of the mast on a ship which moves with constant velocity.



Symmetry transformations of Newton's equation: $m\frac{d^2\boldsymbol{x}}{dt^2} = 0$: (Galilean transformations)

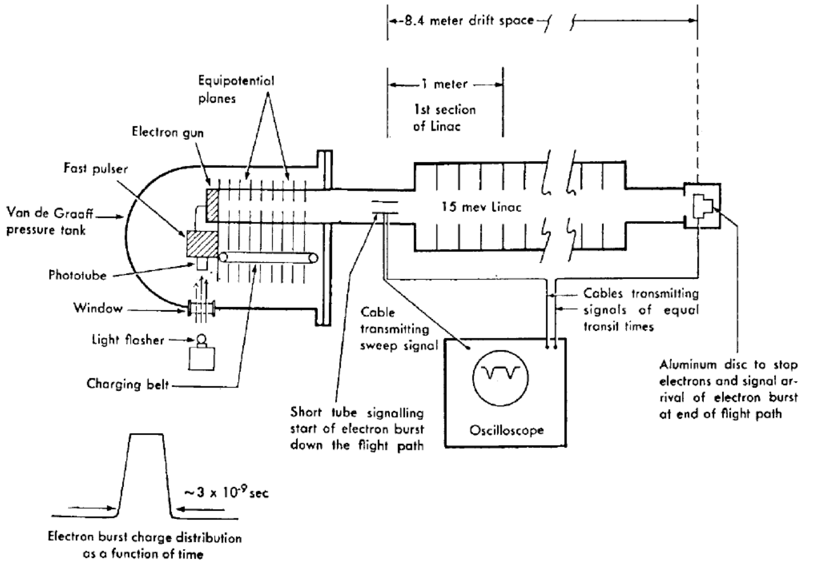
$$\begin{aligned} t &\to t' = t + t_0 \quad \leftarrow \quad \text{absolute time} \\ \boldsymbol{x} &\to \boldsymbol{x}' = R\boldsymbol{x} - t\boldsymbol{v} + \boldsymbol{x}_0 \end{aligned}$$

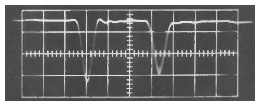
Different reference frames are connected by Galilean transformations.

Addition of the velocity: $\dot{\boldsymbol{x}} \to \dot{\boldsymbol{x}}' = \dfrac{d}{dt}(\boldsymbol{x} - t\boldsymbol{v} + \boldsymbol{x}_0) = \dot{\boldsymbol{x}} - \boldsymbol{v}$ (absolute time)

B. Limiting velocity

W. Bertozzi, Am. J. Phys. **32** 551 (1964)





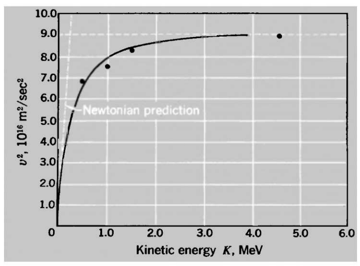
$\leftrightarrow 10^{-8}s$

Energy $K[MeV]$	Time of flight$t[10^{-8}s]$	Velocity $v[10^8 m/s]$	Velocity square $v^2[10^{16}m^2/s^2]$
0.5	3.23	2.60	6.8
1.0	3.08	2.73	7.5
1.5	2.92	2.88	8.3
4.5	2.84	2.96	8.8
15	2.80	3.00	9.0

as the energy increases like $E \to 30E$

energy conservation requires $K \to 30K$

Newton's equation: $v^2 \to 30v^2$?



Limiting velocity: $v < c$ (not enough energy to accelerate beyond c)

C. Particle or wave?

XIX.-th century physics: nature of light

- Thomas Young (1801-04): measurement of interference

- Augustin-Jean Fresnel (1818): explication of interference, diffraction, polarization

- James Clerk Maxwell (1861): electrodynamics

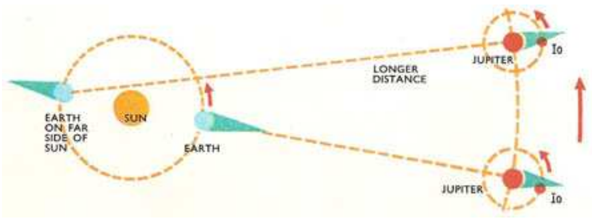
Wave: diffraction, interference, polarization $\leftrightarrow$ Particle: energy-momentum

Mechanical models until 1850 but the speed of light is too high for a particle

Speed of light :

Ole Roemer (1675): eclipse of Jupiter's moons varies in time

$c \approx 2 \times 10^8\ m/s$

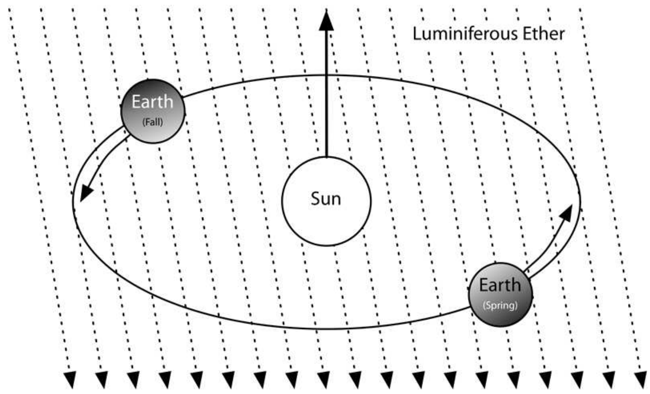


Wave or particle?

Particle: $v_{part} = v_{source} + v_{light}$,

Wave: v_{wave} is fixed within the medium

Ether hypothesis $\rightarrow$

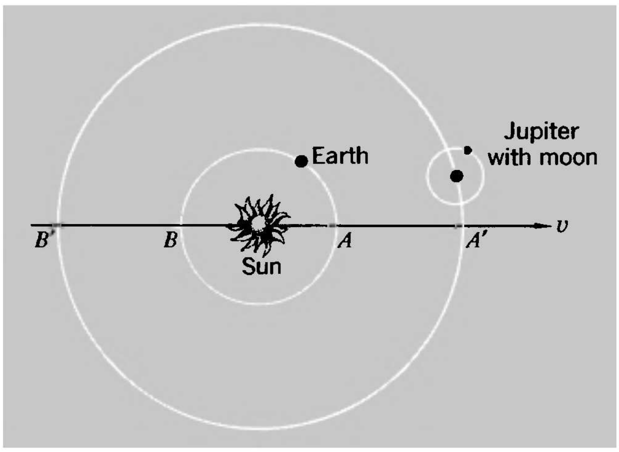


How to find our velocity with respect to the ether?

Maxwell: 1 Jupiter year=12 Earth years,

two measurements, separated by 6 tears

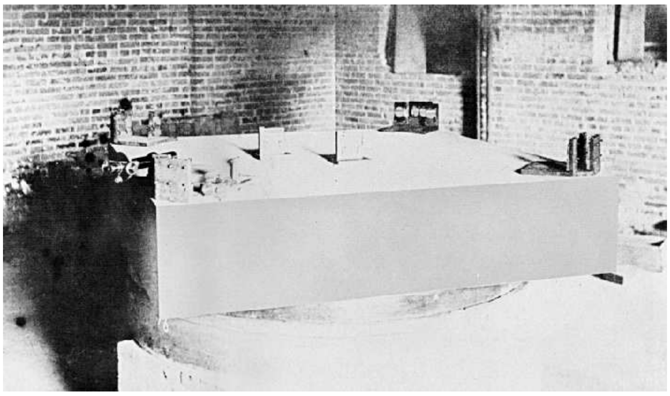
$$
\begin{aligned}
t_A^{écl.} &= \frac{\ell}{c + v_{sol}}, \qquad t_B^{écl.} = \frac{\ell}{c - v_{sol}} \\
t_B^{écl.} - t_A^{écl.} &= \frac{2\ell v_{sol}}{c^2 - v_{sol}^2} = \frac{2\ell v_{sol}}{(c + v_{sol})(c - v_{sol})} \\
&\approx \frac{2\ell v_{sol}}{c^2} = \underbrace{t_0}_{16min} \frac{2v_{sol}}{c}
\end{aligned}
$$

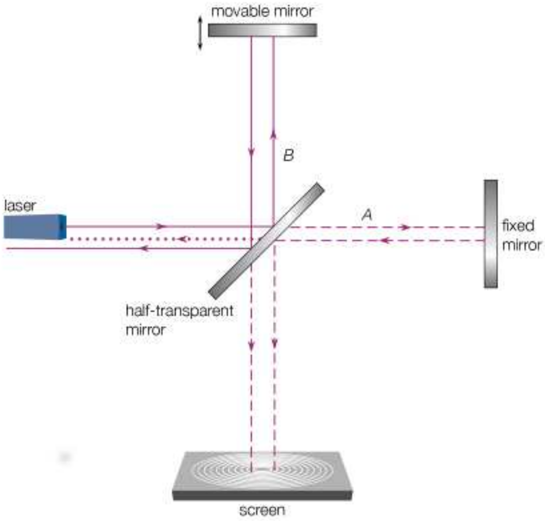


The problem is the repetition of the same experience 6 years later

D. Propagation of the light

Michelson (1881):





$$t_{\parallel} = \frac{\ell_{\parallel}}{c-v} + \frac{\ell_{\parallel}}{c+v} = \frac{2c\ell_{\parallel}}{c^2-v^2} = \frac{\frac{2\ell_{\parallel}}{c}}{1-\frac{v^2}{c^2}}$$

$$\left(\frac{ct_{\perp}}{2}\right)^2 = \ell_{\perp}^2 + \left(\frac{vt_{\perp}}{2}\right)^2, \quad t_{\perp}^2(c^2-v^2) = 4\ell_{\perp}^2, \quad t_{\perp} = \frac{2\ell_{\perp}}{\sqrt{c^2-v^2}}$$

$$\Delta t(\ell_{\parallel},\ell_{\perp}) = t_{\parallel} - t_{\perp} = \frac{2}{c}\left(\frac{\ell_{\parallel}}{1-\frac{v^2}{c^2}} - \frac{\ell_{\perp}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{2}{c}\frac{\ell_{\parallel} - \ell_{\perp}\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}}$$

Rotation by 90^0 within few minutes:

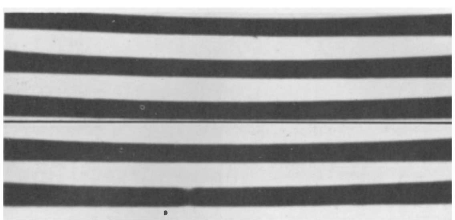
$$\Delta t'(\ell_{\parallel},\ell_{\perp}) = -\Delta t(\ell_{\perp},\ell_{\parallel}) = \frac{2}{c}\frac{\ell_{\parallel}\sqrt{1-\frac{v^2}{c^2}} - \ell_{\perp}}{1-\frac{v^2}{c^2}}$$

$$\Delta t(\ell_{\parallel},\ell_{\perp}) - \Delta t'(\ell_{\parallel},\ell_{\perp}) = \frac{2}{c}\frac{(\ell_{\parallel}+\ell_{\perp})(1-\sqrt{1-\frac{v^2}{c^2}})}{1-\frac{v^2}{c^2}}$$

$$\sqrt{1+\epsilon} \sim 1 + \frac{\epsilon}{2} \quad \rightarrow \quad \approx \frac{2(\ell_{\parallel}+\ell_{\perp})}{c}\frac{\frac{v^2}{2c^2}}{1-\frac{v^2}{c^2}} \approx (\ell_{\parallel}+\ell_{\perp})\frac{v^2}{c^3}$$

The mirrors are not exactly perpendicular $\Longrightarrow$ interference rings

Number of shifted lines: ΔN

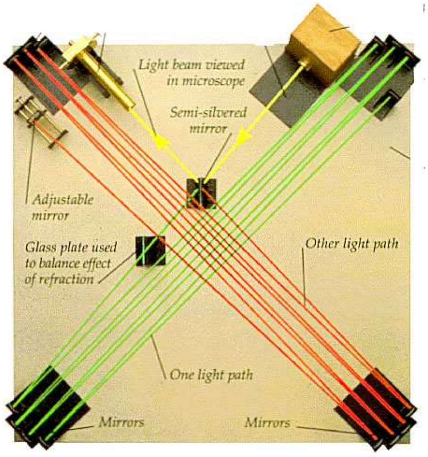


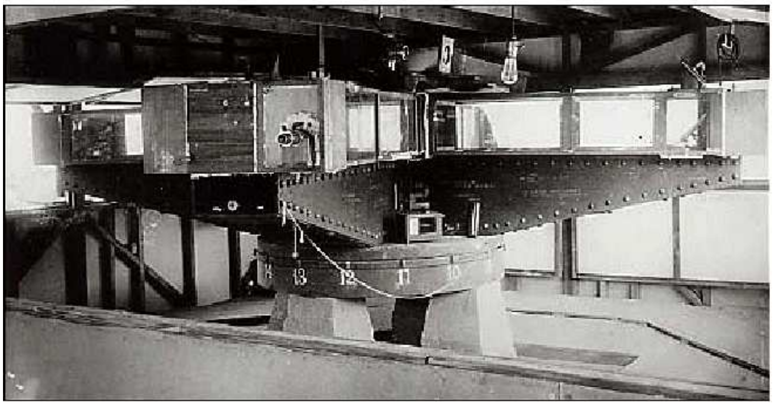
$$\Delta\ell = \lambda\Delta N = (\Delta t - \Delta t')c$$
$$\Delta N = (\Delta t - \Delta t')\frac{c}{\lambda} = \frac{v^2}{c^2}\frac{\ell_{\perp}+\ell_{\parallel}}{\lambda}$$

Assuming $v_{Earth} = 30km/s$, $\frac{v}{c} \approx 10^{-4}$ and
using $\lambda = 6\times10^{-7}m$, $\ell = 1.2m \Longrightarrow \Delta N \approx 0.04$ was not found

$\boxed{\text{No way to measure absolute velocity in physics}}$

Michelson-Morley (1887): $\ell \to 10\ell$, $\Delta N \to 0.4$, $\Delta N_{obs} = 0 \pm 0.005$





Fitzgerald Lorentz (1892): a mysterious contraction of solids in motion:

$$\ell \to \ell\sqrt{1-\frac{v^2}{c^2}},\ \Delta t = 0 \text{ for } \ell_\perp = \ell_\|$$

Einstein (1905): contraction follows form the way one measures the length

Kennedy Thorndike (1932): null result with $\ell_\perp \neq \ell_\|$

E. The problem and its solution

The light is of the nature particle or wave?

