Relativistic Quantum Mechanics

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Video:

https://drive.google.com/drive/folders/1z67BHjRoRKZasGXifJrCF1hmdmHuz9Mg?usp=sharing

Contents

I. RELATIVISTIC CLASSICAL DYNAMICS

A. Special relativity

1. **Postulates:**

   (a) Inertial reference frame: a free point particle moves with constant speed

   (b) The physical laws are equivalent in innertial reference frames

   in particular: the propagation of light takes place with the same velocity, c

2. **Invariant length:** time in length unit: $t \to ct$

   (a) Definition: $x^\mu = (ct, \boldsymbol{x}) = (x^0, \boldsymbol{x})$, $\mu = 0, 1, 2, 3$

$$s^2 = t^2 - \boldsymbol{x}^2 = \sum_{\mu\nu} x^\mu g_{\mu\nu} x^\nu$$

   (b) Metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   (c) Summary of the postulates: s^2 is the same in each inertial frame

3. **Covariant and contravariant vectors:**

   (a) *Scalar product:*

$$xy = \sum_{\mu\nu} x^\mu g_{\mu\nu} y^\nu$$

   (b) *Einstein convention:*

$$\sum_\mu (\cdots)^\mu (\cdots)_\mu \;\to\; (\cdots)^\mu (\cdots)_\mu$$
$$\sum_{\mu\nu} x^\mu g_{\mu\nu} y^\nu \;\to\; x^\mu g_{\mu\nu} y^\nu$$

   (c) *To suppress g:* two representation for each vector u:

   - covariant: u_μ
   - contravariant: u^μ

(d) *Lowering and raising indices by the metric tensor:*

$$u_\mu = g_{\mu\nu} u^\nu, \quad u^\mu = g^{\mu\nu} u_\nu \quad \to \quad xy = x^\mu y_\mu = x_\mu y^\mu$$
$$u_\mu = g_{\mu\nu} g^{\nu\rho} u_\rho \quad \to \quad g_{\mu\nu} g^{\nu\rho} = g_{\mu\cdot}^{\cdot\rho} = \delta_\mu^\rho$$

4. **Lorentz transformations:** Preserves the invariant length

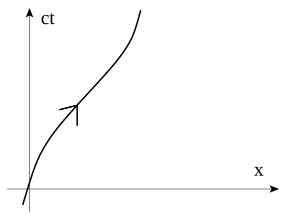
$$\begin{aligned} x^\mu \to x'^\mu &= \Lambda^{\mu}_{\cdot\,\nu}{}^{\!\cdot} x^\nu \\ x^\mu g_{\mu\nu} y^\nu &= \Lambda^{\mu}_{\cdot\,\mu'}{}^{\!\cdot} x^{\mu'} g_{\mu\nu} \Lambda^{\nu}_{\cdot\,\nu'}{}^{\!\cdot} y^{\nu'} = x^{\mu'} \Lambda^{\mu}_{\cdot\,\mu'}{}^{\!\cdot} g_{\mu\nu} \Lambda^{\nu}_{\cdot\,\nu'}{}^{\!\cdot} y^{\nu'} \\ g_{\mu\nu} &= \Lambda^{\mu'}_{\cdot\,\mu}{}^{\!\cdot} g_{\mu'\nu'} \Lambda^{\nu'}_{\cdot\,\nu}{}^{\!\cdot} \\ g &= \Lambda^{\mathrm{tr}} g \Lambda, \qquad\qquad g = g_{\mu\nu}, \quad \Lambda = \Lambda^{\nu}_{\cdot\,\mu}{}^{\!\cdot}, \quad \Lambda^{\mathrm{tr}} = \Lambda_{\mu}^{\cdot}{}^{\nu}_{\cdot} \end{aligned}$$

6 dimensions: 3 spatial rotations and 3 for boosts

5. **World line:**

(a) *Non-relativistic motion:* trajectory: $\boldsymbol{x}(t)$

(b) *Relativistic motion:* world line: $x^\mu(s) = (ct(s), \boldsymbol{x}(s))$, $\mu = 0, 1, 2, 3$.



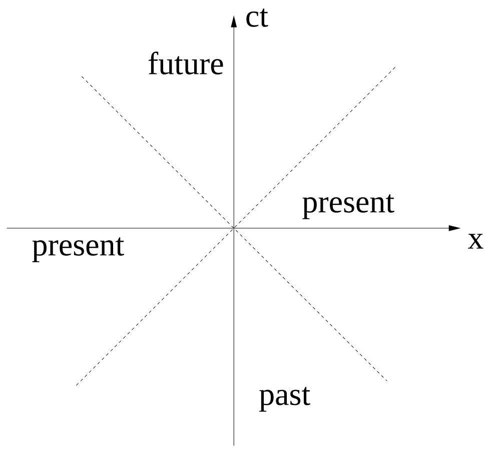
(c) *Intrinsic velocity:*

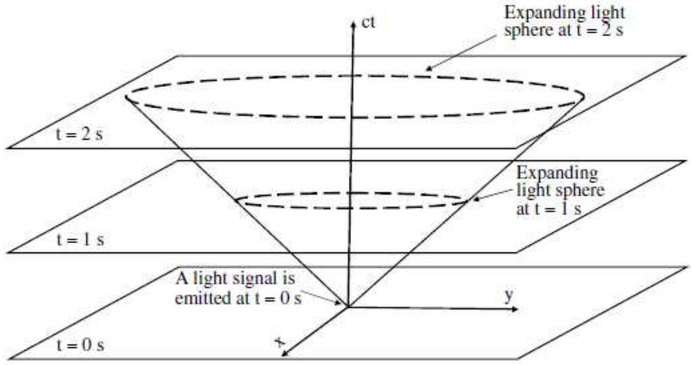
- Non-relativistic physics:$\frac{v}{c} \sim 0$, $\frac{|\boldsymbol{p}|}{mc} \ll 1$ ($\boldsymbol{p} \neq \boldsymbol{v}$)
- Relativistic physics:$\frac{v}{c} \sim 1$, $\frac{|\boldsymbol{p}|}{mc} \gg 1$
- Crossover: $|\boldsymbol{p}| = m$

B. Role of time - Anti-particles

1. **Minkowski geometry:** space-time intervals:

- time-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 > 0$
- space-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 < 0$
- light-like: $c^2(t_1 - t_2)^2 - (\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = 0$





2. **Time:**

   (a) *Similar than the coordinates, except:*

   - Has orientation
   - Remains classical in quantum mechanics
   - Not a fundamental observable (needs periodic motion, memory, irreversibility, etc.)
   - Unique metric signature

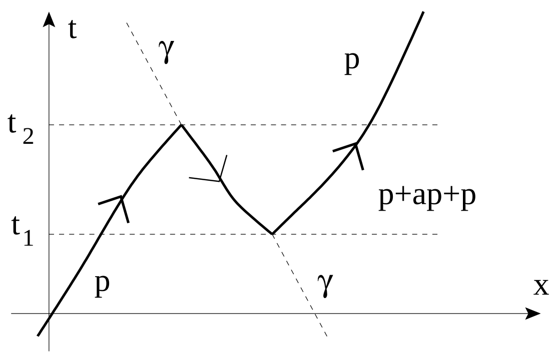
   (b) *Double role:*

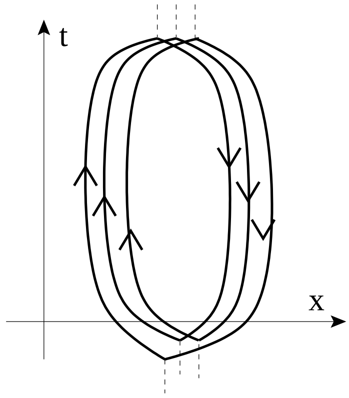
   - a basic parameter of the dynamics
   - introduces causal order

     taken over by the parameter of the world line $\Longrightarrow$anti-particles

3. **New relativistic physics:**

   - Non-relativistic trajectory $\boldsymbol{x}(t)$ $\Longrightarrow$relativistic world line $x^\mu(s) = (t, \boldsymbol{x}(t))$
   - Relativistic world line $x^\mu(s)$ $\Longrightarrow$non-relativistic trajectory $\boldsymbol{x}(t)$?
   - Non-unique t:





- Physical interpretation: creation and annihilation of particle anti particle pairs
- Energy-momentum conservation: Creation or annihilation of another particle ($e^- + e^- \leftrightarrow \gamma$)
- Causality: discontinuous velocities at the creation or annihilation
- Charge:

$$Q_p + Q_{a-p} = 0$$

- Universe of a single fermion

4. **Giving up classical E.O.M.:**

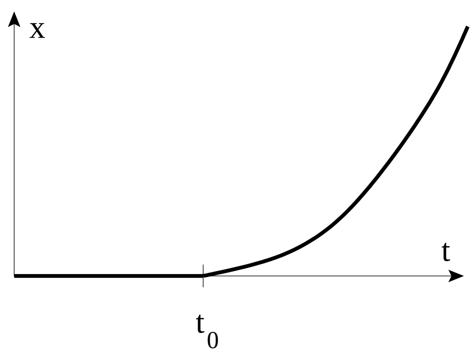
- Sufficent condition for locally unique solution:

$$\dot{x}(t) = f(x,t), \quad x(t_i) = x_i$$

is the continuity of $\partial_x f(x,t)$

- Example:

$$\begin{aligned}
\dot{x} &= g|x|^p, \quad 0 < p < 1 \\
dx x^{-p} &= g dt \quad (x \geq 0) \\
\frac{x^{1-p}}{1-p} &= g(t - t_0) \\
x &= [g(1-p)(t-t_0)]^{\frac{1}{1-p}} \\
x(t) &= \begin{cases} 0 & t < t_0, \\ [g(1-p)(t-t_0)]^{\frac{1}{1-p}} & t \geq t_0, \end{cases}
\end{aligned}$$



$\boxed{t_0 \text{ is not determined by the initial condition}}$

- Quantum mechanics: giving up classical equations of motion at a random time

(change of particle number)

C. Energy of an anti-particle

1. **Anti-particle:** Its proper time ($d\tau = ds$) goes in opposite direction as the standard clocks

2. **Time inversion:** equivalent with $E \to -E$

   - *Classical non-relativistic phyiscs:*

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q},$$

   - *Quantum mechanics:*

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle$$

3. **Ambiguity in classical mechanics:**

$$\begin{aligned} S &= -mc\int ds\sqrt{\dot{x}^\mu(s)g_{\mu\nu}\dot{x}^\mu(s)} \\ p_\mu &= -\frac{\partial L}{\partial \dot{x}^\mu} = mc\dot{x}_\mu \end{aligned}$$

   or

$$\begin{aligned} T^{\mu\nu}(x) &= \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}(x)} = mc\int ds\delta^{(4)}(x - x(s))\dot{x}^\mu(s)\dot{x}^\nu(s) \\ P^\mu &= m\int ds\delta(t - x^0(s))\dot{x}^\mu(s)\dot{x}^0(s) \\ &= m\dot{x}^\mu\frac{\dot{x}^0}{|\dot{x}^0|} \end{aligned}$$

4. **Problems with negative energy:**

   - Free charged particles with $E < 0$
   - Couple to the electronagnetic field
   - Cascading to more negative energy by emitting radiation
   - Instability, no darkness at night, the world as a fire ball

D. Space-time inversions and charge conjugation

1. **Special (inproper) Lorentz transformations:**

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : (x^0, \boldsymbol{x}) \to (x^0, -\boldsymbol{x}), \quad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (x^0, \boldsymbol{x}) \to (-x^0, \boldsymbol{x})$$

2. **Disconnected components of the Lorentz group:**

$$g_{\mu\nu} = \Lambda^{\mu'}_{\cdot\;\mu}{}^{\cdot} g_{\mu'\nu'} \Lambda^{\nu'}_{\cdot\;\nu}{}^{\cdot}, \quad g = \Lambda^{\mathrm{tr}} g \Lambda$$

   (a) *Determinant:*

$$\begin{aligned} \det AB &= \det A \det B \quad \to \quad (\det\Lambda)^2 \det g = \det g \quad \to \quad (\det\Lambda)^2 = 1 \quad \to \quad \det\Lambda = \pm 1 \\ L_\pm &= \{\Lambda | \det\Lambda = \pm 1\}, \quad PL_\pm = L_\mp, \quad TL_\pm = L_\mp \end{aligned}$$

   (b) ${\Lambda^0}_0$*:*

$$\begin{aligned} g_{00} &= 1 = ({\Lambda^0}_0)^2 - \sum_j ({\Lambda^0}_j)^2 \quad \to \quad |{\Lambda^0}_0| \ge \sqrt{1 + \sum_j ({\Lambda^0}_j)^2} \ge 1 \\ L^\uparrow &= \{\Lambda | {\Lambda^0}_0 > 1\}, \quad L^\downarrow = \{\Lambda | {\Lambda^0}_0 < 1\}, \quad TL^\uparrow = L^\downarrow \end{aligned}$$

   (c) *Four disconnected components:*

$$L = L_+^\uparrow \cup L_+^\downarrow \cup L_-^\uparrow \cup L_-^\downarrow$$

$$\uparrow$$

proper Lorentz group

$$L_+^\downarrow = PTL_+^\uparrow, \quad L_-^\uparrow = PL_+^\uparrow, \quad L_-^\downarrow = TL_+^\uparrow$$