Particle: $v_{part} = v_{source} + v_{emission}$, but $c \neq v_{source} + c$?

Wave: v_{onde} est fixed, but Michelson-Morley ?

The addition of the velocity do not apply to light?

One of the following seemingly natural assumptions is wrong:

1. The physical laws assume the same form in each reference frame.
2. The time is absolute.

Einstein (1905): No justification of point 2.

Special relativity: The physical laws assume the same form in each reference frame.

Results:

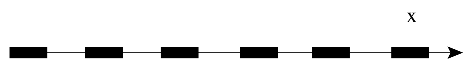
1. The speed of light is fixed by Maxwell's equations.
2. The coordinate and the velocity can be given relative to some reference object.
3. The acceleration and the derivatives $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, $n \geq 2$ are absolute.

General relativity: The coordinates, together with all derivatives, $\frac{d^n \boldsymbol{x}(t)}{dt^n}$, are relative.

A more careful analysis of the measurement of the coordinates and the time is needed.

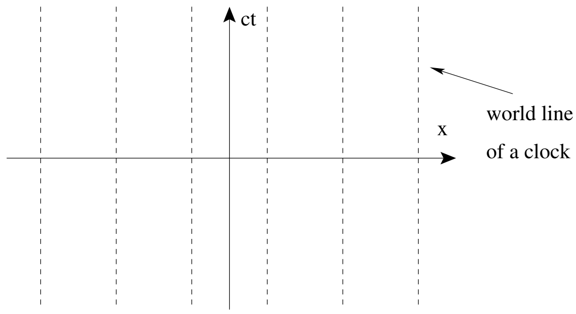
Distance: with a reference meter rod

(macroscopic!)



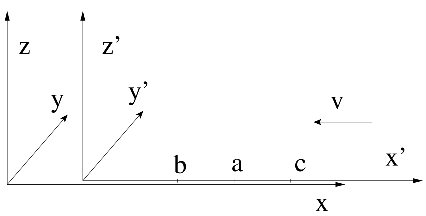
Time: synchronized standard clock

at each space point



Simultaneity is relative:

a light signal $b \leftarrow a \rightarrow c$



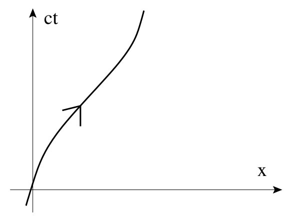
III. SPACE-TIME

A. World line

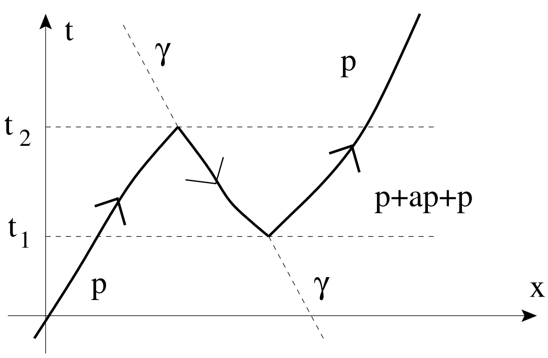
Non-relativistic motion: trajectory: $\boldsymbol{x}(t)$

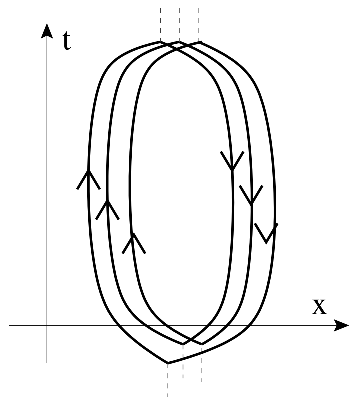
Relativistic motion: world line: $x^\mu(s) = (ct(s), \boldsymbol{x}(s)),\ \mu = 0, 1, 2, 3.$

Non-relativistic motion:



New possibility: world line $\Longrightarrow$ trajectory





Universe of a single fermion

length s: ordering the events

Anti-particle: $t \to -t \ \leftrightarrow \ E \to -E$

Classical mechanics: $\frac{dp}{dt} = -\frac{\partial H(p,q)}{\partial q}$, $\frac{dq}{dt} = \frac{\partial H(p,q)}{\partial p}$

Quantum mechanics: $i\hbar\partial_t\psi = H\psi$

Quantum field theory: $E \geq 0$

Giving up classical E.O.M.:

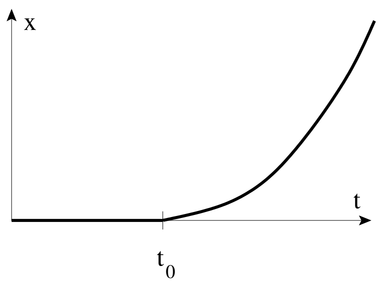
- Sufficent condition for locally unique solution:

$$\dot{x}(t) = f(x,t), \quad x(t_i) = x_i$$

  is the continuity of $\partial_x f(x,t)$

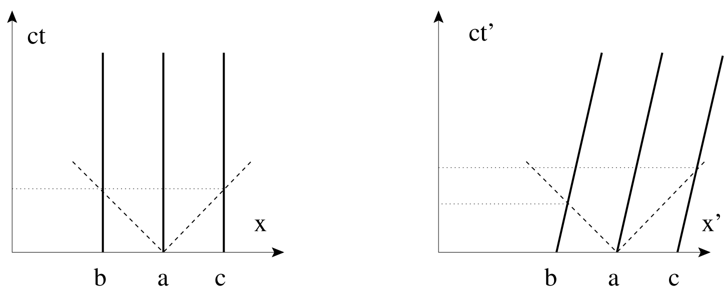
- Example:

$$\begin{aligned}
\dot{x} &= g|x|^p, \quad 0 < p < 1 \\
x(t) &= \alpha t^\beta, \quad \alpha\beta t^{\beta-1} = g(\alpha t^\beta)^p \\
\beta - 1 &= \beta p, \quad \beta = \frac{p}{1-p} \\
\alpha\beta &= g\alpha^p, \quad \frac{p\alpha}{1-p} = g\alpha^p, \quad \frac{p}{g(1-p)} = \alpha^{p-1} \\
x(t) &= \begin{cases} 0 & t < t_0, \\ \left[g\left(\frac{1}{p}-1\right)(t-t_0)\right]^{\frac{1}{1-p}} & t \geq t_0, \end{cases}
\end{aligned}$$



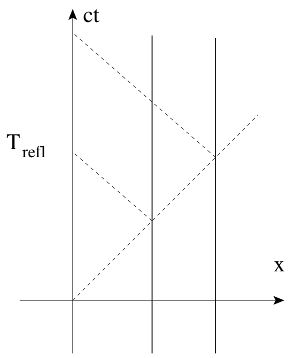
$\boxed{t_0 \text{ is not determined by the initial condition, pair creation is quantum phenomenon}}$

Relative simultaneity:



Clock synchronization:

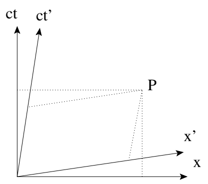
1. reference clock at $\boldsymbol{x} = 0$
2. light signal from $(ct, \boldsymbol{x}) = 0$
3. time at reflection: $t(\boldsymbol{x})$
4. time of return to $\boldsymbol{x} = 0$: $T(\boldsymbol{x})$
5. $\Delta t = \frac{T(\boldsymbol{x})}{2} - t(\boldsymbol{x})$

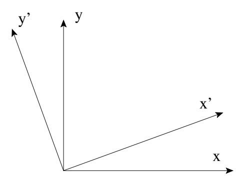


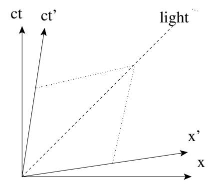
B. Lorentz transformation

Transformation between two reference frame: $(ct, \boldsymbol{x}) \to (ct', \boldsymbol{x}')$

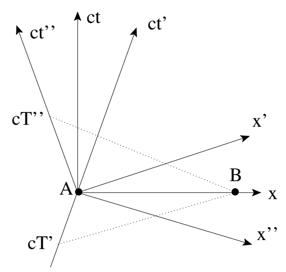
Non-orthogonal axes | Euclidean rotation | Lorentz Transformation







No fixed line | Light: a fixed line



Simultaneity is relative:

$T'(B) - T'(A) < T(B) - T(A) = 0 < T''(B) - T''(A)$

General form:
$$x' = ax - bct$$
boost:
$$= a(x - vt)$$
inverse $(v \to -v)$:
$$x = a(x' + vt')$$

Applied for the propagation of the light $x = ct$, $x' = ct'$

$$ct' = a(c - v)t,\ ct = a(c + v)t' \implies ct = a(c + v)\frac{a}{c}(c - v)t,\ a^2 = \frac{c^2}{c^2 - v^2},\ a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}(\leftarrow\ x = ct), \qquad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$\frac{v}{c} \to 0$: Galilean transformations

No change in the orthogonal directions: $\boldsymbol{v} = (v, 0, 0)$, $y = y'$, $z = z'$

Lorentz (1904) : compensation

Einstein (1905): way of observation

Lorentz symmetry is obeyed by all interactions

C. Addition of the velocity

Two different reference frames: S and S': $x_\parallel = \frac{x'_\parallel + ut'}{\sqrt{1-\frac{u^2}{c^2}}}$, $\quad \boldsymbol{x}_\perp = \boldsymbol{x}'_\perp$, $\quad t = \frac{t' + \frac{ux'_\parallel}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}}$

$$\begin{aligned}
\boldsymbol{v}' &= \frac{d\boldsymbol{x}'}{dt'} \ \to \ \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \\
\Delta t \ \to \ \Delta x_\parallel &= \frac{(v'_\parallel + u)\Delta t'}{\sqrt{1-\frac{u^2}{c^2}}}, \qquad \Delta\boldsymbol{x}_\perp = \boldsymbol{v}'_\perp \Delta t', \qquad \Delta t = \frac{(1+\frac{uv'_\parallel}{c^2})\Delta t'}{\sqrt{1-\frac{u^2}{c^2}}} \\
v_\parallel &= \frac{\Delta x_\parallel}{\Delta t} = \frac{v'_\parallel + u}{1+\frac{uv'_\parallel}{c^2}} = \begin{cases} v'_\parallel + u & u, v'_\parallel \ll c \\ c & v'_\parallel \ll c, u \approx c, \quad \text{ou} \quad u \ll c, v'_\parallel \approx c \end{cases} \\
\boldsymbol{v}_\perp &= \boldsymbol{v}'_\perp \frac{\sqrt{1-\frac{u^2}{c^2}}}{1+\frac{uv'_x}{c^2}} = \begin{cases} \boldsymbol{v}'_\perp & u \ll c \\ 0 & u \approx c \end{cases}
\end{aligned}$$

D. Invariant distance

Euclidean geometry: the invariance of $s^2 = (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$

identifies the symmetries (translations + rotations)

Minkowski geometry: the invariance of $s^2 = c^2(t_2 - t_1)^2 - (\boldsymbol{x}_2 - \boldsymbol{x}_1)^2$

identifies the symmetries (translations + Lorentz transformations)

Proof A: (by Lorentz transformation)

$$\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \\
s^2 &= c^2t^2 - x^2 \to \frac{(ct - \frac{vx}{c})^2 - (x-vt)^2}{1-\frac{v^2}{c^2}} = \frac{(c^2t^2 - x^2)(1-\frac{v^2}{c^2})}{1-\frac{v^2}{c^2}} = c^2t^2 - x^2
\end{aligned}$$

Proof B: (by consistency)

- $s^2 = 0$ the possibility of exchanging light signal is Lorentz invariant

- $s^2 \neq 0$? Three reference frames, S_0 and S_j, $\boldsymbol{v}_{S_0 \to S_j} = \boldsymbol{v}_j$, $j = 1, 2$, $|\boldsymbol{v}_1|, |\boldsymbol{v}_2| \ll c$