3. **Charge conjugation:** $C : s \to -s$ or $t \to -t$ or $E \to -E$

4. **PCT theorem:** Local relativistic field theories are PCT invariant

5. **Parities:**

   - *Space inversion:*

$$P : f(x, \dot{x}) \to f(Px, P\dot{x}) = \pi_f f(x, \dot{x})$$

   - *Time inversion:*

$$T : f(x, \dot{x}) \to f(Tx, T\dot{x}) = \tau_f f(x, \dot{x})$$

   - *Charge conjugation:*

$$C : f(x, \dot{x}) \to f(x, -, \dot{x}) = \gamma_f f(x, \dot{x})$$

   - *Classical physics:*

$$f(z) = \underbrace{\frac{f(z) + f(-z)}{2}}_{+} + \underbrace{\frac{f(z) - f(-z)}{2}}_{-}, \quad P^2 = T^2 = C^2 = \mathbb{1}, \quad \pi, \tau, \gamma \in \{1, -1\}$$

     Examples: $\pi_t = -\pi_{\boldsymbol{x}} = 1$, $-\tau_t = \tau_{\boldsymbol{x}} = 1$ and $\gamma_{x^\mu} t = -\gamma_{\dot{x}^\mu} = 1$

   - *Quantum mechanics:* Wave functions can be multi-valued, $\pi, \tau, \gamma \in U(1)$

II. SCALAR PARTICLE

The relativistic second order generalization of the Schrödinger equation

$$i\hbar\partial_t\psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{x},t)$$

A. Heuristic derivation of the Klein-Gordon equation

1. **Heuristic derivation:**

$$\begin{aligned} p_\mu &= \left(\frac{E}{c}, -\boldsymbol{p}\right) = i\hbar\partial_\mu = i\hbar(\partial_0, \boldsymbol{\nabla}), \quad p^\mu = \left(\frac{E}{c}, \boldsymbol{p}\right) = i\hbar\partial^\mu = i\hbar(\partial_0, -\boldsymbol{\nabla}) \\ p^2 &= m^2c^2\dot{x}^2 = m^2c^2 \end{aligned}$$

$$\boxed{\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi(x) = 0} \quad \leftarrow \quad \text{Klein} - \text{Gordon equation} \quad \Box = \partial_0^2 - \boldsymbol{\nabla}^2$$

$\nearrow$

Compton wavelength $\frac{\hbar}{mc} = \lambda_C$

2. **External elecromagnetic field:**

$$\begin{aligned} p^\mu &\rightarrow p^\mu - \frac{e}{c}A^\mu, \quad \partial_\mu \rightarrow \partial_\mu + i\frac{e}{\hbar c}A_\mu = D_\mu \\ D_\mu\phi &= \partial_\mu\phi + i\frac{e}{\hbar c}A_\mu\phi, \quad D_\mu\phi^* = \partial_\mu\phi - i\frac{e}{\hbar c}A_\mu\phi = (D_\mu\phi)^* \\ D\phi\chi &= (D\phi)\chi + \phi D\chi \\ 0 &= \left(D_\mu D^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi \end{aligned}$$

3. **Klein-Gordon equation as an Euler-Lagrange equation:** ($\hbar = c = 1$)

$$\begin{aligned} L &= (D_\mu\phi)^* D^\mu\phi - m^2\phi^*\phi \\ S[\phi] &= \int dtd^3xL(\phi, \phi^*, D_\mu\phi, D_\mu\phi^*) \end{aligned}$$

4. **Conserved current:**

- *$U(1)$ symmetry:* $\phi(x) \rightarrow e^{i\theta}\phi(x)$, $\phi^*(x) \rightarrow e^{-i\theta}\phi^*(x)$
- *Noether-current:*

$$j_\mu = \frac{i}{2m}(\phi^* D_\mu\phi - (D_\mu\phi)^*\phi) = \frac{i}{2m}\phi^*\overleftrightarrow{\partial_\mu}\phi - \frac{e}{mc}\phi^*\phi A_\mu, \quad f\overleftrightarrow{\partial_\mu}g = g\partial_\mu g - \partial_\mu g f$$

5. **Plane-wave solutions:**

$$\begin{aligned}\phi(x) &= e^{\mp ip_\mu x^\mu}, \quad (\Box + m^2)\phi(x) = (m^2 - p^2)e^{\mp ip_\mu x^\mu} = 0, \quad p_0 = \omega_p, \quad \omega_p = \sqrt{m^2 + \boldsymbol{p}^2}\\ j^\mu &= \pm\frac{p^\mu}{m}\end{aligned}$$

6. **Sign problem:**

   (a) *States with negative energy:* dispersion relation for $E^2 \Longrightarrow E$ is non-definite

   (b) *No probabilistic interpretation:* $p^0 \sim \partial_0$ is non-definite

   (c) *Common source:* Both from the wrong sign of p^0 in the plane wave, the existence of anti-particles

B. First order formalism for scalar particles

1. **Second order E.O.M.:**

- *Initial conditons:* $\phi(t_i, \boldsymbol{x})$ and $\partial_0\phi(t_i, \boldsymbol{x}) \Longrightarrow$two particles ? particle and anti-particle ?
- *Separation of the particle and the anti-particle modes:*

$$\phi(t, \boldsymbol{x}) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\phi(\omega\boldsymbol{x}) \quad \leftarrow \quad \text{positive and negative energy modes}$$

  - Projection onto the positive and negative energy subspaces:

$$\begin{aligned}\delta(t - t') &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\\ \Lambda_\pm(t - t') &= \pm\int_0^{\pm\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t'\mp i\epsilon)} = -\frac{i}{2\pi(t - t' \mp i\epsilon)}\\ \phi^\pm(t, \boldsymbol{x}) &= \Lambda_\pm\phi(t, \boldsymbol{x}) = \int dt'\Lambda_\pm(t - t')\phi(t', \boldsymbol{x})\end{aligned}$$

  - Non-local in time: one needs time to figure out (the sign of) the frequency
  - Possible locally only if we know the solutions of the E.O.M.
  - External field may mix the frequencies in the solution and create particle-anti particle pairs

$$\begin{aligned}[fg](\omega) &= \int dt e^{i\omega t}f(r)g(t)\\ &= \int dt\frac{d\omega' d\omega''}{(2\pi)^2}e^{i\omega t - i(\omega'+\omega'')t}f(\omega')g(\omega'')\\ &= \int\frac{d\omega' d\omega''}{2\pi}\delta(\omega - \omega' - \omega'')f(\omega')g(\omega'')\\ &= \int\frac{d\omega'}{2\pi}f(\omega')g(\omega - \omega')\end{aligned}$$

  (EM field: e^-e^+ pairs, (static!) gravitational field: Hawking radiation)

2. **First order E.O.M.:**

- *Double the number of variables*

$$\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi + \frac{i}{m}\partial_0\phi \\ \phi - \frac{i}{m}\partial_0\phi \end{pmatrix}$$

Slow motion: particle $\sim \chi_1$, anti-particle $\sim \chi_2$

- *Half the order of the equation:* $(\Box + m^2)\phi = (\partial_0^2 - \boldsymbol{\nabla}^2 + m^2)\phi = 0 \Longrightarrow \partial_0^2\phi = (\boldsymbol{\nabla}^2 - m^2)\phi$

$$i\partial_0 \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i\partial_0\phi - \frac{1}{m}(\boldsymbol{\nabla}^2 - m^2)\phi \\ i\partial_0\phi + \frac{1}{m}(\boldsymbol{\nabla}^2 - m^2)\phi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i\partial_0\phi + (m - \frac{1}{m}\boldsymbol{\nabla}^2)\phi \\ i\partial_0\phi + (\frac{1}{m}\boldsymbol{\nabla}^2 - m)\phi \end{pmatrix}$$

$\chi_+ + \chi_- = \phi$, $\chi_+ - \chi_- = \frac{i}{m}\partial_0\phi$:

$$\begin{aligned} i\partial_0 \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} m(\chi_+ - \chi_-) + (m - \frac{1}{m}\boldsymbol{\nabla}^2)(\chi_+ + \chi_-) \\ m(\chi_+ - \chi_-) + (\frac{1}{m}\boldsymbol{\nabla}^2 - m)(\chi_+ + \chi_-) \end{pmatrix} \\ &= \left[-\frac{\boldsymbol{\nabla}^2}{2m} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} \end{aligned}$$

$$\boxed{i\partial_0\chi = H\chi, \quad H = -\frac{\boldsymbol{\nabla}^2}{2m}(\sigma_3 + i\sigma_2) + m\sigma_3}$$

3. **Hermiticity:** $H^\dagger \neq H$ stability of excited states ??

- *Klein-Gordon conjugation:*

$$\begin{aligned} |\psi\rangle &\to \langle\bar{\psi}| = \langle\psi|g \\ \langle\psi|\phi\rangle &\to \langle\bar{\psi}|\phi\rangle = \langle\psi|g|\phi\rangle \\ \langle\phi|A^\dagger|\psi\rangle = \langle\psi|A|\phi\rangle^* &\to \langle\bar{\phi}|\bar{A}|\psi\rangle = \langle\bar{\psi}|A|\phi\rangle^*, \quad \langle\phi|g\bar{A}|\psi\rangle = \langle\psi|gA|\phi\rangle^* \\ g\bar{A} &= (gA)^\dagger \quad \to \quad \bar{A} = g^{-1}A^\dagger g^\dagger \\ g &= \sigma_3, \quad \bar{\chi} = \chi^\dagger\sigma_3, \quad A \to \sigma_3 A^\dagger \sigma_3 = \bar{A}, \quad \bar{H} = H \end{aligned}$$

$$\boxed{\langle\bar{\chi}|A|\chi'\rangle = \int d^3x \bar{\chi}(\boldsymbol{x}) A \chi'(\boldsymbol{x})}$$

- *Non-definite scalar product:* "metric tensor" g, $\langle\psi|$ "covariant", $|\psi\rangle$ "contravariant"

$$\langle\bar{\chi}|\chi'\rangle = \int d^3x \bar{\chi}(t,\boldsymbol{x})\chi'(t,\boldsymbol{x}) = \int d^3x \chi^\dagger(t,\boldsymbol{x})\sigma_3\chi'(t,\boldsymbol{x})$$

- *Non-definite charge density:*

$$\langle\bar{\chi}|\chi\rangle = \langle\chi_+|\chi_+\rangle - \langle\chi_-|\chi_-\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^3x \left[\left(\phi^* - \frac{i}{m}\partial_0\phi^*\right)\left(\phi + \frac{i}{m}\partial_0\phi^*\right) - \left(\phi^* + \frac{i}{m}\partial_0\phi^*\right)\left(\phi - \frac{i}{m}\partial_0\phi^*\right)\right] \\
&= \frac{i}{2m}\int d^3x[\phi^*(\boldsymbol{x})\partial_0\phi'(\boldsymbol{x}) - \partial_0\phi^*(\boldsymbol{x})\phi'(\boldsymbol{x})] = \frac{i}{2m}\int d^3x\phi^*(\boldsymbol{x})\overleftrightarrow{\partial_0}\phi'(\boldsymbol{x}) \\
&= \int d^3x j^0(t,\boldsymbol{x})
\end{aligned}
$$

4. **Lagrangian:**

$$
\begin{aligned}
L &= \frac{i}{2}\bar\chi\partial_0\chi - \frac{i}{2}\partial_0\bar\chi\chi - \bar\chi H\chi \\
0 &= \frac{\partial L}{\partial\bar\chi} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\bar\chi} = -H\chi + \frac{i}{2}\partial_0\chi + \frac{i}{2}\partial_0\chi \quad \to \quad i\partial_0\chi = H\chi \\
0 &= \frac{\partial L}{\partial\chi} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\chi} = -\bar\chi H - \frac{i}{2}\partial_0\chi - \frac{i}{2}\partial_0\chi \quad \to \quad i\partial_0\bar\chi = -\bar\chi H
\end{aligned}
$$

C. Free particle

1. **Plane wave:**

- *Definition:*

$$\chi(x) = u_{\boldsymbol{p}}e^{-ipx}, \quad \chi(x) = v_{\boldsymbol{p}}e^{ipx}$$

- *Mass-shell condition:* $p^2 = m^2 \Longrightarrow p^0 = \omega_p$

$$\omega_p u_{\boldsymbol{p}} = H_{\boldsymbol{p}}u_{\boldsymbol{p}}, \quad -\omega_p v_{\boldsymbol{p}} = H_{\boldsymbol{p}}v_{\boldsymbol{p}}$$

- *Hamiltonian:*

$$H = -\frac{\boldsymbol{\nabla}^2}{2m}(\sigma_3 + i\sigma_2) + m\sigma_3 \quad \to \quad H_{\boldsymbol{p}} = \frac{\boldsymbol{p}^2}{2m}(\sigma_3 + i\sigma_2) + m\sigma_3$$

- *Eigenvectors:*

$$
\begin{aligned}
u_{\boldsymbol{p}} &= \frac{1}{2\sqrt{m\omega_p}}\begin{pmatrix} m+\omega_p \\ m-\omega_p \end{pmatrix} = u_{-\boldsymbol{p}}, \quad v_{\boldsymbol{p}} = \frac{1}{2\sqrt{m\omega_p}}\begin{pmatrix} m-\omega_p \\ m+\omega_p \end{pmatrix} = v_{-\boldsymbol{p}} \\
\bar u_{\boldsymbol{p}}v_{\boldsymbol{p}} &= \bar v_{\boldsymbol{p}}u_{\boldsymbol{p}} = 0, \quad \bar u_{\boldsymbol{p}}u_{\boldsymbol{p}} = -\bar v_{\boldsymbol{p}}v_{\boldsymbol{p}} = 1
\end{aligned}
$$

2. **General solution:**

$$
\begin{aligned}
\chi(x) &= \int_{\boldsymbol{p}}[a_{\boldsymbol{p}}u_{\boldsymbol{p}}e^{-ipx} + b^*_{\boldsymbol{p}}v_{\boldsymbol{p}}e^{ipx}]_{|p^0=\omega_p}, \quad \int_{\boldsymbol{p}} = \int\frac{d^3p}{(2\pi)^3} \\
\phi(x) &= \chi_+(x) + \chi_-(x) \\
&= \frac{1}{2\sqrt{m\omega_p}}\int_{\boldsymbol{p}}[a_{\boldsymbol{p}}(m+\omega_p)e^{-ipx} + a_{\boldsymbol{p}}(m-\omega_p)e^{-ipx} + b^*_{\boldsymbol{p}}(m-\omega_p)e^{ipx} + b^*_{\boldsymbol{p}}(m+\omega_p)e^{ipx}]_{|p^0=\omega_p} \\
&= \int_{\boldsymbol{p}}\sqrt{\frac{m}{\omega_p}}[a_{\boldsymbol{p}}e^{-ipx} + b^*_{\boldsymbol{p}}e^{ipx}]_{|p^0=\omega_p}
\end{aligned}
$$

- *Interpretation:*

  particle: wave function,

  anti-particle: complex conjugate of the wave function ($t \to -t$)

- *Inverse Fourier transformation:*

$$
\begin{aligned}
a_{\boldsymbol{p}} &= \frac{i}{2\sqrt{m\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{x}} e^{ipx}\overleftrightarrow{\partial_0}\phi(x), \quad f\overleftrightarrow{\partial}g = f\partial g - \partial f g \\
&= \frac{i}{2\sqrt{m\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{x}} e^{ipx}\overleftrightarrow{\partial_0}\int_{\boldsymbol{q}}\sqrt{\frac{m}{\omega_q}}[a_{\boldsymbol{q}}e^{-iqx} + b^*_{\boldsymbol{q}}e^{iqx}] \\
&= \frac{i}{2\sqrt{\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{xq}}\frac{1}{\sqrt{\omega_q}}e^{ipx}\overleftrightarrow{\partial_0}[a_{\boldsymbol{q}}e^{-iqx} + b^*_{\boldsymbol{q}}e^{iqx}] \\
&= \frac{1}{2\sqrt{\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{xq}}\frac{1}{\sqrt{\omega_q}}e^{ipx}[\omega_q(a_{\boldsymbol{q}}e^{-iqx} - b^*_{\boldsymbol{q}}e^{iqx}) + \omega_p(a_{\boldsymbol{q}}e^{-iqx} + b^*_{\boldsymbol{q}}e^{iqx})] \\
&= \frac{1}{\sqrt{\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{q}}\frac{1}{\sqrt{\omega_q}}e^{i\omega_p x^0}\omega_q a_{\boldsymbol{q}}e^{-i\omega_q x^0}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q}) \\
&= e^{i\omega_p x^0}a_{\boldsymbol{p}}e^{-i\omega_p x^0} \\
a^*_{\boldsymbol{p}} &= -\frac{i}{2\sqrt{m\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{x}} e^{-ipx}\overleftrightarrow{\partial_0}\phi^*(x), \\
b^*_{\boldsymbol{p}} &= -\frac{i}{2\sqrt{m\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{x}} e^{-ipx}\overleftrightarrow{\partial_0}\phi(x), \\
b_{\boldsymbol{p}} &= \frac{i}{2\sqrt{m\omega_{\boldsymbol{p}}}}\int_{\boldsymbol{x}} e^{ipx}\overleftrightarrow{\partial_0}\phi^*(x).
\end{aligned}
$$

- *Real wave function:* anti-particle = particle for neutral particles

3. **Non-locality in space:** "square root" of the Klein-Gordon equation

$$
\begin{aligned}
0 &= (\Box + m^2)\phi = \partial_0^2\phi - (\boldsymbol{\nabla}^2 - m^2)\phi \\
\partial_0^2\phi &= (\boldsymbol{\nabla}^2 - m^2)\phi \\
i\partial_0\phi^{(\pm)}(x) &= \pm\sqrt{m^2 - \Delta}\phi^{(\pm)}(x) \\
i\partial_0\phi^{(\pm)}(t,\boldsymbol{p}) &= \pm\sqrt{m^2 + \boldsymbol{p}^2}\phi^{(\pm)}(t,\boldsymbol{p})
\end{aligned}
$$

   Difficult mathematical features

4. **Particle and anti-particle projectors:**

- *Momentum space:*

$$
\begin{aligned}
u_{\boldsymbol{p}} &= \frac{1}{2\sqrt{m\omega_p}}\begin{pmatrix} m+\omega_p \\ m-\omega_p \end{pmatrix}, \quad v_{\boldsymbol{p}} = \frac{1}{2\sqrt{m\omega_p}}\begin{pmatrix} m-\omega_p \\ m+\omega_p \end{pmatrix}, \quad \omega_p = \sqrt{m^2 + \boldsymbol{p}^2} \\
\Lambda_{+,\boldsymbol{p}} &= u_{\boldsymbol{p}} \otimes \bar{u}_{\boldsymbol{p}} = \frac{1}{4m\omega_p}\begin{pmatrix} m+\omega_p \\ m-\omega_p \end{pmatrix}(m+\omega_p, -m+\omega_p) = \frac{1}{4m\omega_{\boldsymbol{p}}}\begin{pmatrix} (m+\omega_{\boldsymbol{p}})^2 & \boldsymbol{p}^2 \\ -\boldsymbol{p}^2 & -(m-\omega_{\boldsymbol{p}})^2 \end{pmatrix} \\
\Lambda_{-,\boldsymbol{p}} &= -v_{\boldsymbol{p}} \otimes \bar{v}_{\boldsymbol{p}} = -\Lambda_{+,\boldsymbol{p}}(\omega_p \to -\omega_p)
\end{aligned}
$$

$$\Lambda_{\pm,\boldsymbol{p}} = \pm\frac{1}{4m\omega_{\boldsymbol{p}}}\begin{pmatrix}(m\pm\omega_{\boldsymbol{p}})^2 & \boldsymbol{p}^2\\ -\boldsymbol{p}^2 & -(m\mp\omega_{\boldsymbol{p}})^2\end{pmatrix} = \begin{cases}\pm\frac{|\boldsymbol{p}|}{4m}(\sigma_3+i\sigma_2) & |\boldsymbol{p}|\gg m,\\ \frac{1}{2}(\mathbb{1}\pm\sigma_3) & |\boldsymbol{p}|\ll m,\end{cases}$$

- *Coordinate space:* ϵ: UV cutoff, the limit $\epsilon\to 0$ is sought at the end of the calculation (distributions)

$$\Lambda_{\pm}(\boldsymbol{x},\boldsymbol{y}) = \int_{\boldsymbol{p}} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})-|\boldsymbol{p}|\epsilon}\Lambda_{\pm,\boldsymbol{p}},$$

5. **Charge conjugation:** particle $\longleftrightarrow$ anti-particle

$$C_{\boldsymbol{p}} = v_{\boldsymbol{p}}\otimes\bar{u}_{\boldsymbol{p}} - u_{\boldsymbol{p}}\otimes\bar{v}_{\boldsymbol{p}} = \frac{1}{4m\omega_{\boldsymbol{p}}}\left[\begin{pmatrix}m^2-\omega_{\boldsymbol{p}}^2 & -(m-\omega_{\boldsymbol{p}})^2\\ (m+\omega_{\boldsymbol{p}})^2 & \omega_{\boldsymbol{p}}^2-m^2\end{pmatrix} - \begin{pmatrix}m^2-\omega_{\boldsymbol{p}}^2 & -(m+\omega_{\boldsymbol{p}})^2\\ (m-\omega_{\boldsymbol{p}})^2 & \omega_{\boldsymbol{p}}^2-m^2\end{pmatrix}\right] = \sigma_1,$$

6. **Indefinite norm:**

$$\begin{aligned}\chi(x) &= \int_{\boldsymbol{p}}[a_{\boldsymbol{p}}u_{\boldsymbol{p}}e^{-ipx} + b^*_{\boldsymbol{p}}v_{\boldsymbol{p}}e^{ipx}]_{|p^0=\omega_p}\\ \langle\bar{\chi}|\chi\rangle &= \int d^3x\bar{\chi}(x)\chi(x) = \int_{\boldsymbol{xpq}}[a^*_{\boldsymbol{p}}\bar{u}_{\boldsymbol{p}}e^{ipx} + b_{\boldsymbol{p}}\bar{v}_{\boldsymbol{p}}e^{-ipx}][a_{\boldsymbol{q}}u_{\boldsymbol{q}}e^{-iqx} + b^*_{\boldsymbol{q}}v_{\boldsymbol{q}}e^{iqx}]\\ &= \int_{\boldsymbol{p}}(a^*_{\boldsymbol{p}}a_{\boldsymbol{p}} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p}})\end{aligned}$$

7. **Expectation values:**

$$\langle A\rangle = \frac{\langle\bar{\chi}|A|\chi\rangle}{|\langle\bar{\chi}|\chi\rangle|} \quad \text{or} \quad \langle A\rangle = \frac{\langle\bar{\chi}|A|\chi\rangle}{\langle\bar{\chi}|\chi\rangle}$$

- *Energy-momentum:* $p^\mu = (H, -i\boldsymbol{\nabla})$, is

$$\begin{aligned}\langle\bar{\chi}|p^\mu|\chi\rangle &= \int d^3x\bar{\chi}(t,\boldsymbol{x})(H,-i\boldsymbol{\nabla})\chi(t,\boldsymbol{x})\\ &= \int d^3x\int_{\boldsymbol{pq}}[a^*_{\boldsymbol{p}}\bar{u}_{\boldsymbol{p}}e^{ipx} + b_{\boldsymbol{p}}\bar{v}_{\boldsymbol{p}}e^{-ipx}](\omega_q,\boldsymbol{q})[a_{\boldsymbol{q}}u_{\boldsymbol{q}}e^{-iqx} - b^*_{\boldsymbol{q}}v_{\boldsymbol{q}}e^{iqx}]\\ &= \int_{\boldsymbol{p}}(\omega_p,\boldsymbol{p})[a^*_{\boldsymbol{p}}a_{\boldsymbol{p}} + b_{\boldsymbol{p}}b^*_{\boldsymbol{p}}].\end{aligned}$$

- *Noether current:* $j^\mu = \frac{i}{2m}\phi^*\overleftrightarrow{\partial}^\mu\phi = (j^0, -\boldsymbol{j})$,

$$\begin{aligned}\begin{pmatrix}\chi_+\\ \chi_-\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\phi + \frac{i}{m}\partial_0\phi\\ \phi - \frac{i}{m}\partial_0\phi\end{pmatrix} \quad\to\quad \chi_+ + \chi_- = \phi, \quad \chi_+ - \chi_- = \frac{i}{m}\partial_0\phi\\ j_0 &= \frac{1}{2}(\chi_+ + \chi_-)^*(\chi_+ - \chi_-) + \frac{1}{2}(\chi_+ - \chi_-)^*(\chi_+ + \chi_-) = \chi^*_+\chi_+ - \chi^*_-\chi_- = \bar{\chi}(x)\chi(x)\\ \boldsymbol{j} &= -\frac{i}{2m}(\chi_+ + \chi_-)^*\overleftrightarrow{\boldsymbol{\nabla}}(\chi_+ + \chi_-) = -\frac{i}{2m}\bar{\chi}(x)\overleftrightarrow{\boldsymbol{\nabla}}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\chi(x)\\ &= -\frac{i}{2m}\bar{\chi}(x)\overleftrightarrow{\boldsymbol{\nabla}}(\sigma_3 + i\sigma_2)\chi(x)\end{aligned}$$

$$\langle\bar{\chi}|\chi\rangle = \int_{\boldsymbol{p}}(a^*_{\boldsymbol{p}}a_{\boldsymbol{p}} - b_{\boldsymbol{p}}b^*_{\boldsymbol{p}})$$

$$\begin{aligned}
\langle\bar\chi|\boldsymbol{j}|\chi\rangle &= -\frac{i}{2m}\int d^3x\int_{\boldsymbol{pq}}[a^*_{\boldsymbol p}\bar u_{\boldsymbol p}e^{ipx}+b_{\boldsymbol p}\bar v_{\boldsymbol p}e^{-ipx}]\overleftrightarrow{\boldsymbol{\nabla}}(\sigma_3+i\sigma_2)[a_{\boldsymbol q}u_{\boldsymbol q}e^{-iqx}+b^*_{\boldsymbol q}v_{\boldsymbol q}e^{iqx}]\\
&= \Re\frac{1}{m}\int d^3x\int_{\boldsymbol{pq}}\boldsymbol{q}[a^*_{\boldsymbol p}\bar u_{\boldsymbol p}e^{ipx}+b_{\boldsymbol p}\bar v_{\boldsymbol p}e^{-ipx}](\sigma_3+i\sigma_2)[a_{\boldsymbol q}u_{\boldsymbol q}e^{-iqx}-b^*_{\boldsymbol q}v_{\boldsymbol q}e^{iqx}]\\
&= \int_{\boldsymbol p}\frac{\boldsymbol p}{m}[a^*_{\boldsymbol p}a_{\boldsymbol p}\bar u_{\boldsymbol p}(\sigma_3+i\sigma_2)u_{\boldsymbol p}-b_{\boldsymbol p}b^*_{\boldsymbol p}\bar v_{\boldsymbol p}(\sigma_3+i\sigma_2)v_{\boldsymbol p}].
\end{aligned}$$

$$\begin{aligned}
\bar u_{\boldsymbol p}(\sigma_3+i\sigma_2)u_{\boldsymbol p} &= \frac{1}{4m\omega_{\boldsymbol p}}(m+\omega_{\boldsymbol p},-m+\omega_{\boldsymbol p})\begin{pmatrix}1&1\\-1&-1\end{pmatrix}\begin{pmatrix}m+\omega_{\boldsymbol p}\\m-\omega_{\boldsymbol p}\end{pmatrix}\\
&= \frac{(m+\omega_{\boldsymbol p},-m+\omega_{\boldsymbol p})}{2\omega_{\boldsymbol p}}\begin{pmatrix}1\\-1\end{pmatrix}=\frac{m}{\omega_{\boldsymbol p}}=\bar v_{\boldsymbol p}(\sigma_3+i\sigma_2)v_{\boldsymbol p}
\end{aligned}$$

$$\langle\chi|j^\mu|\chi\rangle=\int_{\boldsymbol p}\left(1,\frac{\boldsymbol p}{\omega_p}\right)[a^*_{\boldsymbol p}a_{\boldsymbol p}-b_{\boldsymbol p}b^*_{\boldsymbol p}].$$