Step 1. $s_j^2 = F(|\boldsymbol{v}_j|, s_0^2)$

Step 2. $F(|\boldsymbol{v}|, 0) = 0$

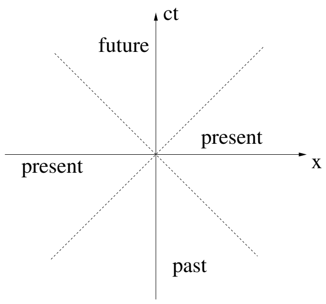
Step 3. continuity of $F(v, s^2)$ in s^2 at $s^2 = 0$: $ds_j^2 = F(|\boldsymbol{v}_j|, ds_0^2) \approx \underbrace{\frac{\partial F(|\boldsymbol{v}_j|, s^2)}{\partial s^2}_{|s^2=0}}_{a(|\boldsymbol{v}_j|)} ds_0^2$

Step 4. $ds_j^2 = a(|\boldsymbol{v}_j|)ds_0^2, \quad ds_2^2 = a(|\boldsymbol{v}_1 - \boldsymbol{v}_2|)ds_1^2, \quad a(|\boldsymbol{v}_1 - \boldsymbol{v}_2|) = \frac{a(|\boldsymbol{v}_2|)}{a(|\boldsymbol{v}_1|)} \to a = 1$

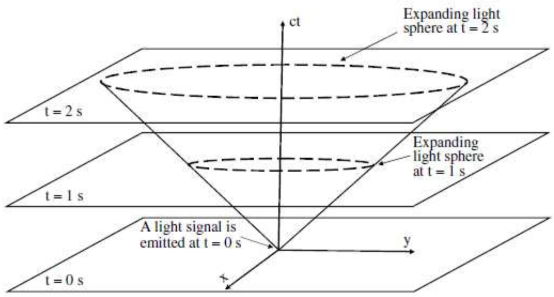
E. Minkowski geometry

Three different kinds of space-time intervals:

- time-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 > 0$
- space-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 < 0$
- light-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = 0$



past, present, future



Light cone

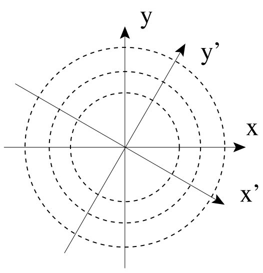
Change of scale:

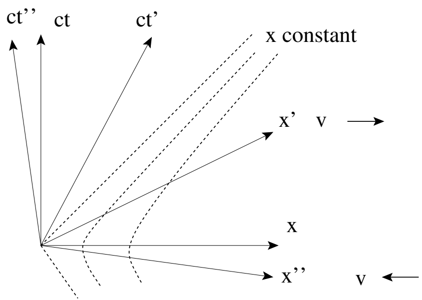
Euclidean:

$x^2 + y^2 = R^2$

Minkowski:

$(ct)^2 - x^2 = s^2$





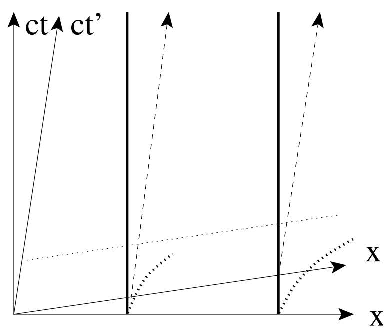
IV. PHYSICAL PHENOMENAS

A. Lorentz contraction of the length

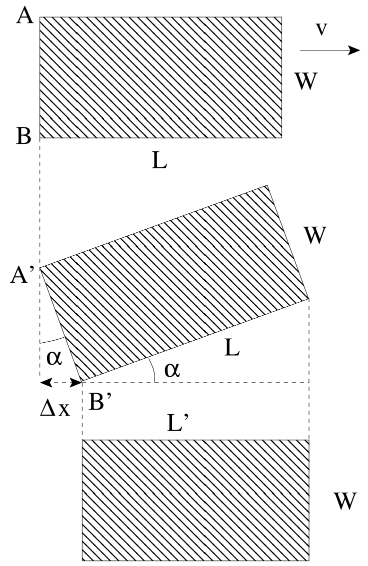
$\ell = x_2 - x_1$

$\ell' = x'_2 - x'_1$ **simultaneous** coordinates

$$
\begin{aligned}
x_a &= \frac{x'_a + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\ell &= \frac{x'_2 - x'_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\ell'}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\ell' &= \ell\sqrt{1 - \frac{v^2}{c^2}}
\end{aligned}
$$



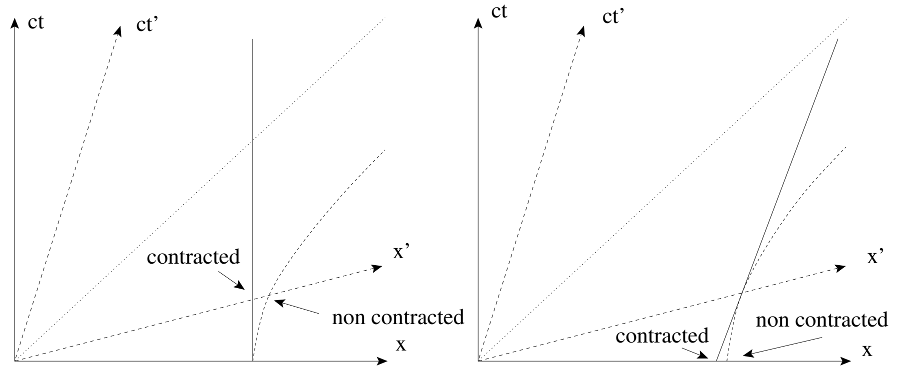
A simple model: apparent rotation of a rectangular board



The light, propagating from A_0 and B_0 arrive at the **same time**:

$$\begin{aligned}\Delta t &= \frac{W}{c}\\ \Delta x &= v\Delta t = W\frac{v}{c}\\ \sin\Theta &= \frac{v}{c}, \quad \cos\Theta = \sqrt{1-\frac{v^2}{c^2}}\\ L' &= L\sqrt{1-\frac{v^2}{c^2}} \leftarrow \text{Lorentz contraction}\end{aligned}$$

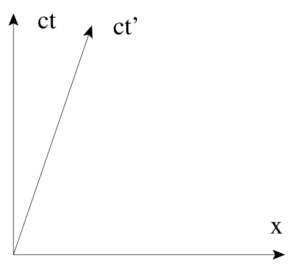
The contraction is relative:



Rod at rest Moving rod

B. Time dilatation

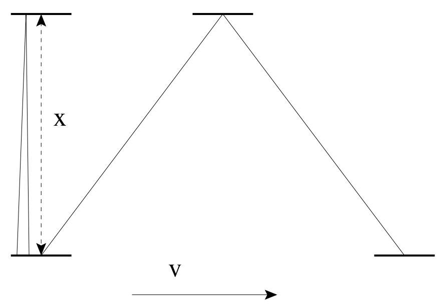
Moving clock: Proper time: $t_0 = \frac{\sqrt{s^2}}{c}$



$$\begin{aligned}s^2 &= c^2t_0^2 = c^2t^2 - x^2 = c^2t^2\left(1-\frac{v^2}{c^2}\right)\\ t &= \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}\end{aligned}$$

$t > t_0$: slowing down

Optical clock:



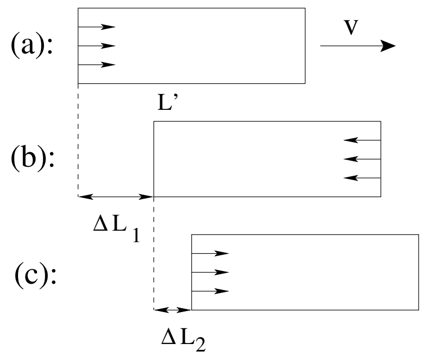
$$\begin{aligned}c^2t_0^2 &= x^2\\ c^2t'^2 &= t'^2v^2 + t_0'^2c^2\\ t' &= \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}\end{aligned}$$

Time dilatation $\Longrightarrow$ Lorentz contraction

Moving rod with mirrors:

$$
\begin{aligned}
c\Delta t_1 &= \ell + v\Delta t_1 \\
c\Delta t_2 &= \ell - v\Delta t_2 \\
2t' &= \Delta t_1 + \Delta t_2 \\
&= \frac{\ell'}{c-v} + \frac{\ell'}{c+v} = \frac{2\ell' c}{c^2 - v^2} \\
t' &= \frac{\ell'}{c}\frac{1}{1-\frac{v^2}{c^2}}
\end{aligned}
$$

$$\ell' = \ell_0\sqrt{1-\frac{v^2}{c^2}} \leftrightarrow t' = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{for} \quad \ell_0 = t_0 c$$

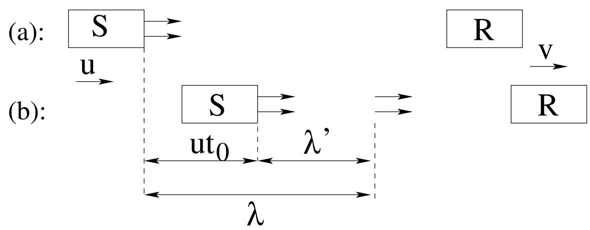


C. Doppler effect

Non-relativistic: sound velocity: w

Following one cycle:

$$
\begin{aligned}
w\Delta t &= \lambda' + u\Delta t \\
\lambda' &= (w-u)\Delta t = \frac{w-u}{\nu} = \frac{w-v}{\nu'} \\
\nu' &= \nu\frac{w-v}{w-u}
\end{aligned}
$$



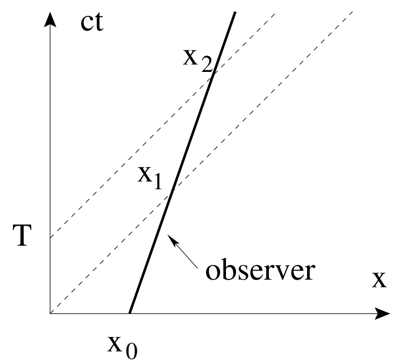
Stationary source: $u = 0$, $\nu' = \nu\left(1 - \frac{v}{w}\right)$, $\nu' = 0$ for $v = w$

Stationary receiver: $v = 0$, $\nu' = \frac{\nu}{1-\frac{u}{w}}$, $\nu' = \infty$ for $u = w$

Relativistic (light): Two pulses with time difference T, speed of the detector is v

$$
\begin{aligned}
x_1 &= ct_1 = x_0 + vt_1 & \rightarrow \quad t_1 &= \frac{x_0}{c-v} \\
x_2 &= c(t_2 - T) = x_0 + vt_2 & \rightarrow \quad t_2 &= \frac{x_0 + cT}{c-v}
\end{aligned}
$$

$$t_2 - t_1 = \frac{T}{1-\frac{v}{c}}, \qquad x_2 - x_1 = \frac{vT}{1-\frac{v}{c}}$$

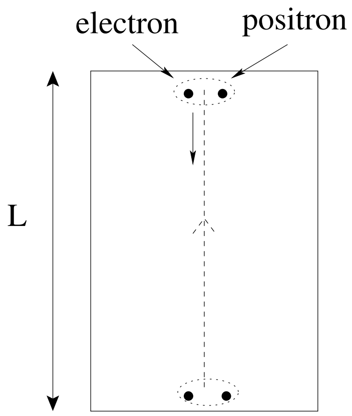


$$
\begin{aligned}
T' = t'_2 - t'_1 &= \frac{t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)}{\sqrt{1-\frac{v^2}{c^2}}} \\
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{T}{1-\frac{v}{c}} - \frac{v}{c^2}\frac{vT}{1-\frac{v}{c}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{T}{1-\frac{v}{c}}\left(1-\frac{v^2}{c^2}\right) = T\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} = T\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \\
\nu' &= \nu\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \neq \underbrace{\nu\left(1-\frac{v}{c}\right)}_{\text{non-relativistic}}
\end{aligned}
$$

Static gravitational field:

$$
\begin{aligned}
E_\downarrow(e^-e^+) &= \underbrace{E_\uparrow(e^-e^+)}_{2mc^2} + 2m\Delta U \\
&= E_\uparrow(e^-e^+)\left(1+\frac{\Delta U}{c^2}\right) \\
E_\uparrow(e^-e^+) &= E_\uparrow(\gamma), \quad E_\downarrow(e^-e^+) = E_\downarrow(\gamma) \\
E &= \hbar\omega \\
\frac{\omega_\downarrow}{\omega_\uparrow} &= \frac{E_\downarrow(e^-e^+)}{E_\uparrow(e^-e^+)} = 1 + \underbrace{\frac{\Delta U}{c^2}}_{z} \\
z &= \frac{\Delta U}{c^2}
\end{aligned}
$$



Time-dependent gravitational field:

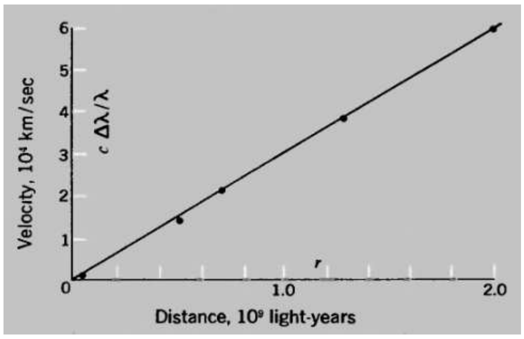
Robertson-Walker model of the Universe

Universe in expansion: $\lambda_{em} < \lambda_{obs}$

Red shift: $\dfrac{\omega_{em}}{\omega_{obs}} = \dfrac{\lambda_{obs}}{\lambda_{em}} = 1 + z > 1$

Hubble constant: $z = \dfrac{H}{c}\ell$

age of the Universe: $T_U = \dfrac{1}{H}$, $z = \dfrac{\frac{\ell}{c}}{T_U}$

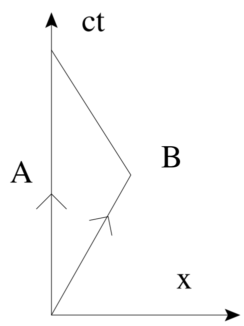


D. Paradoxes

Twins:

A: rests, B: leaves and returns

Which one is older when they meet again?



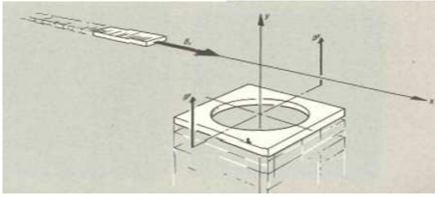
A rod and a circle:

$\ell = 2r$, $\boldsymbol{v}_b = (u, 0, 0)$, $\boldsymbol{v}_c = (0, v, 0)$

$t = 0$: the center of the rod and

the origin of the circle coincide

Can they cross each other?



A spear and a stable:

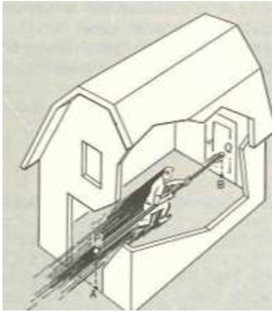
$\ell_{spear} = 20m$, $\ell_{stable} = 10m$

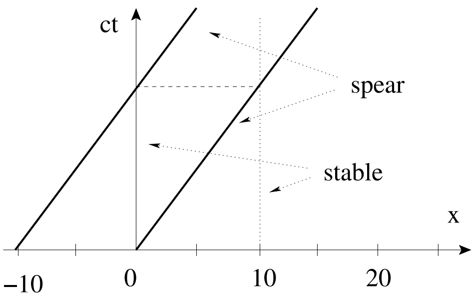
$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2}$

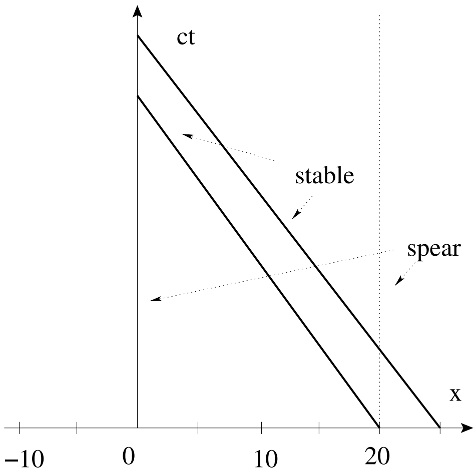
Reference frame of the stable: $\ell'_{spear} = 10m$

Reference frame of the spear: $\ell'_{stable} = 5m$

Can the spear enter into the stable?







V. VARIATIONAL FORMALISM OF A POINT PARTICLE

Need of an equation of motion which is independent of the choice of the coordinate system.

A. A point on a line

Goal: an equation to identify $x_{cl} \in \mathbb{R}$: use a function with extreme at c_{cl} only

$$S(x) \quad \to \quad \frac{dS(x)}{dx}_{|x=x_{cl}} = 0$$

Reparametrization of the line (change of coordinate system): $x \to y$

$$\frac{dS(x(y))}{dy}_{|y=y_{cl}} = \underbrace{\frac{dS(x)}{dx}_{|x=x_{cl}}}_{0} \frac{dx(y)}{dy}_{|y=y_{cl}} = 0$$

Variational principle, an alternative definition: $x \to x + \delta x$

$$\begin{aligned} S(x_{cl} + \delta x) &= S(x_{cl}) + \delta S(x_{cl}) \\ &= S(x_{cl}) + \delta x \underbrace{S'(x_{cl})}_{0} + \frac{\delta x^2}{2} S''(x_{cl}) + \mathcal{O}\left(\delta x^3\right) \\ \Longrightarrow \quad \delta S(x_{cl}) &= \mathcal{O}\left(\delta x^2\right) \qquad \text{(manifestly coordinate independent)} \end{aligned}$$

B. Non-relativistic particle

$x_{cl}(t_i) = x_i$, $x_{cl}(t_f) = x_f \quad \Longrightarrow \quad x_{cl}(t)$

Variational principle: $x(t) \to x(t) + \delta x(t)$, $\delta x(t_i) = \delta x(t_f) = 0$

$$\begin{aligned} S[x(\cdot)] &= \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\ \delta S[x(\cdot)] &= \int_{t_i}^{t_f} dt L\left(x(t) + \delta x(t), \dot{x}(t) + \frac{d}{dt}\delta x(t)\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \\ &= \int_{t_i}^{t_f} dt \Bigg[L(x(t), \dot{x}(t)) + \delta x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial x} \\ &\quad + \frac{d}{dt}\partial x(t) \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} + \mathcal{O}\left(\delta x(t)^2\right) - \int_{t_i}^{t_f} dt L(x(t), \dot{x}(t)) \Bigg] \\ &= \int_{t_i}^{t_f} dt \delta x(t) \underbrace{\left[\frac{\partial L(x(t), \dot{x}(t))}{\partial x} - \frac{d}{dt} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \right]}_{=0} \\ &\quad + \underbrace{\delta x(t)}_{0} \frac{\partial L(x(t), \dot{x}(t))}{\partial \dot{x}} \Bigg|_{t_f}^{t_i} + \mathcal{O}\left(\delta x(t)^2\right) \\ &= \mathcal{O}\left(\delta x(t)^2\right) \end{aligned}$$

Euler-Lagrange equation:

$$\boxed{0 = \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \boldsymbol{x}} - \frac{d}{dt} \frac{\partial L(\boldsymbol{x}, \dot{\boldsymbol{x}})}{\partial \dot{\boldsymbol{x}}}}$$

- Lagrangian:

$$L = T - U = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{x}) \quad \longrightarrow \quad m\ddot{\boldsymbol{x}} = -\boldsymbol{\nabla} U(\boldsymbol{x})$$

- Generalized momentum:

$$\boldsymbol{p} = \frac{\delta L}{\delta \dot{\boldsymbol{x}}}, \qquad \mathrm{E-L\ equation}: \dot{\boldsymbol{p}} = \frac{\delta L}{\delta \boldsymbol{x}}$$

- Cyclic coordinate: $\dfrac{\delta L}{\delta x_{cycl}} = 0$, p_{cycl} conserved

- Energy: Legendre transf.,

$$H(\boldsymbol{p}, \boldsymbol{x}) = \boldsymbol{p}\dot{\boldsymbol{x}} - L(\boldsymbol{x}, \dot{\boldsymbol{x}}), \quad \boldsymbol{p} = \frac{\delta L}{\delta \dot{\boldsymbol{x}}} = m\boldsymbol{v}$$

$$
\begin{aligned}
&= \boldsymbol{p}\frac{\boldsymbol{p}}{m} - \frac{m}{2}\left(\frac{\boldsymbol{p}}{m}\right)^2 + U(\boldsymbol{x}) = \frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{x}) = T + U \\
\dot{H} &= \dot{\boldsymbol{p}}\dot{\boldsymbol{x}} + \boldsymbol{p}\ddot{\boldsymbol{x}} - \dot{\boldsymbol{x}}\frac{\partial L}{\partial \boldsymbol{x}} - \ddot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \dot{\boldsymbol{p}}\dot{\boldsymbol{x}} + \boldsymbol{p}\ddot{\boldsymbol{x}} - \dot{\boldsymbol{x}}\dot{\boldsymbol{p}} - \ddot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} = 0
\end{aligned}
$$

Examples:

1. Particle moving on a curve, $y = f(x)$, on the (x, y) plane: $\dot{y} = f'(x)\dot{x}$

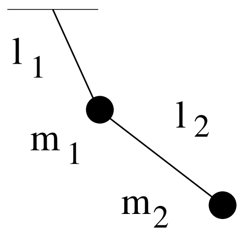
$$
L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgf(x) = \frac{m(x)}{2}\dot{x}^2 - mgf(x), \quad m(x) = m[1 + f'^2(x)]
$$

2. Pendulum:

$$
L = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell\cos\theta
$$

3. Double pendulum:

$$
\begin{aligned}
\boldsymbol{x}_1 &= \ell_1\begin{pmatrix}\sin\theta_1 \\ -\cos\theta_1\end{pmatrix}, \quad \boldsymbol{x}_2 = \boldsymbol{x}_1 + \ell_2\begin{pmatrix}\sin\theta_2 \\ -\cos\theta_2\end{pmatrix} \\
\dot{\boldsymbol{x}}_1 &= \ell_1\dot{\theta}_1\begin{pmatrix}\cos\theta_1 \\ \sin\theta_1\end{pmatrix}, \quad \dot{\boldsymbol{x}}_2 = \dot{\boldsymbol{x}}_1 + \ell_2\dot{\theta}_2\begin{pmatrix}\cos\theta_2 \\ \sin\theta_2\end{pmatrix}
\end{aligned}
$$



$$
\begin{aligned}
L &= \frac{m_1 + m_2}{2}\ell_1^2\dot{\theta}_1^2 + \frac{m_2}{2}\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\
&\quad + g(m_1 + m_2)\ell_1\cos\theta_1 + gm_2\ell_2\cos\theta_2 \\
0 &= m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(-\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - g(m_1 + m_2)\ell_1\sin\theta_1 \\
&\quad + (m_1 + m_2)\ell_1^2\ddot{\theta}_1 - m_2\ell_1\ell_2\frac{d}{dt}[\dot{\theta}_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)] \\
0 &= m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2(-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) - gm_2\ell_2\sin\theta_2 \\
&\quad - m_2\ell_2^2\ddot{\theta}_2 - m_2\ell_1\ell_2\frac{d}{dt}[\dot{\theta}_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)]
\end{aligned}
$$

C. Noether's theorem

$\boxed{\text{Each continuous symmetry generates a conserved quantity}}$

Symmetry: $\boldsymbol{x}(t) \to \boldsymbol{x}'(t')$, $L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x}', \dot{\boldsymbol{x}}') + \dot{\Lambda}(t', \boldsymbol{x}')$

Continuous symmetry: $\exists$ infinitesimal transformations, $\boldsymbol{x} \to \boldsymbol{x} + \epsilon\boldsymbol{f}(t, \boldsymbol{x})$, $t \to t + \epsilon f(t, \boldsymbol{x})$

Conserved quantity: $\dot{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}) = 0$

(a) $\boldsymbol{f} \neq 0$, $f = 0$:

$$
L(\boldsymbol{x}, \dot{\boldsymbol{x}}) = L(\boldsymbol{x} + \epsilon\boldsymbol{f}, \dot{\boldsymbol{x}} + \epsilon\partial_t\boldsymbol{f} + \epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}) + \mathcal{O}\left(\epsilon^2\right)
$$

A particular variation: $\epsilon \to \epsilon(t)$, $\boldsymbol{x} = \boldsymbol{x}_{cl}$, $\delta S[\epsilon] = \mathcal{O}\left(\epsilon^2\right)$

$$\begin{aligned} \tilde{L}(\epsilon,\dot{\epsilon}) &= L(\boldsymbol{x}+\epsilon\boldsymbol{f}, \dot{\boldsymbol{x}}+\epsilon\partial_t\boldsymbol{f}+\epsilon(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}+\dot{\epsilon}\boldsymbol{f}) + \mathcal{O}\left(\epsilon^2\right) \\ &= \epsilon\left[\frac{\partial L}{\partial \boldsymbol{x}}\boldsymbol{f} + \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\partial_t\boldsymbol{f} + \frac{\partial L}{\partial \dot{\boldsymbol{x}}}(\dot{\boldsymbol{x}}\boldsymbol{\partial})\boldsymbol{f}\right] + \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\dot{\epsilon}\boldsymbol{f} + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

Euler-Lagrange equation:

$$\underbrace{\frac{\partial \tilde{L}(\epsilon,\dot{\epsilon})}{\partial \epsilon}}_{0} = \frac{d}{dt}\underbrace{\frac{\partial \tilde{L}(\epsilon,\dot{\epsilon})}{\partial \dot{\epsilon}}}_{p_\epsilon}, \quad p_\epsilon = \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\boldsymbol{f}$$

Translations: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{n}$, $\boldsymbol{n}^2 = 1$. $L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(\boldsymbol{T}\boldsymbol{x})$, $T = \mathbb{1} - \boldsymbol{n}\otimes\boldsymbol{n}$, $p_\epsilon = m\dot{\boldsymbol{x}}\boldsymbol{n}$

Rotations: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{n}\times\boldsymbol{x}$, $\boldsymbol{n}^2 = 1$. $L = \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(|\boldsymbol{x}|)$, $p_\epsilon = m\dot{\boldsymbol{x}}(\boldsymbol{n}\times\boldsymbol{x}) = \boldsymbol{n}(\boldsymbol{x}\times m\dot{\boldsymbol{x}}) = \boldsymbol{n}\boldsymbol{L}$

(b) $\boldsymbol{f} = 0$, $f \neq 0$: $t \to t' = t+\epsilon$, $\boldsymbol{x}(t) \to \boldsymbol{x}(t-\epsilon) = \boldsymbol{x}(t) - \epsilon\dot{\boldsymbol{x}}(t)$, $\delta\boldsymbol{x} = -\epsilon\dot{\boldsymbol{x}}$