- *Sign problem:* Either choice of the expectation value yields non-definite energy or charge density in the presence of particle and anti-particle

D. Localization

1. **Problems:**
   (a) *Spread of the wave packet:*
      i. Non-relativistic wave packet of width Δx
      ii. The speed v_{spr} of the spread: $v_{spr}\sim\frac{\Delta p}{m}\sim\frac{\hbar}{m\Delta x}$ (Heisenberg)

   (b) *Pair creation:*
      i. One-dimensional particle is an interval $\Delta x \Longrightarrow p_n=\frac{2\pi n\hbar}{\Delta x}$
      ii. Strong localization:

$$\begin{aligned}
\Delta_n E &= c(\sqrt{m^2c^2+p^2_{n+1}}-\sqrt{m^2c^2+p^2_n})\\
&= mc^2\left(\sqrt{1+\frac{4\pi^2(n+1)^2\hbar}{\Delta^2xm^2c^2}}-\sqrt{1+\frac{4\pi^2n^2\hbar}{\Delta^2xm^2c^2}}\right)>3mc^2
\end{aligned}$$

      iii. Particle-anti particle radiation for $\Delta x\approx\frac{\hbar}{mc}=\lambda_C$

2. **Decoupling of the particle and anti-particle modes:**
   (a) *Coupled $\to$ decoupled basis:*

$$S_{\boldsymbol p} = \frac{m+\omega_p-\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}}$$

$$
\begin{aligned}
S_{\boldsymbol{p}} u_{\boldsymbol{p}} &= \frac{m+\omega_p-\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \frac{1}{2\sqrt{m\omega_p}} \begin{pmatrix} m+\omega_p \\ m-\omega_p \end{pmatrix} \\
&= \frac{m+\omega_p}{4m\omega_p} \begin{pmatrix} m+\omega_p \\ m-\omega_p \end{pmatrix} - \frac{m-\omega_p}{4m\omega_p} \begin{pmatrix} m-\omega_p \\ m+\omega_p \end{pmatrix} \\
&= \frac{1}{4m\omega_p} \begin{pmatrix} (m+\omega_p)^2-(m-\omega_p)^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = w_+ \\
S_{\boldsymbol{p}} v_{\boldsymbol{p}} &= \frac{m+\omega_p-\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \frac{1}{2\sqrt{m\omega_p}} \begin{pmatrix} m-\omega_p \\ m+\omega_p \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = w_-.
\end{aligned}
$$

(b) *Unitary transformation:*

$$
\bar{S}_{\boldsymbol{p}} = \sigma_3 S^\dagger \sigma_3 = \frac{m+\omega_p+\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} = S_{\boldsymbol{p}}^{-1},
$$

(c) *Hamiltonian in the Feshbach-Villars (decoupled) basis:*

$$
\begin{aligned}
H_{FV} &= S_{\boldsymbol{p}} H_{\boldsymbol{p}} S_{\boldsymbol{p}}^{-1} \\
&= \frac{m+\omega_p-\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \left[\frac{\boldsymbol{p}^2}{2m}(\sigma_3+i\sigma_2)+m\sigma_3\right] \frac{m+\omega_p+\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \\
&= \left[\frac{\boldsymbol{p}^2}{2m}(\sigma_3+i\sigma_2)+m\sigma_3\right] \left(\frac{m+\omega_p+\sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}}\right)^2 \\
&= \left[\frac{\boldsymbol{p}^2}{2m}(\sigma_3+i\sigma_2)+m\sigma_3\right] \left(\frac{2m^2+\boldsymbol{p}^2}{2m\omega_p} - \sigma_1 \frac{\boldsymbol{p}^2}{2m\omega_p}\right) \\
&= \sigma_3 \left[\left(\frac{\boldsymbol{p}^2}{2m}+m\right)\frac{2m^2+\boldsymbol{p}^2}{2m\omega_p} - \frac{\boldsymbol{p}^2}{2m}\frac{\boldsymbol{p}^2}{2m\omega_p}\right] \\
&\quad + i\sigma_2 \left[\frac{\boldsymbol{p}^2}{2m}\frac{2m^2+\boldsymbol{p}^2}{2m\omega_p} - \left(\frac{\boldsymbol{p}^2}{2m}+m\right)\frac{\boldsymbol{p}^2}{2m\omega_p}\right] \\
&= \sigma_3 \frac{(\boldsymbol{p}^2+2m^2)(2m^2+\boldsymbol{p}^2)-\boldsymbol{p}^4}{4m^2\omega_p} + i\sigma_2 \frac{\boldsymbol{p}^2(2m^2+\boldsymbol{p}^2)-(\boldsymbol{p}^2+2m^2)\boldsymbol{p}^2}{4m^2\omega_p} \\
&= \sigma_3 \omega_p \\
&= \omega_{\boldsymbol{p}}(w_+ \otimes \bar{w}_+ + w_- \otimes \bar{w}_-) \\
H_{\boldsymbol{p}} &= S_{\boldsymbol{p}}^{-1} H_{FV} S_{\boldsymbol{p}}
\end{aligned}
$$

3. **Momentum-dependence:** $\chi_\pm(x) = (1 \pm \frac{\omega_p}{m})e^{-ipx}$ $\Longrightarrow$momentum-dependent basis transformation

4. **Full Hilbert space:**

$$
\begin{aligned}
\langle \boldsymbol{p}|S_{aa'}|\boldsymbol{p}'\rangle &= (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{p}')S_{\boldsymbol{p}aa'}, \\
\langle \boldsymbol{x}|S_{aa'}|\boldsymbol{x}'\rangle &= \int_{\boldsymbol{p}\boldsymbol{p}'} \langle \boldsymbol{x}|\boldsymbol{p}\rangle\langle \boldsymbol{p}|S_{aa'}|\boldsymbol{p}'\rangle\langle \boldsymbol{p}'|\boldsymbol{x}'\rangle = \int_{\boldsymbol{p}} S_{\boldsymbol{p}aa'} e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')},
\end{aligned}
$$

5. **Coordinate operator:** c.f. Newton, Wigner

$$
\hat{\boldsymbol{x}}_{KG} = i\boldsymbol{\nabla}_{p_{KG}}
$$

$$
\begin{aligned}
\hat{\boldsymbol{x}}_{FV} &= S\hat{\boldsymbol{x}}_{KG}S^{-1} = Si\boldsymbol{\nabla}_p S^{-1} = i(\boldsymbol{\nabla}_p + \boldsymbol{a}_p) \\
\boldsymbol{a}_p &= S(\boldsymbol{\nabla}_p S^{-1}) \\
&= \frac{m+\omega_p - \sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}}\boldsymbol{\nabla}_p \frac{m+\omega_p + \sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \\
&= \frac{m+\omega_p - \sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}}\boldsymbol{\nabla}\omega_p \frac{2\omega_p(1-\sigma_1) - m - \omega_p - \sigma_1(m-\omega_p)}{4\sqrt{m}\omega_p^{3/2}} \\
&= \boldsymbol{\nabla}\omega_p \frac{m+\omega_p - \sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}} \frac{\omega_p - m - \sigma_1(m+\omega_p)}{4\sqrt{m}\omega_p^{3/2}} \\
&= \boldsymbol{\nabla}\omega_p \frac{\omega_p^2 - m^2 + m^2 - \omega_p^2 - \sigma_1[(m+\omega_p)^2 - (m-\omega_p)^2]}{8m\omega_p^2} \\
&= \boldsymbol{c}_{\boldsymbol{p}}\sigma_1, \quad \boldsymbol{c}_{\boldsymbol{p}} = -\frac{\boldsymbol{\nabla}_p\omega_p}{2\omega_p}
\end{aligned}
$$

$$
\boxed{\hat{\boldsymbol{x}}_{FV} = \hat{\boldsymbol{x}}_{KG} + i\boldsymbol{c}_{\boldsymbol{p}}\sigma_1}
$$

Hermitian but spin dependent

6. **$\hat{\boldsymbol{x}}_{KG}$ mixes the particle and anti-article states:**

   creation and annihilation pairs by the electromagnetic field

7. **Coordinate and momentum eigenvalues:**

   - *Spectrum:*

$$
\hat{\boldsymbol{x}}_{KG}|\boldsymbol{x},a\rangle = \boldsymbol{x}|\boldsymbol{x},a\rangle, \quad \hat{\boldsymbol{p}}_{KG}|\boldsymbol{p},a\rangle = \boldsymbol{p}|\boldsymbol{p},a\rangle, \quad \hat{\boldsymbol{x}}_{FV}|\dot{\boldsymbol{x}},\dot{a}\rangle = \dot{\boldsymbol{x}}|\dot{\boldsymbol{x}},\dot{a}\rangle, \quad \hat{\boldsymbol{p}}_{FV}|\dot{\boldsymbol{p}},\dot{a}\rangle = \dot{\boldsymbol{p}}|\dot{\boldsymbol{p}},\dot{a}\rangle
$$

   - *Basis transformation:*

$$
\begin{aligned}
\langle \boldsymbol{x},a|\boldsymbol{p},b\rangle &= a\delta_{ab}e^{i\boldsymbol{x}\boldsymbol{p}}, \quad \langle \dot{\boldsymbol{x}},\dot{a}|\dot{\boldsymbol{p}},\dot{b}\rangle = \dot{a}\delta_{\dot{a}\dot{b}}e^{i\dot{\boldsymbol{x}}\dot{\boldsymbol{p}}} \\
\langle \dot{\boldsymbol{p}},\dot{a}|\boldsymbol{p},a\rangle &= (2\pi)^3\delta(\dot{\boldsymbol{p}}-\boldsymbol{p})S_{\boldsymbol{p}\dot{a}a}, \quad \langle \boldsymbol{p},a|\dot{\boldsymbol{p}},\dot{a}\rangle = (2\pi)^3\delta(\dot{\boldsymbol{p}}-\boldsymbol{p})\bar{S}_{\boldsymbol{p}a\dot{a}}
\end{aligned}
$$

   - *Overlap:*

$$
\begin{aligned}
\mathbb{1} &= \sum_a \int d^3x|\boldsymbol{x},a\rangle a\langle \boldsymbol{x},a| = \sum_a \int_{\boldsymbol{p}} |\boldsymbol{p},a\rangle a\langle \boldsymbol{p},a| = \sum_{\dot{a}} \int_{\dot{\boldsymbol{p}}} |\dot{\boldsymbol{p}},\dot{a}\rangle \dot{a}\langle \dot{\boldsymbol{p}},\dot{a}| \\
|\dot{\boldsymbol{x}},\dot{a}\rangle &= \sum_a \int_{\boldsymbol{p}\dot{\boldsymbol{p}}} d^3x|\boldsymbol{x},a\rangle a\langle \boldsymbol{x},a|\boldsymbol{p},a\rangle a\langle \boldsymbol{p},a|\dot{\boldsymbol{p}},\dot{a}\rangle \dot{a}\langle \dot{\boldsymbol{p}},\dot{a}|\dot{\boldsymbol{x}},\dot{a}\rangle \\
&= \sum_a \int_{\boldsymbol{p}} d^3x|\boldsymbol{x},a\rangle \bar{S}_{\boldsymbol{p}a\dot{a}}\dot{a}e^{i(\boldsymbol{x}-\dot{\boldsymbol{x}})\boldsymbol{p}} \\
\langle \dot{\boldsymbol{x}},\dot{a}|\boldsymbol{x},a\rangle &= \int_{\boldsymbol{p}} \dot{a}S_{\boldsymbol{p}\dot{a}a}e^{i(\dot{\boldsymbol{x}}-\boldsymbol{x})\boldsymbol{p}} \\
&= \sigma_3 \int_{\boldsymbol{p}} e^{i(\dot{\boldsymbol{x}}-\boldsymbol{x})\boldsymbol{p}} \frac{m+\omega_p - \sigma_1(m-\omega_p)}{2\sqrt{m\omega_p}}
\end{aligned}
$$

$$= \int_{\boldsymbol{p}} \frac{e^{i(\dot{\boldsymbol{x}}-\boldsymbol{x})\boldsymbol{p}}}{2\sqrt{m\omega_p}} \begin{pmatrix} \omega_p + m & \omega_p - m \\ m - \omega_p & -\omega_p - m \end{pmatrix}$$

$$\langle \boldsymbol{x}, a | \dot{\boldsymbol{x}}, \dot{a} \rangle = \int_{\boldsymbol{p}} \bar{S}_{\boldsymbol{p}a\dot{a}} \dot{a} e^{i(\boldsymbol{x}-\dot{\boldsymbol{x}})\boldsymbol{p}}$$

8. **Wave function:**

$$\chi_{KG}(t, \boldsymbol{x}) = {}_{KG}\langle \boldsymbol{x} | \chi(t) \rangle_{KG} = \int_{\boldsymbol{p}} [a_{\boldsymbol{p}} u_{\boldsymbol{p}} e^{-ipx} + b^*_{\boldsymbol{p}} v_{\boldsymbol{p}} e^{ipx}]_{|p^0=\omega_p}$$

$$\chi_{FV}(t, \dot{\boldsymbol{x}}) = {}_{FV}\langle \dot{\boldsymbol{x}} | \chi(t) \rangle_{FV} = \langle \dot{\boldsymbol{x}} | S | \chi \rangle = \int_{\boldsymbol{p}} [a_{\boldsymbol{p}} e^{-ip\dot{x}} w_+ + b^*_{\boldsymbol{p}} e^{ip\dot{x}} w_-]$$

E. The birth of relativistic quantum field theory

1. **Occupation number representation:** $n(\boldsymbol{p})$ particles with momentum $\boldsymbol{p}$

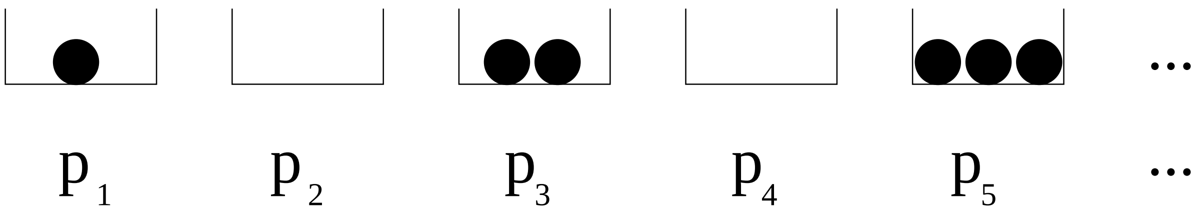
2. **Energy-momentum spectrum of free particles:** $(\hbar = c = 1)$

$$E(\boldsymbol{p}) = \begin{cases} \frac{\boldsymbol{p}^2}{2m} & \text{non} - \text{relativistic} \\ \sqrt{m^2 + \boldsymbol{p}^2} & \text{relativistic} \end{cases}$$

$$E = \sum_{j=1}^{N} E(\boldsymbol{p}_j) = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) E(\boldsymbol{p}), \qquad \boldsymbol{P} = \sum_{j=1}^{N} \boldsymbol{p}_j = \sum_{\boldsymbol{p}} n(\boldsymbol{p}) \boldsymbol{p}$$

3. **Harmonic oscillator for each momentum:** equidistant spectrum $\Longrightarrow$harmonic oscillator

- *Example:* $n(\boldsymbol{p}_1) = 1$, $n(\boldsymbol{p}_2) = 0$, $n(\boldsymbol{p}_3) = 2$, $n(\boldsymbol{p}_4) = 0$, $n(\boldsymbol{p}_5) = 3$, ...,



- *Canonical operators:*

$$[\hat{X}_{\dot{\boldsymbol{p}},\dot{a}}, \hat{P}_{\dot{\boldsymbol{p}}',\dot{a}'}] = i\delta_{\dot{a},\dot{a}'} \delta(\dot{\boldsymbol{p}} - \dot{\boldsymbol{p}}')$$

- *Creation and annihilation operators:*

$$\hat{a}_{\dot{\boldsymbol{p}}} = \frac{\omega_{\dot{\boldsymbol{p}}} \hat{X}_{\dot{\boldsymbol{p}},+} + i\hat{P}_{\dot{\boldsymbol{p}},+}}{\sqrt{2\omega_{\dot{\boldsymbol{p}}}}}, \quad \hat{b}_{\dot{\boldsymbol{p}}} = \frac{\omega_{\dot{\boldsymbol{p}}} \hat{X}_{\dot{\boldsymbol{p}},-} + i\hat{P}_{\dot{\boldsymbol{p}},-}}{\sqrt{2\omega_{\dot{\boldsymbol{p}}}}}.$$

- *Fock space:* harmonic oscillator of $\boldsymbol{p} \Longrightarrow \mathcal{H}_{\boldsymbol{p}}$, $\mathcal{H}_F = \otimes \prod_{\boldsymbol{p}} \mathcal{H}_{\boldsymbol{p}}$, span by $\{|n_{\dot{\boldsymbol{p}}}, \bar{n}_{\dot{\boldsymbol{p}}}\rangle\}$

  No need of (anti)symmetrization

- *Hamiltonian:* $E_n = \hbar\omega(n+\frac{1}{2})$

$$\hat{H} = \sum_{\dot{a}} \int_{\dot{\boldsymbol{p}}} \left(\frac{1}{2}\hat{P}^2_{\dot{\boldsymbol{p}},\dot{a}} + \frac{\omega^2_{\dot{\boldsymbol{p}}}}{2}\hat{X}^2_{\dot{\boldsymbol{p}},\dot{a}} \right) = \hbar \int_{\dot{\boldsymbol{p}}} \omega_{\dot{\boldsymbol{p}}} (\hat{a}^\dagger_{\dot{\boldsymbol{p}}}\hat{a}_{\dot{\boldsymbol{p}}} + \hat{b}^\dagger_{\dot{\boldsymbol{p}}}\hat{b}_{\dot{\boldsymbol{p}}} + 1)$$

4. **Relativistic quantum field:**

$$\hat{\phi}(\dot{x}) = \int_{\dot{\boldsymbol{p}}} [\hat{a}_{\dot{\boldsymbol{p}}} e^{-i\dot{p}\dot{x}} + \hat{b}^\dagger_{\dot{\boldsymbol{p}}} e^{i\dot{p}\dot{x}}],$$

- *Second quantization:* wave function $\Longrightarrow$operator
- *Field operator:* removes a particle ($\hat{a}_{\dot{\boldsymbol{p}}}$) and creates and anti-particle ($\hat{b}^\dagger_{\dot{\boldsymbol{p}}}$)
- *New space cordinate variable* $\boldsymbol{x} \to \dot{\boldsymbol{x}}$
- *Field as a coordinate variable:* $\boldsymbol{x} \to \Delta x \boldsymbol{n}$ spatial lattice (regularization of UV and IR divergences)

  UV regulator: minimal distance Δx

  IR regulator: maximal distance $L = N\Delta x$, $n_j = 1, \ldots, N$

  – QM: $\hat{x}_j$, $j = 1, \ldots, D$

  – QFT: $\hat{\phi}(\boldsymbol{x}) \to \hat{\phi}(\Delta x \boldsymbol{n}) = \hat{\phi}_{\boldsymbol{n}}$

  QFT is like QM in $D = N^3$ dimensions

- *Internal and external spaces:*

  – $\phi(x) : R^4 \to C$, R^4 : external space (what?) C: internal space (when, where?)

  – External space: first quantization

  – Internal space: second quantization

5. **Scalar product:**

- *No single particle wave function*
- *New scalar product is inferred from the harmonic oscillators*
- *New scalar product $\Longrightarrow$new physical content*

6. **No problematic -1 factor:**

- *Positive definite scalar product for harmonic oscillators*
- *Positive definite energy for harmonic oscillators*

7. **What is QFT?**

$\boxed{\text{Representation of exernal symmetries in the Fock space in terms of elementary particles}}$

F. Spread of the wave packet

1. **Non-relativistic wave-packet:** $E_{\boldsymbol{p}} = \frac{\boldsymbol{p}^2}{2m}$

(a) *Coordinate:*

$$
\begin{aligned}
\psi(t,\boldsymbol{x}) &= \int_{\boldsymbol{p}} \psi_{\boldsymbol{p}} e^{-itE_p + i\boldsymbol{x}\boldsymbol{p}} \\
\langle \boldsymbol{x} \rangle &= \int_{\boldsymbol{pq}} d^3x \boldsymbol{x} \psi^*_{\boldsymbol{p}} \psi_{\boldsymbol{q}} e^{-it(E_q - E_p) + i\boldsymbol{x}(\boldsymbol{q}-\boldsymbol{p})} \\
&= \int_{\boldsymbol{pq}} d^3x \psi^*_{\boldsymbol{p}} \psi_{\boldsymbol{q}} e^{it(E_p - E_q)} (-i\boldsymbol{\nabla}_q) e^{i\boldsymbol{x}(\boldsymbol{q}-\boldsymbol{p})} \\
&= \int_{\boldsymbol{pq}} d^3x e^{i\boldsymbol{x}(\boldsymbol{q}-\boldsymbol{p})} \psi^*_{\boldsymbol{p}} e^{itE_p} i\boldsymbol{\nabla}_q (\psi_{\boldsymbol{q}} e^{-itE_q}) \\
&= \int_{\boldsymbol{pq}} (2\pi)^3 \delta(\boldsymbol{q}-\boldsymbol{p}) \psi^*_{\boldsymbol{p}} e^{itE_p} i\boldsymbol{\nabla}_q (\psi_{\boldsymbol{q}} e^{-itE_q}) \\
&= \int_{\boldsymbol{p}} \psi^*_{\boldsymbol{p}} (i\boldsymbol{\nabla}_p + t\boldsymbol{\nabla} E_p) \psi_{\boldsymbol{p}} \\
&= \langle \psi | \boldsymbol{x} | \psi \rangle_{|t=0} + t \langle \psi | \boldsymbol{v}_{gr} | \psi \rangle_{|t=0}
\end{aligned}
$$

$\nearrow$ $\qquad\qquad$ $\nwarrow$

$\hat{\boldsymbol{x}} = i\boldsymbol{\nabla}_p$ $\qquad$ group velocity: $\boldsymbol{v}_{gr}(\boldsymbol{p}) = \boldsymbol{\nabla} E_p = \frac{\boldsymbol{p}}{m}$

(b) *Coordinate square:*

$$
\begin{aligned}
\langle \boldsymbol{x}^2 \rangle &= -\int_{\boldsymbol{pq}} d^3x \psi^*_{\boldsymbol{p}} \psi_{\boldsymbol{q}} e^{it(E_p - E_q)} \boldsymbol{\nabla}_q^2 e^{-i\boldsymbol{x}(\boldsymbol{p}-\boldsymbol{q})} \\
&= -\int_{\boldsymbol{pq}} d^3x e^{-i\boldsymbol{x}(\boldsymbol{p}-\boldsymbol{q})} \psi^*_{\boldsymbol{p}} \psi_{\boldsymbol{q}} e^{itE_p} \boldsymbol{\nabla}_q^2 (\psi_{\boldsymbol{q}} e^{-itE_q}) \\
&= -\int_{\boldsymbol{p}} [\psi^*_{\boldsymbol{p}} e^{itE_p} \boldsymbol{\nabla}^2 (\psi_{\boldsymbol{p}} e^{-itE_p})] \\
\boldsymbol{\nabla}(\psi_{\boldsymbol{p}} e^{-itE_p}) &= (\boldsymbol{\nabla}\psi_{\boldsymbol{p}} - it\boldsymbol{\nabla}E_p \psi_{\boldsymbol{p}}) e^{-itE_p} \\
\boldsymbol{\nabla}^2(\psi_{\boldsymbol{p}} e^{-itE_p}) &= [\boldsymbol{\nabla}^2\psi_{\boldsymbol{p}} - it\boldsymbol{\nabla}^2 E_p \psi_{\boldsymbol{p}} - it\boldsymbol{\nabla}E_p \boldsymbol{\nabla}\psi_{\boldsymbol{p}} - it\boldsymbol{\nabla}E_p (\boldsymbol{\nabla}\psi_{\boldsymbol{p}} - it\boldsymbol{\nabla}E_p \psi_{\boldsymbol{p}})] e^{-itE_p} \\
\langle \boldsymbol{x}^2 \rangle &= -\int_{\boldsymbol{p}} \psi^*_{\boldsymbol{p}} [\boldsymbol{\nabla}^2 - it\boldsymbol{\nabla}E_p \boldsymbol{\nabla} - it\boldsymbol{\nabla}E_p \boldsymbol{\nabla} - it\boldsymbol{\nabla}^2 E_p - t^2 (\boldsymbol{\nabla}E_p)^2] \psi_{\boldsymbol{p}} \\
&= \langle \psi | \boldsymbol{x}^2 + t\boldsymbol{v}_{gr}\boldsymbol{x} + \boldsymbol{x}t\boldsymbol{v}_{gr} + t^2 \boldsymbol{v}_{gr}^2 | \psi \rangle_{|t=0} \\
&= \langle \psi | (\boldsymbol{x} + t\boldsymbol{v}_{gr})^2 | \psi \rangle_{|t=0}
\end{aligned}
$$

(c) *Second moment of the coordinate:* $\sigma_x^2(t) = \langle x^2 \rangle - \langle x \rangle^2$

$$
\begin{aligned}
\sigma_x^2(t) &= \langle \psi | (\boldsymbol{x} + t\boldsymbol{v}_{gr})^2 | \psi \rangle_{|t=0} - (\langle \psi | \boldsymbol{x} | \psi \rangle_{|t=0} + t \langle \psi | \boldsymbol{v}_{gr} | \psi \rangle_{|t=0})^2 \\
&= \sigma_x^2(0) + t[\langle \psi | \boldsymbol{x}\boldsymbol{v}_{gr} + \boldsymbol{v}_{gr}\boldsymbol{x} | \psi \rangle_{|t=0} - 2\langle \psi | \boldsymbol{x} | \psi \rangle_0 \langle \psi | \boldsymbol{v}_{gr} | \psi \rangle_{|t=0}] + t^2 \sigma_v^2(0) \\
\sigma_v^2(0) &= \langle \psi | \boldsymbol{v}_{gr}^2 | \psi \rangle_{|t=0} - \langle \psi | \boldsymbol{v}_{gr} | \psi \rangle^2_{|t=0}
\end{aligned}
$$

- $\sigma_x^2(0)$ can be arbitrarily small
- $\partial_t^2 \sigma_x^2(t)$ can be arbitrarily large