A reparametrization of the time integral of the action with $\epsilon \to \epsilon(t)$:

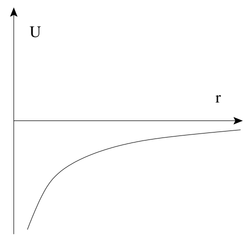
$$\begin{aligned} S[\boldsymbol{x}] &= \int_{t_i+\epsilon(t_i)}^{t_f+\epsilon(t_f)} \underbrace{\frac{dt'}{1+\dot{\epsilon}}}_{dt=\frac{dt'}{(\frac{dt'}{dt})}} L(\boldsymbol{x}(t'-\epsilon), \dot{\boldsymbol{x}}(t'-\epsilon)) \\ 0 &= -\int_{t_i}^{t_f} dt\left(\epsilon\dot{\boldsymbol{x}}\frac{\partial L}{\partial \boldsymbol{x}} + \frac{d}{dt}\epsilon\dot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}} + \dot{\epsilon}L\right) + \epsilon L(\boldsymbol{x}(t),\dot{\boldsymbol{x}}(t))\Big|_{t_i}^{t_f} \\ &= -\int_{t_i}^{t_f} dt\left[\epsilon\dot{\boldsymbol{x}}\left(\frac{\partial L}{\partial \boldsymbol{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) + \dot{\epsilon}L\right] + \epsilon\left(L - \dot{\boldsymbol{x}}\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right)_{t_i}^{t_f} \\ \epsilon(t) \to \epsilon \qquad H &= \frac{\partial L}{\partial \dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} - L \quad \text{is conserved} \end{aligned}$$

D. Examples

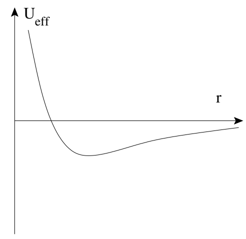
1. Spherically symmetric potential:

$$\begin{aligned} L &= \frac{m}{2}\dot{\boldsymbol{x}}^2 - U(|\boldsymbol{x}|) \\ \boldsymbol{x} &= r\begin{pmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{pmatrix}, \quad \dot{\boldsymbol{x}} = \dot{r}\begin{pmatrix}\sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta\end{pmatrix} + r\begin{pmatrix}\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi \\ \dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi \\ -\dot{\theta}\sin\theta\end{pmatrix} \\ L &= \frac{m}{2}[\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta] - U(r) \\ 0 &= \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \\ 0 &= r^2\dot{\phi}^2\sin\theta\cos\theta - \frac{d}{dt}r^2\dot{\theta} \quad \to \quad \text{planar motion}: \ \ \theta = \frac{\pi}{2} \\ 0 &= \frac{d}{dt}\frac{\delta L}{\delta\dot{\phi}} = \frac{d}{dt}p_\phi = \frac{d}{dt}mr^2\dot{\phi}\sin^2\theta \quad \to \quad \text{conservation of } p_\phi = mr^2\dot{\phi} = L_z = \ell \\ 0 &= -U'(r) + mr(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - m\ddot{r} = -U'(r) + mr\dot{\phi}^2 - m\ddot{r} \\ &= -U'(r) + \frac{\ell^2}{mr^3} - m\ddot{r} = -U'(r) - \frac{d}{dr}\frac{\ell^2}{2mr^2} - m\ddot{r} \\ m\ddot{r} &= -U'_{eff}(r), \quad U_{eff}(r) = U(r) + \frac{\ell^2}{2mr^2} \end{aligned}$$

One-dimensional motion in an effective potential with an centrifugal barrier:

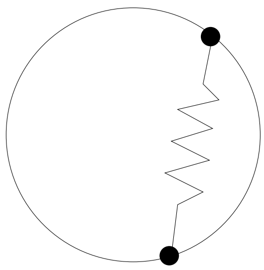


$$L = \frac{m}{2}\dot{r}^2 - U_{eff}(r)$$
$$U(r) = -\frac{e^2}{r}$$



- What can one say about the possibility of falling into the center for $U(r) = -\frac{g}{r^n}$ with $n = 2$ or 3?
- Is there circular orbit for $U(r) = -\frac{g_1}{r} - \frac{g_3}{r^3}$?
- What are the stable orbits for $U(r) = -g\frac{e^{-kr}}{r}$?

2. Two particles on a ring: Two point particles of mass m move on a ring of radius r without friction. They are connected by a massless spring of length $\ell < 2r$ and spring constant k.



(a): Find a continuous symmetry and the corresponding conserved quantity.
(b): Solve the equation of motion for small oscillations.

- Lagrangian:

$$L = \frac{m}{2}r^2(\dot{\alpha}^2 + \dot{\beta}^2) - \frac{k}{2}\left(2r\sin\frac{\alpha-\beta}{2} - \ell\right)^2$$

- Symmetry: $\alpha \to \alpha + \phi$, $\beta \to \beta + \phi$ (rotation)
- New coordinates: $\Theta = \frac{\alpha+\beta}{2}$, $\chi = \alpha - \beta$, $\Theta \to \Theta + \phi$, $\chi \to \chi$

$$\begin{aligned}
\alpha &= \Theta + \frac{\chi}{2}, \quad \beta = \Theta - \frac{\chi}{2} \\
L &= \frac{m}{2}r^2\left[\left(\dot{\Theta} + \frac{\dot{\chi}}{2}\right)^2 + \left(\dot{\Theta} - \frac{\dot{\chi}}{2}\right)^2\right] - \frac{k}{2}\left(2r\sin\frac{\chi}{2} - \ell\right)^2 \\
&= mr^2\left(\dot{\Theta}^2 + \frac{\dot{\chi}^2}{4}\right) - \frac{k}{2}\left(2r\sin\frac{\chi}{2} - \ell\right)^2
\end{aligned}$$

- Conservation law:

$$p_\Theta = \frac{\partial L}{\partial \dot\Theta} = 2mr^2\dot\Theta = L_z = mr^2(\dot\alpha + \dot\beta)$$

- E.O.M.

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot x} \\ \delta\Theta : \ 2mr^2\ddot\Theta &= 0, \quad \Theta = \Theta_0 + \Omega t \\ \delta\chi : \ \frac{mr^2}{2}\ddot\chi &= -kr\left(2r\sin\frac{\chi}{2} - \ell\right)\cos\frac{\chi}{2} \end{aligned}$$

- Small oscillations: $2r\sin\frac{\chi}{2} \approx \ell$, $2r\sin\frac{\chi_0}{2} = \ell$, $\chi = \chi_0 + \epsilon$

$$\begin{aligned} \frac{mr^2}{2}\ddot\epsilon &= -kr\left(2r\sin\frac{\chi_0+\epsilon}{2} - \ell\right)\cos\frac{\chi_0+\epsilon}{2} \approx -kr^2\epsilon\cos^2\frac{\chi_0}{2} \\ \omega^2 &= 2\frac{k}{m}\cos^2\frac{\chi_0}{2} = 2\frac{k}{m}\left[1 - \left(\frac{\ell}{2r}\right)^2\right] \end{aligned}$$

Special cases:

$$\begin{aligned} \ell \approx 2r : \quad \omega &\approx 0 \quad \text{Why?} \\ \ell \approx 0 : \quad \omega &\approx \sqrt{\frac{2k}{m}} \end{aligned}$$

VI. RELATIVISTIC MECHANICS

A. Vectors and tensors

The components of the four-vector, $x^\mu = (ct, \boldsymbol{x}) = (x^0, \boldsymbol{x})$, $\mu = 0, 1, 2, 3$,
are mixed by Lorentz transformations

Invariant length: to characterize the Lorentz transformations

$$s^2 = x^{02} - \boldsymbol{x}^2 = x^\mu g_{\mu\nu} x^\nu = xx$$

Metric tensor: to define the scalar product

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Scalar product:

$$xy = \sum_{\mu\nu} x^\mu g_{\mu\nu} y^\nu$$

To suppress g one uses two representation for each vector u:

- a covariant, u_μ, and

- a contravariant, u^μ:

$$\begin{aligned} u_\mu &= g_{\mu\nu}u^\nu, \quad u^\mu = g^{\mu\nu}u_\nu \quad \to \quad xy = x^\mu y_\mu = x_\mu y^\mu \\ u_\mu &= g_{\mu\nu}g^{\nu\rho}u_\rho \quad \to \quad g_{\mu\nu}g^{\nu\rho} = g_{\mu\cdot}^{\cdot\rho} = \delta_\mu^\rho \end{aligned}$$

Einstein convention:

$$\sum_\mu (\cdots)^\mu(\cdots)_\mu \to (\cdots)^\mu(\cdots)_\mu$$

Lorentz group: Family of linear transformations

$$x^\mu \to x'^\mu = \Lambda^{\mu}_{\cdot\,\nu}{}^{\cdot} x^\nu,$$

preserving the invariant length,

$$xy = x^{\mu'}\Lambda^{\mu}_{\cdot\,\mu'}{}^{\cdot} g_{\mu\nu}\Lambda^{\nu}_{\cdot\,\nu'}{}^{\cdot} y^{\nu'}, \qquad g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\,\mu}{}^{\cdot} g_{\mu'\nu'}\Lambda^{\nu'}_{\cdot\,\nu}{}^{\cdot}.$$

It has 6 dimensions, 3 for spatial rotation,

$$\Lambda = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R \end{pmatrix}, \qquad R^{\mathrm{tr}}R = \mathbb{1}$$

and another 3 for boosts: $\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$, $\boldsymbol{x}_\parallel\boldsymbol{x}_\perp = \boldsymbol{v}\boldsymbol{x}_\perp = 0$.

$$\boldsymbol{x}' = \alpha(\boldsymbol{x}_\parallel - \boldsymbol{v}t) + \gamma\boldsymbol{x}_\perp, \qquad t' = \beta\left(t - \frac{\boldsymbol{x}_\parallel\boldsymbol{v}}{\tilde{c}^2}\right)$$

Invariance:

$$\begin{aligned} c^2t^2 - \boldsymbol{x}^2 &= c^2\beta^2\left(t - \frac{\boldsymbol{x}_\parallel\boldsymbol{v}}{\tilde{c}^2}\right)^2 - \alpha^2(\boldsymbol{x}_\parallel - \boldsymbol{v}t)^2 - \gamma\boldsymbol{x}_\perp^2 \\ &= c^2t^2\left(\beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2}\right) - \boldsymbol{x}_\parallel^2\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2\right) + 2\boldsymbol{v}\boldsymbol{x}_\parallel t\left(\alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2\right) \\ \mathcal{O}\left(\boldsymbol{x}_\perp^2\right):\ \gamma &= \pm 1 \to 1 \quad \leftarrow \quad v = 0 \\ \mathcal{O}\left(\boldsymbol{x}_\perp\boldsymbol{v}\right):\ 0 &= \alpha^2 - \frac{c^2}{\tilde{c}^2}\beta^2 \\ \mathcal{O}\left(\boldsymbol{x}_\parallel^2\right):\ -1 &= \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{\tilde{c}^2}\beta^2 - \alpha^2 \ \to\ \beta^2 = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{\boldsymbol{v}^2}{\tilde{c}^2}} \\ \mathcal{O}\left(c^2t^2\right):\ 1 &= \beta^2 - \alpha^2\frac{\boldsymbol{v}^2}{c^2} = \frac{\tilde{c}^2}{c^2}\frac{1}{1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}}\left(1 - \frac{c^2}{\tilde{c}^2}\frac{\boldsymbol{v}^2}{c^2}\right) \ \to\ \tilde{c}^2 = c^2 \\ \alpha &= \beta = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \leftarrow \quad v = 0 \end{aligned}$$

$$\boldsymbol{x}'_\parallel = \frac{\boldsymbol{x}_\parallel - \boldsymbol{v}t}{\sqrt{1-\frac{v^2}{c^2}}}, \quad ct' = \frac{ct - \frac{\boldsymbol{v}\boldsymbol{x}_\parallel}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Inverse: $\boldsymbol{v} \to -\boldsymbol{v}$

$$\boldsymbol{x}_\parallel = \frac{\boldsymbol{x}'_\parallel + \boldsymbol{v}t'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad ct = \frac{ct' + \frac{\boldsymbol{v}\boldsymbol{x}'_\parallel}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Exercise: internal consistency of the notation, $\Lambda = \Lambda^{\mu}_{\cdot\,\nu}{}^{\cdot}$, $g = g_{\mu\nu}$, $g^{\mathrm{tr}} = g$,

$$\begin{aligned}
g &= \Lambda^{\mathrm{tr}} g \Lambda, \quad \leftrightarrow \quad g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\,\mu}{}^{\cdot} g_{\mu'\nu'} \Lambda^{\nu'}_{\cdot\,\nu}{}^{\cdot} \\
\Lambda^{-1} &= g^{-1}\Lambda^{\mathrm{tr}} g = (g\Lambda g^{-1})^{\mathrm{tr}}, \quad \leftrightarrow \quad \Lambda^{-1\mu}_{\cdot\,\nu}{}^{\cdot} = g^{\mu\mu'}\Lambda^{\nu'}_{\cdot\,\mu'}{}^{\cdot} g_{\nu'\nu} = \Lambda_{\nu}{}^{\cdot}{}^{\mu}_{\cdot} \quad \text{(orth.)} \\
x'^{\mu} &= (\Lambda x)^{\mu} = \Lambda^{\mu}_{\cdot\,\nu}{}^{\cdot} x^{\nu} \\
x^{\mu} &= (g\Lambda g^{-1})_{\nu}{}^{\cdot}{}^{\mu}_{\cdot} x'^{\nu} = x'^{\nu}\Lambda_{\nu}{}^{\cdot}{}^{\mu}_{\cdot} = (x'\Lambda)^{\mu} \\
x'_{\mu} &= (g\Lambda x)_{\mu} = (g\Lambda g^{-1} g x)_{\mu} = \Lambda_{\mu}{}^{\cdot}{}^{\nu}_{\cdot} x_{\nu} = (\Lambda x)_{\mu} \\
x_{\mu} &= x'^{\nu} g_{\nu\lambda} g^{\lambda\rho}\Lambda_{\rho\mu} = x'_{\lambda}\Lambda^{\lambda}_{\cdot\,\mu}{}^{\cdot} = (x'\Lambda)_{\mu} \\
\text{N.B. rotations :} \quad \boldsymbol{x}' &= R\boldsymbol{x}, \quad \boldsymbol{x} = R^{\mathrm{tr}}\boldsymbol{x}' = \boldsymbol{x}'R
\end{aligned}$$

$$\begin{aligned}
\text{contravariant tensor} &: T^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}_{\ \nu_1}\cdots\Lambda^{\mu_n}_{\ \nu_n} T^{\nu_1\cdots\nu_n} \\
\text{covariant tensor} &: T_{\mu_1\cdots\mu_n} = \Lambda_{\mu_1}^{\ \nu_1}\cdots\Lambda_{\mu_n}^{\ \nu_n} T_{\nu_1\cdots\nu_n} \\
\text{mixed tensor} &: T^{\rho_1\cdots\rho_m}_{\mu_1\cdots\mu_n} = \Lambda^{\rho_1}_{\ \kappa_1}\cdots\Lambda^{\rho_m}_{\ \kappa_m}\Lambda_{\mu_1}^{\ \nu_1}\cdots\Lambda_{\mu_n}^{\ \nu_n} T^{\kappa_1\cdots\kappa_m}_{\nu_1\cdots\nu_n}
\end{aligned}$$

Invariant tensor:

$$g_{\mu\nu} = \Lambda^{\mu'}_{\ \mu} g'_{\mu'\nu'}\Lambda^{\nu'}_{\ \nu}$$

B. Relativistic generalization of the Newtonian mechanics

Four-velocity: $ds = dx^0\sqrt{1-\frac{v^2}{c^2}}$

$$\begin{aligned}
ds^2 &= dx^{02} - d\boldsymbol{x}^2 = dx^{02}\left[1 - \left(\frac{d\boldsymbol{x}}{dx^0}\right)^2\right] = dx^{02}\left(1 - \frac{\dot{\boldsymbol{x}}^2}{c^2}\right) \\
ds &= dx^0\sqrt{1-\frac{v^2}{c^2}} \\
u^{\mu} &= \frac{dx^{\mu}(s)}{ds} = \dot{x}(s) = \left(\frac{dx^0}{ds}, \frac{dx^0}{ds}\frac{\boldsymbol{v}}{c}\right) = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\frac{\boldsymbol{v}}{c}}{\sqrt{1-\frac{v^2}{c^2}}}\right)
\end{aligned}$$

Four-acceleration:

$$\dot{u}^{\mu} = \frac{du^{\mu}}{ds}, \qquad u^2 = 1 \quad \longrightarrow \quad \dot{u}u = 0$$

Four-momentum:

$$p^\mu = mcu^\mu = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}} \right) = \left(\frac{E}{c}, \boldsymbol{p} \right), \qquad p^2 = m^2c^2$$

N.B. (i) $c \to \infty$, $\boldsymbol{p} \to m\boldsymbol{v}$, (ii) $v \to c$, $\boldsymbol{p} \to \infty$

Four-force:

$$\frac{dp^\mu}{ds} = \frac{d}{ds}\left(mc\frac{dx^\mu}{ds} \right) = K^\mu$$

Spatial components:

$$\begin{aligned}
\frac{dt}{ds}\frac{d}{dt}\left(mc\frac{dt}{ds}\frac{d\boldsymbol{x}}{dt} \right) &= \boldsymbol{K} \\
m(v) &= m\frac{dx^0}{ds} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} \\
\frac{d}{dt}\left(m(v)\frac{d\boldsymbol{x}}{dt} \right) &= \frac{d}{dt}\boldsymbol{p} = \frac{ds}{dt}\boldsymbol{K} = \boldsymbol{F}
\end{aligned}$$

Temporal component: (energy equation)

$$\begin{aligned}
\frac{d}{ds}\left(mc\frac{dx^0}{ds} \right) &= \frac{d}{ds}\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = K^0 \\
Ku &= 0 \qquad \longrightarrow \qquad \underbrace{K^0}_{\frac{dt}{ds}\frac{d}{dt}cm(v)}\frac{dx^0}{ds} = \underbrace{\boldsymbol{K}}_{\frac{dt}{ds}\boldsymbol{F}}\underbrace{\boldsymbol{u}}_{\frac{dt}{ds}\boldsymbol{v}} \\
\frac{dm(v)c^2}{dt} &= \boldsymbol{F}\boldsymbol{v} \qquad \longrightarrow \qquad m(v)c^2 = \int \boldsymbol{F}\boldsymbol{v}dt = \int \boldsymbol{F}d\boldsymbol{r} \\
\text{let} \quad \boldsymbol{F} &= -\boldsymbol{\nabla}\phi \qquad \longrightarrow \qquad E = \underbrace{m(v)c^2}_{cp^0} + \phi \quad \text{is constant}
\end{aligned}$$

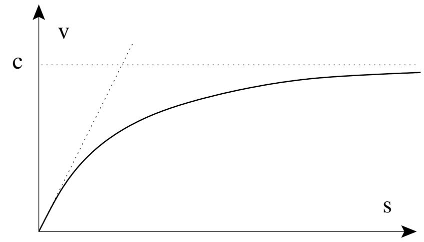
Energy:

$$\begin{aligned}
p^\mu &= \left(\frac{E}{c}, m(v)\boldsymbol{v} \right), \qquad \frac{E^2}{c^2} = \boldsymbol{p}^2 + m^2c^2, \qquad E(\boldsymbol{p}) = \pm c\sqrt{\boldsymbol{p}^2 + m^2c^2} \\
\frac{\partial |E(\boldsymbol{p})|}{\partial \boldsymbol{p}} &= \frac{\boldsymbol{p}c}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} = \frac{\frac{\boldsymbol{p}}{m}}{\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}} = \frac{\frac{\boldsymbol{v}}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}}{\sqrt{1+\frac{\boldsymbol{v}^2}{c^2(1-\frac{\boldsymbol{v}^2}{c^2})}}} = \boldsymbol{v}, \quad |\boldsymbol{v}| \le c
\end{aligned}$$

Ex. 1: Constant non-relativistic force

$$F = \frac{d}{dt}p = \frac{d}{dt}\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x_i = v_i = 0 \quad \text{(non Lorentz inv.!)}$$

$$\begin{aligned} Ft &= \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \quad 1-\frac{v^2}{c^2} = \left(\frac{mv}{Ft}\right)^2 \\ c^2 - v^2 &= v^2\left(\frac{mc}{Ft}\right)^2 \rightarrow c^2 = v^2\left[1+\left(\frac{mc}{Ft}\right)^2\right] \\ v &= \frac{c}{\sqrt{1+\left(\frac{mc}{Ft}\right)^2}} \approx \begin{cases} t\frac{F}{m} & Ft \ll mc \\ c & Ft \gg mc \end{cases} \end{aligned}$$



Ex. 2: Constant relativistic force

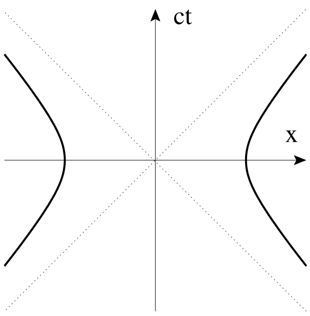
$$\frac{dp}{ds} = K$$

Initial conditions in a co-moving reference frame:

$$x_i = x_0, \quad v_i = 0,, \quad x_i = x_0,, \quad v_i = 0, \quad u = \dot{x} = (1, \mathbf{0}), \quad \dot{u}u = 0 \rightarrow \dot{u} = (0, \frac{\boldsymbol{a}}{c^2}), \quad \dot{u}^2 = -\frac{\boldsymbol{a}^2}{c^4}$$

Solution of $u^2(s) = 0$:

$$\begin{aligned} u^\mu &= \left(\cosh\frac{a}{c^2}s, \sinh\frac{a}{c^2}s\right) \\ x^\mu &= \frac{c^2}{a}\left(\sinh\frac{a}{c^2}s, \cosh\frac{a}{c^2}s - 1\right) + (0, x_0) \end{aligned}$$



Rindler geometry:

Half of the space-time is unreachable

Thermal radiation of particle-anti particle pairs on the horizon

$\Longrightarrow$ Hawking radiation of black holes

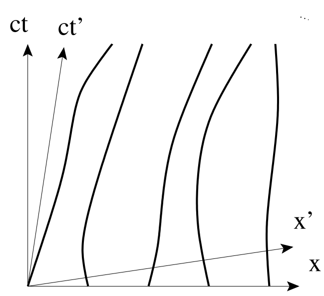
C. Interactions

$$\begin{aligned} E.M. \quad \ddot{x}_a^\mu &= F_a^\mu(x_1, \ldots, x_n), \quad a = 1, \ldots, n \\ C.I. \quad x_a(s_i) &= x_{ia}, \ \dot{x}_a(s_i) = u_a \end{aligned}$$

Problems:

1. The initial conditions are imposed on the spatial hyper-surface, $t = t_i$ and the Lorentz boost mixes into the dynamics:

$$\dot{x}\ddot{x} = 0 \quad \rightarrow \quad u_a F_a = 0$$



2. The four-force, $F_a^\mu(x_1,\ldots,x_n)$, represents a superluminal effect

No-Go theorem: there is no relativistic interaction, mediated by a force of the type

$$F_a^\mu(x_1,\ldots,x_n)\neq 0$$

e.g.

$$F_a^\mu(x_1,\ldots,x_n)=\sum_{b\neq a}(x_a^\mu-x_b^\mu)f((x_a-x_b)^2)\quad\to\quad(x_a-x_b)\dot x_a\neq 0$$

Solution: - distribute degrees of freedom at each space point

- transfer the excitations with a limited speed

$\boxed{\Longrightarrow\ \text{fields, } \phi(t,\boldsymbol{x}), \text{ appear in physics}}$

D. Variational principle of relativistic mechanics

- Lorentz invariant action:

$$\begin{aligned}[S] &= TML^2T^{-2}=ML^2T^{-1}=[\boldsymbol{x}\times\boldsymbol{p}]=[\hbar]\\ S &= -mc\int_{s_i}^{s_f}ds=-mc\int_{s_i}^{s_f}ds\sqrt{\dot x^\mu g_{\mu\nu}\dot x^\mu}\end{aligned}$$

Reparametrization invariance: $s\to\tau$:

$$S_\tau = -mc\int_{\tau_{min}}^{\tau_{max}}d\tau\left|\frac{ds}{d\tau}\right|\sqrt{\frac{d\tau}{ds}\frac{dx^\mu}{d\tau}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\tau}{ds}}=S_s$$

- E.O.M.: $\dot x^2=1\quad\Longrightarrow\ \tau\neq s$ to allow independent variation of $x^\mu(\tau)$, $x^\mu(\tau)\to x^\mu(\tau)+\delta x^\mu(\tau)$

$$\begin{aligned}0 &= -mc\frac{d}{d\tau}\frac{\frac{dx^\mu}{d\tau}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}}}=-mc\frac{\frac{d^2x^\mu}{d\tau^2}}{\sqrt{\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}}}+mc\frac{\frac{dx^\mu}{d\tau}\frac{d^2x^\rho}{d\tau^2}g_{\rho\nu}\frac{dx^\mu}{d\tau}}{\sqrt{\left(\frac{dx^\rho}{d\tau}g_{\rho\nu}\frac{dx^\mu}{d\tau}\right)^3}}\\ (\tau=s)\quad &= -mc\ddot x^\mu\end{aligned}$$

- The four-momentum:

$$p^\mu=-\frac{\partial L}{\partial\dot x_\mu}=mc\dot x^\mu$$

- Nonrelativistic notation: $x^\mu=(ct,\boldsymbol{x})$,

$$\begin{aligned}S &= \int_{t_i}^{t_f}dt\,\underbrace{(-mc^2)\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}_{L_t}=-mc^2(t_f-t_i)+\int_{x_i}^{x_f}dt\left[\frac{m}{2}\boldsymbol{v}^2+\mathcal{O}\left(\frac{v^4}{c^2}\right)\right]\\ \boldsymbol{p} &= \frac{\partial L}{\partial\boldsymbol{v}}=\frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}\\ E &= \frac{\partial L}{\partial\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}-L=\boldsymbol{p}\boldsymbol{v}-L\\ &= \frac{m\left[v^2+c^2\left(1-\frac{v^2}{c^2}\right)\right]}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}=mc^2+\frac{v^2}{2m}+\mathcal{O}\left(\frac{v^4}{c^2}\right)\end{aligned}$$