2. **Relativistic wave-packet:**

(a) *Decoupled basis:*

- Coordinate:

$$\begin{aligned}
\langle\chi|\boldsymbol{x}_{FV}|\chi\rangle &= i\int_{\dot{\boldsymbol{p}}\dot{\boldsymbol{q}}} d^3\dot{x}[a^*_{\dot{\boldsymbol{p}}}\bar{w}_+e^{it\omega_{\dot{p}}-i\dot{\boldsymbol{x}}\dot{\boldsymbol{p}}}+b_{\dot{\boldsymbol{p}}}\bar{w}_-e^{-it\omega_{\dot{p}}+i\dot{\boldsymbol{x}}\dot{\boldsymbol{p}}}] \\
&\quad [-a_{\dot{\boldsymbol{q}}}w_+e^{-it\omega_{\dot{q}}}\boldsymbol{\nabla}_{\dot{q}}e^{i\dot{\boldsymbol{x}}\dot{\boldsymbol{q}}}+b^*_{\dot{\boldsymbol{q}}}w_-e^{it\omega_{\dot{p}}}\boldsymbol{\nabla}_{\dot{q}}e^{-i\dot{\boldsymbol{x}}\dot{\boldsymbol{q}}}] \\
&= \int_{\dot{\boldsymbol{q}}}[a^*_{\dot{\boldsymbol{q}}}(i\boldsymbol{\nabla}+t\boldsymbol{\nabla}\omega_{\dot{q}})a_{\dot{\boldsymbol{q}}}+b_{\dot{\boldsymbol{q}}}(i\boldsymbol{\nabla}-t\boldsymbol{\nabla}\omega_{\dot{q}})b^*_{\dot{\boldsymbol{q}}}] \\
&= \int_{\dot{\boldsymbol{q}}}[a^*_{\dot{\boldsymbol{q}}}(i\boldsymbol{\nabla}+t\boldsymbol{\nabla}\omega_{\dot{q}})a_{\dot{\boldsymbol{q}}}-b^*_{\dot{\boldsymbol{q}}}(i\boldsymbol{\nabla}+t\boldsymbol{\nabla}\omega_{\dot{q}})b_{\dot{\boldsymbol{q}}}] \\
&= \langle a|\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr}|a\rangle_{|t=0}-\langle b|\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr}|b\rangle_{|t=0} \\
\boldsymbol{v}_{gr} &= c\boldsymbol{\nabla}_p\sqrt{m^2c^2+\boldsymbol{p}^2}=c\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}} \quad \leftarrow \quad \text{bounded by } c
\end{aligned}$$

- Coordinate square:

$$\begin{aligned}
\langle\chi|\boldsymbol{x}^2_{FV}|\chi\rangle &= -\int_{\dot{\boldsymbol{p}}\dot{\boldsymbol{q}}} d^3\dot{x}[a^*_{\dot{\boldsymbol{p}}}\bar{w}_+e^{it\omega_{\dot{p}}-i\dot{\boldsymbol{x}}\dot{\boldsymbol{p}}}+b_{\dot{\boldsymbol{p}}}\bar{w}_-e^{-it\omega_{\dot{p}}+i\dot{\boldsymbol{x}}\dot{\boldsymbol{p}}}] \\
&\quad [w_+\boldsymbol{\nabla}^2_{\dot{q}}e^{i\dot{\boldsymbol{q}}\dot{\boldsymbol{x}}}a_{\dot{\boldsymbol{q}}}e^{-it\omega_{\dot{q}}}+w_-\boldsymbol{\nabla}^2_{\dot{q}}|e^{-i\dot{\boldsymbol{q}}\dot{\boldsymbol{x}}}b^*_{\dot{\boldsymbol{q}}}e^{it\omega_{\dot{q}}}] \\
&= -\int_{\dot{\boldsymbol{q}}}[a^*_{\dot{\boldsymbol{q}}}[\boldsymbol{\nabla}^2-2it\boldsymbol{\nabla}\omega_{\dot{q}}\boldsymbol{\nabla}-it\boldsymbol{\nabla}^2\omega_{\dot{q}}-t^2(\boldsymbol{\nabla}\omega_{\dot{p}})^2]a_{\dot{\boldsymbol{q}}} \\
&\quad -b_{\dot{\boldsymbol{q}}}[\boldsymbol{\nabla}^2+2it\boldsymbol{\nabla}\omega_{\dot{p}}\boldsymbol{\nabla}+it\boldsymbol{\nabla}^2\omega_{\dot{q}}-t^2(\boldsymbol{\nabla}\omega_{\dot{q}})^2]b^*_{\dot{\boldsymbol{q}}}] \\
&= \langle a|(\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr})^2|a\rangle_{|t=0}-\langle b|(\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr})^2|b\rangle_{|t=0}
\end{aligned}$$

- Second moment:

$$\begin{aligned}
\sigma^2_{\dot{x}}(t) &= \langle a|(\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr})^2|a\rangle_0-\langle b|(\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr})^2|b\rangle_0-(\langle a|\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr}|a\rangle_0-\langle b|\boldsymbol{x}_{FV}+t\boldsymbol{v}_{gr}|b\rangle_0)^2 \\
&= \sigma^2_{\dot{x}}(0)+ta+t^2b
\end{aligned}$$

with

$$\begin{aligned}
a &= (\langle a|\boldsymbol{x}_{FV}\boldsymbol{v}_{gr}+\boldsymbol{v}_{gr}\boldsymbol{x}_{FV}|a\rangle_0-2\langle a|\boldsymbol{x}_{FV}|a\rangle_0\langle a|\boldsymbol{v}_{gr}|a\rangle_{|t=0} \\
&\quad -\langle b|\boldsymbol{x}_{FV}\boldsymbol{v}_{gr}+\boldsymbol{v}_{gr}\boldsymbol{x}_{FV}|b\rangle_0-2\langle b|\boldsymbol{x}_{FV}|b\rangle_0\langle b|\boldsymbol{v}_{gr}|b\rangle_{|t=0} \\
&\quad +2\langle a|\boldsymbol{x}_{FV}|a\rangle_0\langle b|\boldsymbol{v}_{gr}|b\rangle_0+2\langle a|\boldsymbol{v}_{gr}|a\rangle_0\langle b|\boldsymbol{x}_{FV}|b\rangle_{|t=0} \\
b &= \langle a|\boldsymbol{v}^2_{gr}|a\rangle_{|t=0}-\langle a|\boldsymbol{v}_{gr}|a\rangle^2_{|t=0}-\langle b|\boldsymbol{v}^2_{gr}|b\rangle_{|t=0}-\langle b|\boldsymbol{v}_{gr}|b\rangle^2_{|t=0}+2\langle a|\boldsymbol{v}_{gr}|a\rangle_{|t=0}\langle b|\boldsymbol{v}_{gr}|b\rangle_{|t=0}.
\end{aligned}$$

- Arbitrarily narrow wave-packet
- Particles and anti-particles remain decoupled
- Velocity of spread bounded by c

(b) *Coupled basis:*

- Coordinate: $\hat{\boldsymbol{x}}_{KG}=\hat{\boldsymbol{x}}_{FV}-i\boldsymbol{c}_{\boldsymbol{p}}\sigma_1$, $\boldsymbol{c}_{\boldsymbol{p}}=-\frac{\boldsymbol{\nabla}_p\omega_p}{2\omega_p}$

$$\langle\chi|\boldsymbol{x}_{KG}|\chi\rangle = i\int_{\boldsymbol{pq}} d^3\dot{x}[a^*_{\boldsymbol{p}}\bar{w}_+e^{it\omega_p-i\boldsymbol{x}_{FV}\boldsymbol{p}}+b_{\boldsymbol{p}}\bar{w}_-e^{-it\omega_p+i\boldsymbol{x}_{FV}\boldsymbol{p}}]$$

$$
\begin{aligned}
&\times[a_{\boldsymbol q}e^{-it\omega_q}(-\boldsymbol\nabla_q-\boldsymbol c_q\sigma_1)w_+e^{i\boldsymbol q\boldsymbol x_{FV}}+b^*_{\boldsymbol q}e^{it\omega_q}(\boldsymbol\nabla_q-\boldsymbol c_q\sigma_1)w_-e^{-i\boldsymbol q\boldsymbol x_{FV}}]\\
=\ &\langle\chi|\boldsymbol x_{FV}|\chi\rangle+i\int_{\boldsymbol q}\boldsymbol c_q[b_{-\boldsymbol q}a_{\boldsymbol q}e^{-2it\omega_q}-a^*_{-\boldsymbol q}b^*_{\boldsymbol q}e^{2it\omega_q}]\\
=\ &\langle a|\boldsymbol x+t\boldsymbol v_{gr}|a\rangle_{|t=0}-\langle b|\boldsymbol x+t\boldsymbol v_{gr}|b\rangle_{|t=0}-2\mathrm{Im}(e^{-2it\omega_{\boldsymbol p}}\langle b^*|\boldsymbol c_{\boldsymbol p}|a\rangle_{|t=0})
\end{aligned}
$$

$\nearrow$

Zitterbewegung

- Coordinate square:

$$
\begin{aligned}
\langle\chi|\boldsymbol x^2_{FV\leftarrow KG}|\chi\rangle =\ &-\int_{\boldsymbol p\boldsymbol q}[a^*_{\boldsymbol p}\bar w_+e^{it\omega_p-i\boldsymbol x\boldsymbol p}+b_{\boldsymbol p}\bar w_-e^{-it\omega_p+i\boldsymbol x\boldsymbol p}]\\
&\times\{a_{\boldsymbol q}e^{-it\omega_q}(\boldsymbol\nabla_q^2+2\boldsymbol c_q\boldsymbol\nabla_q\sigma_1+\boldsymbol c_q^2)w_+e^{i\boldsymbol q\boldsymbol x}+b^*_{\boldsymbol q}e^{it\omega_q}(\boldsymbol\nabla_q^2+2\boldsymbol c_q\boldsymbol\nabla_q\sigma_1+\boldsymbol c_q^2)w_-e^{-i\boldsymbol q\boldsymbol x}\}\\
=\ &-\int_{\boldsymbol p\boldsymbol q}[a^*_{\boldsymbol p}\bar w_+e^{it\omega_p-i\boldsymbol x\boldsymbol p}+b_{\boldsymbol p}\bar w_-e^{-it\omega_p+i\boldsymbol x\boldsymbol p}]\\
&\times\{[(\boldsymbol\nabla_q^2-2\boldsymbol c_q\boldsymbol\nabla_q\sigma_1+\boldsymbol c_q^2)a_{\boldsymbol q}e^{-it\omega_q}]w_+e^{i\boldsymbol q\boldsymbol x}+[(\boldsymbol\nabla_q^2-2\boldsymbol c_q\boldsymbol\nabla_q\sigma_1+\boldsymbol c_q^2)b^*_{\boldsymbol q}e^{it\omega_q}]w_-e^{-i\boldsymbol q\boldsymbol x}\}\\
=\ &-\int_{\boldsymbol q}\{a^*_{\boldsymbol q}[\boldsymbol\nabla^2-2it\boldsymbol\nabla\omega_{\boldsymbol q}\boldsymbol\nabla-it\boldsymbol\nabla^2\omega_{\boldsymbol q}-t^2(\boldsymbol\nabla\omega_{\boldsymbol q})^2+\boldsymbol c_q^2]a_{\boldsymbol q}\\
&-b_{\boldsymbol q}[\boldsymbol\nabla^2+2it\boldsymbol\nabla\omega_{\boldsymbol q}\boldsymbol\nabla+it\boldsymbol\nabla^2\omega_{\boldsymbol q}-t^2(\boldsymbol\nabla\omega_{\boldsymbol q})^2+\boldsymbol c_q^2]b^*_{\boldsymbol q}\\
&-2e^{2it\omega_q}a^*_{-\boldsymbol q}\boldsymbol c_q(\boldsymbol\nabla+it\boldsymbol\nabla\omega_{\boldsymbol q})b^*_{\boldsymbol q}+e^{-2it\omega_q}b_{-\boldsymbol q}\boldsymbol c_q(\boldsymbol\nabla-it\boldsymbol\nabla\omega_{\boldsymbol q})a_{\boldsymbol q}\}\\
=\ &\langle a|(\boldsymbol x+t\boldsymbol v_{gr})^2|a\rangle_0-\langle b|(\boldsymbol x-t\boldsymbol v_{gr})^2|b\rangle_0+2\mathrm{Im}(e^{-2it\omega_{\boldsymbol p}}\langle b^*|\boldsymbol c_{\boldsymbol p}(\boldsymbol x+t\boldsymbol v_{gr})|a\rangle_0)
\end{aligned}
$$

$\nearrow$

limit of localization

(c) *Advantage of the decoupled basis:* absence of the non-physical Zitterbewegung

(d) *Disadvantage of the decoupled basis:* local gauge transformations must be redefined

$\uparrow$

The "true" UV gauge transformations

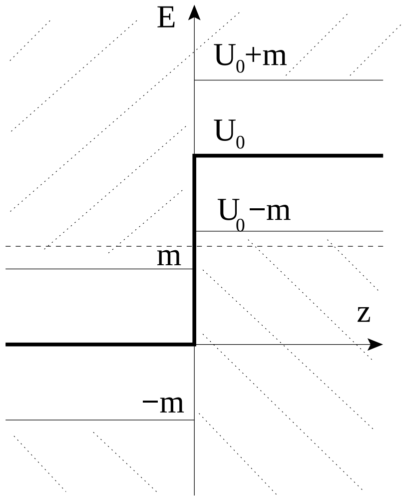
G. External field

1. **Electromagnetic field:**

$$
\begin{aligned}
\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} &= \frac12\begin{pmatrix}\phi+\frac{i}{m}D_0\phi\\ \phi-\frac{i}{m}D_0\phi\end{pmatrix},\quad D_0=\partial_0+ie\varphi\\
iD_0\chi &= H\chi,\quad H=-\frac{\boldsymbol D^2}{2m}(\sigma_3+i\sigma_2)+m\sigma_3,\quad \boldsymbol D=\boldsymbol\nabla-ie\boldsymbol A\\
i\partial_t\chi &= \left[-\frac{\boldsymbol\nabla^2}{2m}(\sigma_3+i\sigma_2)+m\sigma_3-\left(\frac{ie}{m}\boldsymbol A\boldsymbol\nabla+\frac{ie}{2m}\boldsymbol\nabla\boldsymbol A\right)(\sigma_3+i\sigma_2)+e\phi\right]\chi
\end{aligned}
$$