Motion in a spherically symmetric potential:

- Coordinates: $x^\mu = (ct, r, \theta, \phi)$
- Action:

$$\begin{aligned} S &= -mc\int ds\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} - \int dtU(r) = -\int ds[mc\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} + \dot{t}U(r)] \\ L &= -mc\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} - \dot{t}U(r) \\ &= -mc\sqrt{c^2\dot{t}^2 - \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)} - \dot{t}U(r) \end{aligned}$$

- $\delta\theta$:

$$r^2\sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{ds}r^2\dot{\theta} \quad\to\quad \theta(s) = \frac{\pi}{2}$$

- Cyclic coordinates: t, ϕ:

$$\begin{aligned} \frac{\partial L}{\partial \dot{t}} &= -mc^3\dot{t} - U(r) = -mc^2 - E \quad\to\quad \dot{t} = \frac{(mc^2+E-U)^2}{m^2c^6}, \\ \frac{\partial L}{\partial \dot{\phi}} &= mcr^2\dot{\phi} = \ell, \end{aligned}$$

- Energy equation: $1 = \dot{x}^2$

$$1 = c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\phi}^2 = \frac{(mc^2+E-U)^2}{m^2c^4} - \dot{r}^2 - \frac{\ell^2}{m^2c^2r^2}$$

- Radial equation of motion:

$$\begin{aligned} \dot{r}^2 &= \dot{t}^2\left(\frac{dr}{dt}\right)^2 = \frac{(mc^2+E-U)^2}{m^2c^4} - \frac{\ell^2}{m^2c^2r^2} - 1 \\ \left(\frac{dr}{dt}\right)^2 &= \frac{m^2c^6}{(mc^2+E-U)^2}\left[\frac{(mc^2+E-U)^2}{m^2c^4} - \frac{\ell^2}{m^2c^2r^2} - 1\right] \\ &= c^2\left[1 - \frac{\frac{\ell^2c^2}{r^2} + m^2c^4}{(mc^2+E-U)^2}\right] \\ &= c^2\left[1 - \frac{\frac{\ell^2}{m^2c^2r^2} + 1}{(1+\frac{E-U}{mc^2})^2}\right] \end{aligned}$$

- Nonrelativistic limit, $\frac{v}{c} \to 0$, in the leading $\mathcal{O}\left((\frac{v}{c})^0\right)$ order: $(U, E = \mathcal{O}\left((\frac{v}{c})^0\right))$:

$$\begin{aligned} \left(\frac{dr}{dt}\right)^2 &\approx c^2\left[1 - \left(\frac{\ell^2}{m^2c^2r^2} + 1\right)\left(1 - 2\frac{E-U}{mc^2}\right)\right] \approx c^2\left(-\frac{\ell^2}{m^2c^2r^2} + 2\frac{E-U}{mc^2}\right) \\ &= -\frac{\ell^2}{m^2r^2} + \frac{2}{m}(E-U) \\ E &\approx \frac{m}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\ell^2}{2mr^2} + U \end{aligned}$$

- Further problems:

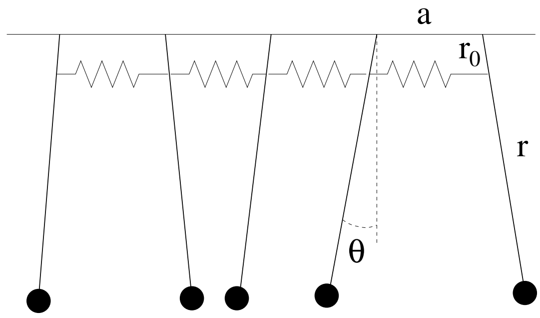
  1. $\mathcal{O}\left((\frac{v}{c})^2\right)$?
  2. Find the equation of motion of the action

$$S = -mc\int ds\sqrt{\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu} - \frac{1}{c}\int dsU(r)$$

VII. FIELD THEORIES

Mechanical analogy: $\theta_n(t) \to \Phi\theta_n(t) = \phi(t, x_n)$, $\Phi = r_0\sqrt{ak}$, $c = a\frac{r_0}{r}\sqrt{\frac{k}{m}}$, $\lambda = \frac{gr}{a}$

$$\begin{aligned}
L &= \sum_n \left[\frac{mr^2}{2}\dot\theta_n^2 - \frac{kr_0^2}{2}(\theta_{n+1}-\theta_n)^2 + gr\cos\theta_n\right] \\
&= a\sum_n \left[\frac{1}{2c^2}(\partial_t\phi_n)^2 - \frac{1}{2}\left(\frac{\phi_{n+1}-\phi_n}{a}\right)^2 + \lambda\cos\frac{\phi_n}{\Phi}\right] \\
&\to \int dx \left[\frac{1}{2c^2}(\partial_t\phi(x))^2 - \frac{1}{2}(\partial_x\phi(x))^2 + \lambda\cos\frac{\phi(x)}{\Phi}\right] \\
S &= \int dt dx \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + \lambda\cos\frac{\phi(x)}{\Phi}\right]
\end{aligned}$$



A. Equation of motion

Auxiliary conditions: $\phi(t_i, \boldsymbol{x}) = \phi_i(\boldsymbol{x})$, $\phi(t_f, \boldsymbol{x}) = \phi_f(\boldsymbol{x})$

Action:

$$S[\phi(\cdot)] = \int_V \underbrace{dt d^3x}_{\frac{1}{c}d^4x} L(\phi, \partial\phi)$$

Variation:

$$\phi(x) \to \phi(x) + \delta\phi(x), \qquad \delta\phi(t_i, \boldsymbol{x}) = \delta\phi(t_f, \boldsymbol{x}) = 0$$

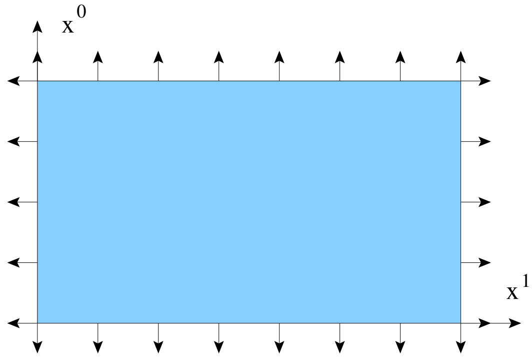
$$\begin{aligned}
\delta S &= \int_V dt d^3x \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\delta\partial_\mu\phi\right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dt d^3x \left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\partial_\mu\delta\phi\right) + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_V dt d^3x \left[\frac{\partial L(\phi,\partial\phi)}{\partial\phi}\delta\phi + \partial_\mu\left(\delta\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\delta\phi\right] + \mathcal{O}\left(\delta^2\phi\right) \\
&= \int_{\partial V} ds_\mu \delta\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi} + \int_V dt d^3x \delta\phi\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) + \mathcal{O}\left(\delta^2\phi\right)
\end{aligned}$$

$\nearrow$ $\qquad\qquad\qquad \nwarrow$

$\int_V dx\partial_\mu j^\mu = \int_{\partial V} ds_\mu j^\mu$ $\qquad\qquad$ 0 (local Lagrangian)

Euler-Lagrange equation:

$$\frac{\partial L(\phi,\partial\phi)}{\partial\phi_a} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi_a} = 0.$$



B. Wave equations for a scalar particle

Schrödinger equation:

$$i\hbar\partial_t\psi(t,\boldsymbol{x}) = \left[-\frac{\hbar^2}{2m}\Delta + U(\boldsymbol{x})\right]\psi(t,\boldsymbol{x}), \quad L = \psi^*\left[i\hbar\partial_t + \frac{\hbar^2}{2m}\Delta - U(\boldsymbol{x})\right]\psi$$

Relativistic generalisation: $\hat{p}_\mu = (\frac{E}{c}, -\boldsymbol{p}) = -\frac{\hbar}{i}\partial_\mu$, $\hat{p}^\mu = (\frac{E}{c}, \boldsymbol{p}) = i\hbar\partial^\mu$,

Klein-Gordon equation: $p^\mu = mc\dot{x}^\mu$, $p^2 = m^2c^2$:

$$\begin{aligned} 0 &= (\hat{p}^2 - m^2c^2)\phi = -\hbar^2\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi, \\ 0 &= \left(\Box + \frac{1}{\lambda_C^2}\right)\phi, \quad \lambda_C = \frac{\hbar}{mc} \quad \text{(Compton wavelength)} \end{aligned}$$

with interaction:

$$\begin{aligned} \text{real} \quad L &= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\lambda_C^2}\phi^2 - U(\phi) \\ \text{complex} \quad L &= \partial\phi^*\partial\phi - \frac{1}{\lambda_C^2}\phi^*\phi - V(\phi^*\phi) \end{aligned}$$

Free particle: $U(\phi) = \frac{m^2c^2}{2\hbar^2}\phi^2$, $V(\phi^*\phi) = \frac{m^2c^2}{\hbar^2}\phi^*\phi$, $\phi(x) = e^{-\frac{i}{\hbar}p_\mu x^\mu}$,

$$\begin{aligned} 0 &= \left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)e^{-\frac{i}{\hbar}p_\mu x^\mu} \\ 0 &= p^2 - \frac{m^2c^2}{\hbar^2} \\ \phi(x) &= e^{\mp ip_\mu x^\mu}, \quad p_0 = \omega_p = \sqrt{\frac{m^2c^2}{\hbar^2} + \boldsymbol{p}^2} \end{aligned}$$

C. Electrodynamics

Degrees of freedom: charge, $x_a^\mu(s)$, $a = 1, \ldots, n$, EM field, $A^\mu(x)$

Gauge invariance: $A_\mu \to A_\mu + \partial_\mu\alpha$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\mu\partial_\nu\alpha - \partial_\nu A_\mu - \partial_\nu\partial_\mu\alpha = F_{\mu\nu}$

EM field:

Lagrangian:

1. $L_A = \mathcal{O}\left((\partial_0 A_\mu)^2\right)$

2. Lorentz invariance
3. gauge invariance

$$L_A = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

Full action:

$$\begin{aligned} S &= -c\sum_n m_n \int ds_n - \frac{e}{c}\int dt d^3x j^\mu(x) A_\mu(x) - \frac{1}{16\pi}\int dt d^3x F_{\mu\nu}(x) F^{\mu\nu}(x) \\ &= -c\sum_n m_n \int ds_n - \frac{e}{c^2}\int d^4x j^\mu(x) A_\mu(x) - \frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x), \end{aligned}$$

Electric current:

$$\begin{aligned} j^\mu(x) &= c\sum_a \int ds \delta(x - x_a(s)) \dot{x}^\mu \\ &= c\sum_a \int ds \delta(\boldsymbol{x} - \boldsymbol{x}_a(s)) \delta(x^0 - x^0_a(s)) \dot{x}^\mu \\ &= c\sum_a \delta(\boldsymbol{x} - \boldsymbol{x}_a(s)) \frac{\dot{x}^\mu}{|\dot{x}^0|} \\ &= \underbrace{\sum_a \delta(\boldsymbol{x} - \boldsymbol{x}_a(s))}_{\rho(\boldsymbol{x})} \frac{dx^\mu}{dt} \\ &= (c\rho, \boldsymbol{j}) = (c\rho, \rho\boldsymbol{v}) = \rho\frac{ds}{dt}\dot{x}^\mu \end{aligned}$$

Current conservation:

$$\begin{aligned} \partial_\mu j^\mu &= \partial_t \rho + \nabla \cdot \boldsymbol{j} \\ &= \sum_a [-\boldsymbol{v}_a(t) \boldsymbol{\nabla} \delta(\boldsymbol{x} - \boldsymbol{x}_a(t)) + \boldsymbol{\nabla}\delta(\boldsymbol{x} - \boldsymbol{x}_a(t)) \boldsymbol{v}_a(t)] = 0 \end{aligned}$$

$$\begin{aligned} S &= -c\sum_n m_n \int ds_n - \frac{e}{c^2}\int d^4x j^\mu(x) A_\mu(x) - \frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) \\ &\to -c\sum_n m_n \int ds_n - \frac{e}{c^2}\int d^4x j^\mu(x) [A_\mu(x) + \partial_\mu \alpha(x)] - \frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) \\ &= -c\sum_n m_n \int ds_n - \frac{e}{c^2}\int d^4x [j^\mu(x) A_\mu(x) - \partial_\mu j^\mu(x) \alpha(x)] - \frac{1}{16\pi c}\int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x) = S \end{aligned}$$

Maxwell's equation:

$$\begin{aligned} S_A &= -\frac{e}{c^2}\int d^4x j^\mu A_\mu - \frac{1}{16\pi c}\int d^4x (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ c\frac{\delta}{\delta A_\nu}: \quad \frac{e}{c} j^\nu &= \frac{1}{4\pi}\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \frac{1}{4\pi}\partial_\mu F^{\mu\nu} \end{aligned}$$