$\uparrow$ $\uparrow$

mixing of particle and anti-particle modes



2. **One dimensional potential barrier:** (Klein) $U(z) = U_0\Theta(z)$, $U_0 > 2m$, $m < E < U_0 - m$

- *Wave function:*

$$\begin{aligned}\psi(t,z) &= \chi(z)e^{-itE}\\ 0 &= [(E-U(z))^2 + \nabla_z^2 - m^2]\chi(z)\\ \chi(z) &= \Theta(-z)[\chi_i(z)+\chi_r(z)] + \Theta(z)\chi_t(z)\\ \chi_i(z) &= e^{ipz}, \quad \chi_r(z) = be^{-ipz}, \quad \chi_t(z) = de^{ip'z}, \quad p = \sqrt{E^2-m^2}, \quad p' = \sqrt{(E-U_0)^2 - m^2}\end{aligned}$$

- *Matching conditions:*

(a) continuity of the wave function:

$$1 + b = d$$

(b) continuity of its first derivative:

$$1 - b = d\xi, \quad \xi = \frac{p'}{p}$$

(c) Solution:

$$\begin{aligned}1-b &= (1+b)\xi\\ b &= \frac{1-\xi}{1+\xi}, \quad d = \frac{2}{1+\xi}.\end{aligned}$$

- *Current:*

$$\begin{aligned}j^z &= \frac{1}{2im}(\chi^*\nabla_z\chi - \nabla_z\chi^*\chi)\\ j_i^z(0) &= p, \quad j_r^z(0) = -|b|^2 p, \quad j_t^z(0) = |d|^2 p'\end{aligned}$$

- *Reflection and transmission coefficients:*

$$R = |b|^2 = \frac{(1-\xi)^2}{(1+\xi)^2}, \quad T = |d|^2\xi = \frac{4\xi}{(1+\xi)^2}$$

- *Current conservation:*

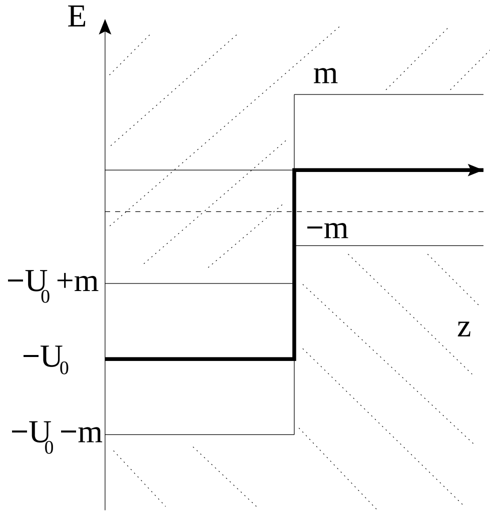
$$j_i^z(0) + j_r^z(0) = j_t^z(0) \quad \longrightarrow \quad R + T = 1$$

- *Mixing of the particle-anti particle modes:*

$$j^0(z) = \begin{cases} \frac{E}{m} > 0 & z < 0 \\ \frac{E-U}{m} < 0 & z > 0 \end{cases}$$

- *Positivity of the coefficients:* $\xi > 0 \Longrightarrow 0 \le R, T \le 1$ (no Klein paradox)

3. **Spherical potential well:** $U(r) = -U_0\Theta(R-r)$, $-m < E < m$ and $E > m - U_0$



(Oppenheimer-Weinberg-Schiff-Snyder)

- *Klein-Gordon equation:*

$$\begin{aligned} 0 &= [(\partial_0 + iU(r))^2 - \Delta + m^2]\phi(x) \\ \phi_{lm}(x) &= \eta_\ell(r) Y_m^\ell(\theta,\phi) e^{-itE} \\ 0 &= \left[(E - U(r))^2 + \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{l(l+1)}{r^2} - m^2\right]\eta_\ell(r) \\ 0 &= \left[\partial_r^2 - \frac{l(l+1)}{r^2} + (E - U(r))^2 - m^2\right] u_\ell(r), \quad \eta(r) = \frac{u(r)}{r} \\ \ell = 0: \quad u_0'' &= [m^2 - (E - U(r))^2]u_0 \end{aligned}$$

- *Wave function:*

  (a) Inside: $\sin\kappa r$ and $\cos\kappa r$

$$u_0 = \sin\kappa r, \quad \kappa = \sqrt{(E+U_0)^2 - m^2}$$

$\cos\kappa r$ suppressed by the regularity condition, $u_\ell(0) = 0$

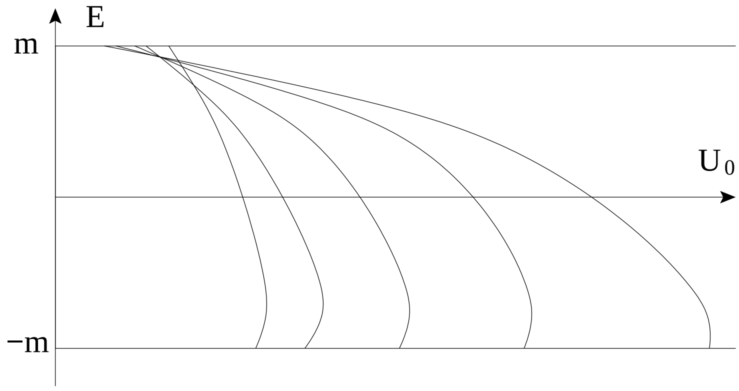
(b) Outside:

$$u_0 = ae^{-kr}, \quad k = \sqrt{m^2 - E^2}$$

(c) Matching conditions:

$$\begin{aligned} \sin \kappa R &= ae^{-kR}, \quad \kappa \cos \kappa R = -kae^{-kR} \\ \rightarrow \quad \tan R\kappa &= -\frac{\kappa}{k} \\ \tan R\sqrt{(E+U_0)^2 - m^2} &= -\sqrt{\frac{(E+U_0)^2 - m^2}{m^2 - E^2}} \end{aligned}$$

Graphical solution:



- more and more states dive into the gap
- bound states for both e^- and e^+
- at certain values of U_0 a pair of bound states disappears
- energy spectrum displays a complex conjugate pair at such U_0
- virtual particle-anti particle pairs
- radiating potential well

III. FERMIONS

A. Heuristic derivation of Dirac equation

1. **"Square root" of the Klein-Gordon equation:** $\partial_0^2 \phi = \Delta\phi - m^2\phi$

$$i\partial_0 \psi = H\psi, \quad H = \boldsymbol{\alpha p} + \beta m$$

- *Conditions:* $\{A, B\} = AB + BA$

$$\alpha_j \alpha_k \partial_j \partial_k = \frac{1}{2}(\{\alpha_j, \alpha_k\} + [\alpha_j, \alpha_k])\frac{1}{2}(\{\partial_j, \partial_k\} + [\partial_j, \partial_k])$$

$$
\begin{aligned}
&= \frac{1}{4}(\{\alpha_j,\alpha_k\}+[\alpha_j,\alpha_k])\{\partial_j,\partial_k\} \\
&= \frac{1}{2}\{\alpha_j,\alpha_k\}\partial_j\partial_k \\
-\partial_0^2\psi &= [-\{\alpha_j,\alpha_k\}\partial_j\partial_k+\beta^2m^2-m\{\alpha_j,\beta\}+\partial_j]\psi = -\Delta\phi+m^2\phi
\end{aligned}
$$

$$
\boxed{\{\alpha_j,\alpha_k\}=2\delta_{j,k},\quad \beta^2=\mathbb{1},\quad \{\boldsymbol{\alpha},\beta\}=0}
$$

- *Covariant notation:* $\gamma^\mu=(\beta,\beta\boldsymbol{\alpha})$,

$$
\begin{aligned}
0 &= (i\gamma^\mu\partial_\mu-m)\psi(x)=(i\gamma^0\partial_0+i\boldsymbol{\gamma}\boldsymbol{\nabla}-m)\psi(x)=(i\partial\!\!\!/-m)\psi(x) \\
\{\gamma^\mu,\gamma^\nu\} &= 2g^{\mu\nu}=2\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}
\end{aligned}
$$

γ^μ transforms as contravariant four-vector.

- *Useful equation:*

$$
(i\partial\!\!\!/+m)(i\partial\!\!\!/-m)=(i\partial\!\!\!/-m)(i\partial\!\!\!/+m)=-\partial^2-m^2.
$$

- *Dirac matrices in the standard representation:* simplest realization in terms of 4×4 matrices

$$
\beta=\gamma^0=\begin{pmatrix}\mathbb{1}_2&0\\0&-\mathbb{1}_2\end{pmatrix},\qquad \boldsymbol{\alpha}=\begin{pmatrix}0&\boldsymbol{\sigma}\\\boldsymbol{\sigma}&0\end{pmatrix},\qquad \boldsymbol{\gamma}=\begin{pmatrix}0&\boldsymbol{\sigma}\\-\boldsymbol{\sigma}&0\end{pmatrix},
$$

2. **Dirac conjugate:**

- $\psi^\dagger$ *belongs to a non-covariant equation:*

$$
0=i\partial_\mu\psi^\dagger(x)\gamma^{\mu\dagger}+\psi(x)m,\qquad\qquad \gamma^{0\dagger}=\gamma^0,\quad \gamma^{j\dagger}=-\gamma^j
$$

- *Dirac conjugate:* $\bar\psi=\psi^\dagger\gamma^0$, $\gamma^0\gamma^\mu\gamma^0=\gamma^{\mu\dagger}$, $\gamma^0\gamma^{\mu\dagger}\gamma^0=\gamma^\mu$

$$
0=[i\partial_\mu\psi^\dagger(x)\gamma^{\mu\dagger}+\psi(x)m]\gamma^0=[i\partial_\mu\psi^\dagger(x)\gamma^{02}\gamma^{\mu\dagger}+\psi(x)m]\gamma^0\quad\to\quad i\partial_\mu\bar\psi\gamma^\mu+\bar\psi m=0
$$

  - Klein-Gordon: Hermitian Hamiltonian,
  - Dirac: covariant E.O.M.

- *Electric current:* Noether current of the symmetry $\psi(x)\to e^{i\theta}\psi(x)$, $\bar\psi(x)\to e^{-i\theta}\bar\psi(x)$

$$
\begin{aligned}
L &= \frac{i}{2}[\bar\psi\gamma^\mu(\partial_\mu\psi)-(\partial_\mu\bar\psi)\gamma^\mu\psi]-m\bar\psi\psi \\
j^\mu &= \bar\psi\gamma^\mu\psi
\end{aligned}
$$

- *Positive definite probability density and scalar product:*

$$
\begin{aligned}
j^0(x) &= \psi^\dagger(x)\psi(x) \geq 0 \\
\langle\psi|\psi'\rangle &= \int d^3x\psi^\dagger(\boldsymbol{x})\psi'(\boldsymbol{x}) \geq 0
\end{aligned}
$$

B. Non-relativistic spinors

TABLE I: Real classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N)$	general linear group	$\det A \neq 0^a$	N^2	$\{\tau$: real $N \times N$ matrices$\}$
$SL(N)$	special linear group	$\det A = 1$	$N^2 - 1$	$\mathrm{tr}\tau = 0^b$
$O(N)$	orthogonal group	$A^{tr}A = \mathbb{1}^c$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$
$SO(N)$	special orthogonal group	$A^{tr}A = \mathbb{1}$, $\det A = 1$	$\frac{1}{2}N(N-1)$	$\tau^{tr} = -\tau$, $\mathrm{tr}\tau = 0$

[a]The matrix A is supposed to be an element of the group in question.
[b]$\det(\mathbb{1} + \epsilon\tau) = 1 + \epsilon\mathrm{tr}\tau + \mathcal{O}\left(\epsilon^2\right)$
[c]$\det A^{tr}A = (\det A)^2 = 1$ and $\det A = \pm 1$.

TABLE II: Complex classical matrix groups.