Mechanical E.O.M.: $x(s) \to x(\tau)$ to avoid $\dot{x}^2(s) = 1$

$$
\begin{aligned}
S_{ch} &= -\sum_n \int d\tau \left[m_n c \sqrt{\dot{x}^\mu(\tau) g_{\mu\nu} \dot{x}^\nu(\tau)} + \frac{e}{c} \dot{x}_n^\mu(\tau) A_\mu(x_n(\tau)) \right] \\
\frac{\delta}{\delta x^\mu}: \quad 0 &= -\frac{e}{c} \dot{x}_n^\nu(\tau) \partial_\mu A_\nu(x_n(\tau)) - \frac{d}{d\tau} \left[-mc \frac{\dot{x}_n^\mu(\tau)}{\sqrt{\dot{x}^\mu(\tau) g_{\mu\nu} \dot{x}^\nu(\tau)}} - \frac{e}{c} A_\mu(x_n(\tau)) \right] \\
&= mc \frac{\ddot{x}_n^\mu(\tau)}{\sqrt{\dot{x}^2(\tau)}} - \frac{e}{c} \dot{x}_n^\nu(\tau) [\partial_\mu A_\nu(x_n(\tau)) - \partial_\nu A_\mu(x_n(\tau))] + mc \frac{\dot{x}_n^\mu(\tau)}{[\dot{x}^2(\tau)]^{3/2}} \ddot{x}^\mu(\tau) \dot{x}_\mu(\tau) \\
\tau \to s: \quad mc \ddot{x}_n^\mu(s) &= \frac{e}{c} F_{\mu\nu} \dot{x}_n^\nu(s)
\end{aligned}
$$

Bianchi identity : $\partial_\mu \partial_\nu A_\rho = \partial_\nu \partial_\mu A_\rho$

$$
\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0
$$

Maxwell: $(\boldsymbol{E}, \boldsymbol{H}) \to A_\mu$, the first unification in physics, $A^\mu = (\phi, \boldsymbol{A})$, $A_\mu = (\phi, -\boldsymbol{A})$

$$
\begin{aligned}
\boldsymbol{E} &= -\partial_0 \boldsymbol{A} - \nabla\phi = -\frac{1}{c} \partial_t \boldsymbol{A} - \nabla\phi, \\
\boldsymbol{H} &= \nabla \times \boldsymbol{A}.
\end{aligned}
$$

Inversion:

$$
\epsilon_{jk\ell} H_\ell = \epsilon_{jk\ell} \epsilon_{\ell mn} \nabla_m A_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \nabla_m A_n = \nabla_j A_k - \nabla_k A_j = -F_{jk}
$$

Field strength:

$$
F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}
$$

Dual field strength:

$$
\begin{aligned}
\tilde{F}_{\mu\nu} &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \\
\tilde{F}_{0j} &= -\frac{1}{2} \epsilon_{jk\ell} F^{k\ell} = \frac{1}{2} \epsilon_{jk\ell} \epsilon_{k\ell m} H_m = H_j \quad (\epsilon_{123} = -\epsilon^{123} = -1) \\
\tilde{F}_{jk} &= -\epsilon_{jk\ell} F^{\ell 0} = \epsilon_{jk\ell} E_\ell \\
\tilde{F}_{\mu\nu} &= \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix}
\end{aligned}
$$

Inhomogeneous Maxwell equations:

$$
\frac{4\pi}{c} j^0 = \nabla_j F^{j0}, \quad \to \quad 4\pi\rho = \boldsymbol{\nabla E}
$$

$$\frac{4\pi}{c}j^k = \partial_0 F^{0k} + \nabla_j F^{jk} \quad \to \quad \frac{4\pi}{c}\boldsymbol{j} = -\frac{1}{c}\partial_t \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{H}$$

Homogeneous Maxwell equations:

$$\begin{aligned} 0 &= \partial_\mu \tilde{F}^{\mu\nu} \\ 0 &= \nabla_j \tilde{F}^{j0} = \boldsymbol{\nabla H}, \\ 0 &= \partial_0 \tilde{F}^{0k} + \nabla_j \tilde{F}^{jk} = -\left(\frac{1}{c}\partial_t \boldsymbol{H} + \boldsymbol{\nabla} \times \boldsymbol{E}\right) \end{aligned}$$

D. Noether theorem

$$\boxed{\text{There is a conserved current for each continuous symmetry}}$$

Symmetry: invariance of the E.O.M. which is derived from the action

$$x^\mu \to x'^\mu, \quad \phi_a(x) \to \phi'_a(x) \quad \to \quad L(\phi, \partial\phi) \to L(\phi', \partial'\phi') + \partial'_\mu \Lambda^\mu$$

External and internal spaces:

$$\phi_a(x) : \underbrace{\mathbb{R}^4}_{\text{external space}} \to \underbrace{\mathbb{R}^n}_{\text{internal space}} .$$

Continuous symmetry: $\exists$ infinitesimal symmetry transformations

- External symmetry: $x^\mu \to x^\mu + \delta x^\mu$, e.g. Poincare group
- Internal symmetry: $\phi_a(x) \to \phi_a(x) + \delta\phi_a(x)$, e.g. $\phi(x) \to e^{i\alpha}\phi(x)$ for a complex field

Conserved current: $\partial_\mu j^\mu = 0$, conserved charge: $Q(t)$:

$$\partial_0 Q(t) = \partial_0 \int_V d^3x j^0 = -\int_V d^3x \boldsymbol{\partial} v j = -\int_{\partial V} d\boldsymbol{s} \cdot \boldsymbol{j}$$

Internal symmetry:

$$\delta x^\mu = 0, \quad \delta\phi_a(x) = \epsilon\tau_{ab}\phi_b(x).$$

Symmetry:

$$L(\phi, \partial\phi) = L(\phi + \epsilon\tau\phi, \partial\phi + \epsilon\tau\partial\phi) + \mathcal{O}\left(\epsilon^2\right).$$

New field variable: $\epsilon(x)$, $\phi(x) = \phi_{cl}(x) + \epsilon(x)\tau\phi_{cl}(x)$, $\frac{\delta S[\phi_{cl}]}{\delta\phi} = 0$

Linearized Lagrangian for $\epsilon(x)$ ($\epsilon = 0$ is a solution!):

$$\begin{aligned} L(\epsilon, \partial\epsilon) &= L(\phi_{cl} + \epsilon\tau\phi, \partial\phi_{cl} + \partial\epsilon\tau\phi + \epsilon\tau\partial\phi) \\ &= \frac{\partial L(\phi_{cl}, \partial\phi_{cl})}{\partial\phi}\epsilon\tau + \frac{\partial L(\phi_{cl}, \partial\phi_{cl})}{\partial\partial_\mu\phi}[\partial_\mu\epsilon\tau\phi + \epsilon\tau\partial_\mu\phi] + \mathcal{O}\left(\epsilon^2\right) \end{aligned}$$

Symmetry: $\frac{\partial L}{\partial \phi}\epsilon\tau\phi + \frac{\partial L}{\partial\partial_\mu\phi}\epsilon\tau\partial_\mu\phi = 0 \to \epsilon$ is a cyclic field:

$$\begin{aligned}
0 &= \frac{\partial L(\epsilon,\partial\epsilon)}{\partial\epsilon} - \partial_\mu\frac{\partial L(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon} \\
J^\mu_\epsilon &= \frac{\partial L(\epsilon,\partial\epsilon)}{\partial\partial_\mu\epsilon} = \frac{\partial L(\phi_{cl},\partial\phi_{cl})}{\partial\partial_\mu\phi}\tau\phi \\
\partial_\mu J^\mu_\epsilon &= 0
\end{aligned}$$

1. Independent conserved current for each independent direction in the symmetry group

2. Defined up to a multiplicative constant

Examples:

1. n-component real scalar field: ϕ_a, $a = 1,\cdots,n$, $G = O(n)$,

$$\begin{aligned}
L &= \frac{1}{2}(\partial\phi)^2 - V(\phi^2) \\
\delta\phi &= \epsilon^a\tau^a\phi \\
J^a_\mu &= -\partial_\mu\phi\tau^a\phi
\end{aligned}$$

2. Single complex scalar field: $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $G = U(1)$, $\phi(x) \to e^{i\alpha}\phi(x)$

$$\begin{aligned}
L &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}(\phi_1^2+\phi_2^2) - V\left(\frac{1}{2}(\phi_1^2+\phi_2^2)\right) \\
&= \partial_\mu\phi^*\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^\dagger\phi - V(\phi^\dagger\phi)
\end{aligned}$$

Field variable:

(a) $\binom{\phi}{\phi^*}$:

$$\begin{aligned}
\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} &: \begin{pmatrix}\phi\\ \phi^*\end{pmatrix} \to \begin{pmatrix}e^{i\alpha}\phi\\ e^{-i\alpha}\phi^*\end{pmatrix}, \quad \delta\begin{pmatrix}\phi\\ \phi^*\end{pmatrix} = i\alpha\begin{pmatrix}\phi\\ -\phi^*\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi\\ \phi^*\end{pmatrix}, \quad \tau = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \\
J &= -\frac{\partial L}{\partial\partial_\mu\phi}\tau\phi = -i\left(\frac{\partial L}{\partial\partial_\mu\phi}\phi - \frac{\partial L}{\partial\partial_\mu\phi^*}\phi^*\right) = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi) = i\phi^*\overleftrightarrow{\partial}_\mu\phi
\end{aligned}$$

(b) $\binom{\phi_1}{\phi_2}$:

$$\begin{aligned}
\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} &: \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}[\cos\alpha\phi_1 - \sin\alpha\phi_2 + i(\cos\alpha\phi_2 + \sin\alpha\phi_1)] \\
&\delta\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \alpha\begin{pmatrix}-\phi_2\\ \phi_1\end{pmatrix} = \alpha\tau\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \quad \tau = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \\
J &= -\frac{\partial L(\phi,\partial\phi_1)}{\partial\partial_\mu\phi}\tau\phi = -\left(-\frac{\partial L}{\partial\partial_\mu\phi_1}\phi_2 + \frac{\partial L}{\partial\partial_\mu\phi_2}\phi_1\right) = \partial_\mu\phi_1\phi_2 - \partial_\mu\phi_2\phi_1 \\
&= \frac{i}{2}[\partial_\mu(\phi_1+i\phi_2)^*(\phi_1+i\phi_2) - (\phi_1+i\phi_2)^*\partial_\mu(\phi_1+i\phi_2)] = -i(\partial_\mu\phi^*\phi - \phi^*\partial_\mu\phi)
\end{aligned}$$

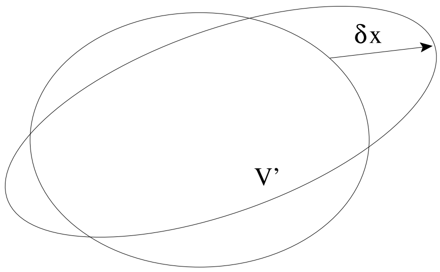
3. n-component complex scalar field: ϕ_a, $a = 1, \cdots, n$, $G = U(n)$

$$
\begin{aligned}
L &= \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi) \\
\delta\phi &= \epsilon^a\tau^a\phi, \quad \delta\phi^\dagger = \epsilon^a(\phi\tau^a)^\dagger = -\epsilon^a\phi^\dagger\tau^a \\
J^a_\mu &= -\partial_\mu\phi^\dagger\tau^a\phi + \partial_\mu\phi(\tau^a)^{\mathrm{tr}}\phi^\dagger = -\partial_\mu\phi^\dagger\tau^a\phi + \phi^\dagger\tau^a\partial_\mu\phi = \phi^\dagger\tau^a\overleftrightarrow{\partial}_\mu\phi
\end{aligned}
$$

External symmetry:

Translations:

- Action is rewritten in terms of $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$
- $x \to x' + \epsilon$, $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$, $\delta\phi(x) = -\epsilon^\mu\partial_\mu\phi(x)$, $S \to S' = S$



$$
\begin{aligned}
0 &= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{V'-V} dx L(\phi(x), \partial\phi(x)) \\
&= \int_V \delta L(\phi(x), \partial\phi(x)) + \int_{\partial V} dS_\nu \epsilon^\nu L(\phi(x), \partial\phi(x)) \\
&= -\int_V dx\epsilon^\nu\partial_\nu\phi\left(\frac{\partial L(\phi,\partial\phi)}{\partial\phi} - \partial_\mu\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi}\right) \quad \leftarrow \quad 0 \quad \text{E.O.M.} \\
&\int_{\partial V} dS_\mu\left[-\epsilon^\nu\partial_\nu\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi} + \epsilon^\mu L(\phi,\partial\phi)\right]
\end{aligned}
$$

- Translation in direction ν:

$$
0 = \int_{\partial V} dS_\mu\left[-\partial_\nu\phi\frac{\partial L(\phi,\partial\phi)}{\partial\partial_\mu\phi} + g^\mu_\nu L(\phi,\partial\phi)\right]
$$

- V is arbitrary: The energy momentum tensor

$$
T^{\mu\nu} = \frac{\partial L}{\partial\partial_\nu\phi}\partial^\mu\phi - g^{\mu\nu}L
$$

is conserved

$$
\partial_\mu T^{\mu\nu} = 0
$$

- "Charge" of the translation ϵ^ν: energy momentum

$$
P^\mu = \int d^3x T^{0\mu}
$$

- Parameterization:

$$
\begin{aligned}
T^{\mu\nu} &= \begin{pmatrix} \epsilon & c\boldsymbol{p} \\ \frac{1}{c}\boldsymbol{s} & \sigma \end{pmatrix} \\
\epsilon &= \text{energy density} \\
\boldsymbol{p} &= \text{momentum density} \\
\boldsymbol{S} &= \text{energy flux density} \\
\sigma^{jk} &= \text{momentum flux } p^k \text{ in the direction } j
\end{aligned}
$$

(c is restored).