Symbol	Name	Definition	Dimension	Generators
$GL(N,C)$	complex general linear group	$\det A \neq 0$	$2N^2$	$\{\tau$: complex $N \times N$ matrices$\}$
$SL(N,C)$	complex special linear group	$\det A = 1$	$2N^2 - 2$	$\mathrm{tr}\tau = 0$
$U(N)$	unitary group	$A^\dagger A = \mathbb{1}^a$	N^2	$\tau^\dagger = -\tau$
$SU(N)$	special unitary group	$A^\dagger A = \mathbb{1}$, $\det A = 1$	$N^2 - 1$	$\tau^\dagger = -\tau$, $\mathrm{tr}\tau = 0$

[a]$\det A^\dagger A = (\det A)^* \det A = |\det A|^2 = 1$

- **Elementary systems:** Irreducible representations of the external symmetries

  1. Representation: $U(g) : G \to \mathcal{H}$, $U(gg') = U(g)U(g')$
  2. Irreducible (star) representation: $\exists\ |\psi_0\rangle \in \mathcal{H}$, $\forall\ |\psi\rangle \in \mathcal{H}$, $\exists\ g_\psi \in G$, $|\psi\rangle = U(g_\psi)|\psi_0\rangle$, $g \in G$
  3. Symmetry group: Connected subgroup and disconnected cosets

     first find the representation first the connected subgroup

     next for the cosets

- $SU(2)$**:**

1. Definition:

$$A(a,\boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a}\boldsymbol{\sigma} = \begin{pmatrix} a+ia_3 & ia_1+a_2 \\ ia_1-a_2 & a-ia_3 \end{pmatrix}, \quad a,\boldsymbol{a}\in R, \quad \det A(a,\boldsymbol{a}) = a^2+\boldsymbol{a}^2 = 1$$

2. $\dim(SU(2)) = 3$

3. Quaternion (Pauli matrices) algebra:

$$\boxed{\sigma_a\sigma_b = \mathbb{1}\delta_{ab} + i\epsilon_{abc}\sigma_c, \qquad \boldsymbol{\sigma}^* = -\sigma_2\boldsymbol{\sigma}\sigma_2}$$

4. Multiplication:

$$\begin{aligned} A(a,\boldsymbol{a})A(b,\boldsymbol{b}) &= (a\mathbb{1}+i\boldsymbol{a}\boldsymbol{\sigma})(b\mathbb{1}+i\boldsymbol{b}\boldsymbol{\sigma}) \\ &= (ab-\boldsymbol{a}\boldsymbol{b}+i(a\boldsymbol{b}+b\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b})\boldsymbol{\sigma} \\ &= A(ab-\boldsymbol{a}\boldsymbol{b}, a\boldsymbol{b}+b\boldsymbol{a}-\boldsymbol{a}\times\boldsymbol{b}). \end{aligned}$$

5. Inverse: $A^{-1}(a,\boldsymbol{a}) = A(a,-\boldsymbol{a}) = A^\dagger(a,\boldsymbol{a}) \Longrightarrow \{A\} = SU(2)$

6. Another parametrization: spin $s=\frac{1}{2}$, $\boldsymbol{n}^2=1$,

$$A_{\boldsymbol{n}}(\alpha) = e^{-\frac{i}{2}\alpha\boldsymbol{n}\boldsymbol{\sigma}} = \mathbb{1}\cos\frac{\alpha}{2} - i\boldsymbol{n}\boldsymbol{\sigma}\sin\frac{\alpha}{2} = A\left(\cos\frac{\alpha}{2}, -\boldsymbol{n}\sin\frac{\alpha}{2}\right)$$

- **Fundamental representation:** $\psi \to A\psi$,
- **Complex conjugate fundamental representation:** $\eta \to A^*\eta$

1. Unitary equivalent representations:

$$U'(g) = V^\dagger U(g)V$$

2. The fundamental and the complex conjugate fundamental representations are unitary equivalent:

$$(i\boldsymbol{\sigma})^* = \sigma_2 i\boldsymbol{\sigma}\sigma_2 = \sigma_2^\dagger i\boldsymbol{\sigma}\sigma_2 \quad \Longrightarrow \quad A^* = \sigma_2 A\sigma_2 = \sigma_2^\dagger A\sigma_2$$

- **Adjoint representation:** X_{ab}, $a,b=1,2$,

$$X \to AXA^\dagger,$$

1. Reducible representation: 8 dimensions, $8=4+4$
2. $X^\dagger = \pm X \Longrightarrow AXA^\dagger = A(\pm X)A^\dagger = \pm(AXA^\dagger)^\dagger$
3. Choose the 4 dimensional representation by Hermitean matrices:

$$X(x^\mu) = x^0\mathbb{1} + \boldsymbol{x}\boldsymbol{\sigma} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix},$$

Spin wave function of two spin half particles, $\frac{1}{2}\otimes\frac{1}{2} = 0\oplus 1 = a\oplus\boldsymbol{a}$

- **Relation between $SO(3)$ and $SU(2)$, Part I.:** The triplet multiplet is a three-vector

  1. Disconnected components of $O(3)$:

$$\begin{aligned} A^{\rm tr}A &= 1\!\!1 \quad \to \quad \det(A^{\rm tr}A) = \det(A^{\rm tr})\det(A) = [\det(A)]^2 = 1 \quad \to \quad \det(A) = \pm 1 \\ \mathrm{SO}(3) &= \{\det(A) = 1 | A^{\rm tr}A = 1\!\!1\} \quad \leftarrow \quad \text{connected, rotations without inversion} \end{aligned}$$

  2. length $\det X = -\boldsymbol{x}^2$ is preserved: $\det X \to \det AXA^\dagger = \det A \det X \det A^\dagger = \det X$
  3. infinitesimal $SU(2)$ transformations are infinitesimal $SO(3)$ rotations:

$$\begin{aligned} A_{\boldsymbol{n}}(\alpha)X(0,\boldsymbol{x})A^\dagger_{\boldsymbol{n}}(\alpha) &\approx \left(1\!\!1 - i\frac{\alpha}{2}\boldsymbol{n\sigma}\right)\boldsymbol{x\sigma}\left(1\!\!1 + i\frac{\alpha}{2}\boldsymbol{n\sigma}\right) \\ &\approx \boldsymbol{x\sigma} - i\frac{\alpha}{2}[\boldsymbol{n\sigma},\boldsymbol{x\sigma}] \\ &= X(0,\boldsymbol{x} + \alpha\boldsymbol{n}\times\boldsymbol{x}), \end{aligned}$$

  4. The repeated application for finite α: the adjoint representation is a mapping $SU(2) \to SO(3)$.
  5. The representation of inversion needs further steps.

- **Fundamental group:** Group over the homotopy classes of closed paths: Γ is a topological space (each point is contained in a neighborhood, i.e. an open set)

  1. $\gamma : [0,1] \to \Gamma$, $\gamma(0) = \gamma(1)$
  2. γ_1 homotopic with γ_2, $\gamma_1 \approx \gamma_2$, if
     (a) γ_1 can continuously be deformed to γ_2 within Γ, or
     (b) $\exists\ f : [0,1] \otimes [0,1] \to \Gamma$ continuous function such that

$$f(s,0) = \gamma_1(s), \quad f(s,1) = \gamma_2(s)$$

  3. Equivalence classes of homotopically equivalent loops $G_\gamma = \{\gamma' | \gamma' \approx \gamma\}$.
  4. Multiplication of loops:

$$\gamma_2 \otimes \gamma_1(s) = \begin{cases} \gamma_1(2s) & 0 < s < \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} < s < 1 \end{cases}$$

     e.g. paths on the circle with a given winding number
  5. $\gamma_2 \otimes \gamma_1 = \gamma_3 \quad \leftrightarrow \quad G_{\gamma_2} \otimes G_{\gamma_1} = G_{\gamma_3}$
  6. Fundamental group, $\pi_1(\Gamma)$:
     (a) Homotopy classes with the multiplication and $\gamma^{-1}(s) = \gamma(1-s)$.
     (b) e.g. $\pi_1(U(1)) = Z$.

- **Relation between $SO(3)$ and $SU(2)$, Part II.:**

  1. The mapping $A \to AXA^\dagger$, $SU(2) \to SO(3) \subset O(3)$ is a two-to-one,

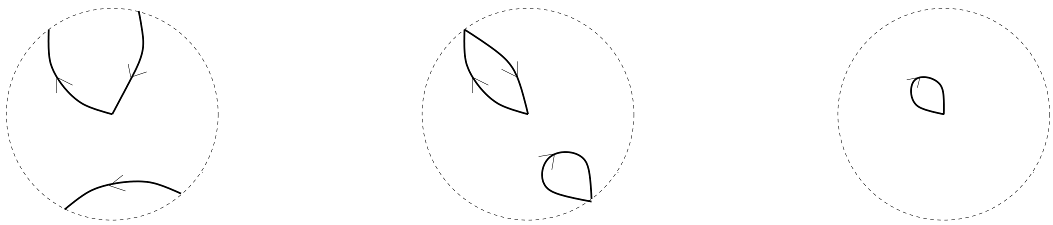
  $$(-A)X(-A)^\dagger = AXA^\dagger$$

  2. The group manifold $SU(2)$ is simply connected, $\pi_1(SU(2)) = \pi_1(S_3) = \mathbb{1}$.
  3. The group manifold $O(3)$ is doubly connected, $\pi_1(O(3)) = Z_2 = \{1, -1\}$.

     Proof:

     (a) $R_{\boldsymbol{n}}(\pi), R_{\boldsymbol{n}}(-\pi) \in SO(3)$, are identical, $R_{\boldsymbol{n}}(\pi) = R_{\boldsymbol{n}}(-\pi)$ because a rotation by 2π leaves a vector unchanged.

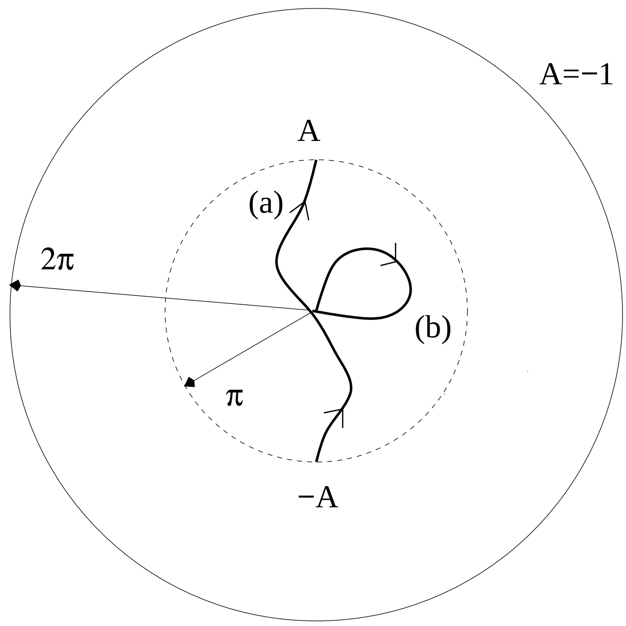
     (b) Continuous deformation of the loop can change n_d, the number of diametrically opposite points of $SO(3)$ visited in units of 2.



     (c) The parity $n_d \mod 2$ is topological invariant.

  4. $SO(3): \ |\alpha| < \pi \ (AXA^\dagger = (-A)X(-A)^\dagger)$, $SU(2): \ |\alpha| < 2\pi$

  $$A_{\boldsymbol{n}}(\alpha) = e^{-\frac{i}{2}\alpha \boldsymbol{n}\boldsymbol{\sigma}} = \mathbb{1}\cos\frac{\alpha}{2} - i\boldsymbol{n}\boldsymbol{\sigma}\sin\frac{\alpha}{2}$$



C. Multi-valued wave functions

1. **Wave function:**

   (a) $\psi(\boldsymbol{x}) : R^3 \to C$ possible Riemann-sheets, e.g. $\ln z$.

   (b) Do they play role in physics?

   (c) There might be several Riemann-sheets if the particle is excluded from some region, $\psi(\boldsymbol{x}) = 0$

2. **Particle on the circle:** $\boldsymbol{x} = r(\cos\phi, \sin\phi)$, $\psi(\phi) = \langle\phi|\psi\rangle$

   - Hermitian extension of the momentum operator $p = \frac{\hbar}{ir}\partial_\phi = p^\dagger$

     (a) $-\infty < x < \infty$, $\psi \in L_2$:

$$\begin{aligned}\langle\psi_1|p|\psi_2\rangle &= \int_{-\infty}^{\infty} dx\psi_1^*(x)\frac{\hbar}{ir}\partial_x\psi_2(x)\\ &= \int_{-\infty}^{\infty} d\phi\left(\frac{\hbar}{ir}\partial_x\psi_1(x)\right)^*\psi_2(x) + \underbrace{\frac{\hbar}{ir}\psi_1^*(x)\psi_2(x)\Big|_{-\infty}^{\infty}}_{0}\\ &= \langle\psi_1|p^\dagger|\psi_2\rangle\end{aligned}$$

     (b) $0 < x < \infty$, $\nexists$

     (c) $-\pi < x < \pi \Longrightarrow$different Hilbert spaces:

$$\begin{aligned}\langle\psi_1|p|\psi_2\rangle &= \int_{-\pi}^{\pi} d\phi\psi_1^*(\phi)\frac{\hbar}{ir}\partial_\phi\psi_2(\phi)\\ &= \int_{-\pi}^{\pi} d\phi\left(\frac{\hbar}{ir}\partial_\phi\psi_1(\phi)\right)^*\psi_2(\phi) + \frac{\hbar}{ir}\psi_1^*(\phi)\psi_2(\phi)\Big|_{-\pi}^{\pi}\\ &= \langle\psi_1|p^\dagger|\psi_2\rangle\\ \Longrightarrow \mathcal{H}_\theta &= \{\psi(\phi)|\psi(\phi+2\pi) = e^{i\theta}\psi(\phi)\}\end{aligned}$$

- $\mathcal{H}_\theta$:
  - Basis:

$$\psi_n(\phi) = e^{i(n+\frac{\theta}{2\pi})\phi}$$

  - Spectrum:

$$\begin{aligned} p_\theta\psi_n &= \frac{\hbar}{r}\left(n+\frac{\theta}{2\pi}\right)\psi_v \\ H_\theta\psi_n &= \frac{\hbar^2}{2mr^2}\left(n+\frac{\theta}{2\pi}\right)^2\psi_n \end{aligned}$$

  - Periodic θ-dependence:

$$\theta \to \theta + 2\pi \quad \to \quad \psi_n(\phi) \to \psi_{n+1}(\phi)$$

- Physical origin: The particle interferes with itself as it turns around the circle. No classical analogy.
- What matters
  (a) in classical mechanics: where you are
  (b) in quantum mechanics: where you are and how did you get there
- Quantum symmetry: $\phi \to \phi + 2\pi$, irreducible representations: $\psi(\phi+2\pi) = e^{i\theta}\psi(\phi)$ (like translations)
- θ: new topological quantum number

3. **Aharonov-Bohm effect:**

- Particle on a ring which encircles a magnetic flux

$$\Phi = \int_\Sigma d\boldsymbol{n}\boldsymbol{B}(\boldsymbol{x}) = \oint_\mathcal{R} d\boldsymbol{x}\boldsymbol{A}(\boldsymbol{x})$$

- The magnetic field is vanishing on the circle, $A = A_\phi = \frac{\Phi}{2\pi r} = \partial_{r\phi}\alpha(\phi)$ $\alpha(\phi) = \frac{\phi\Phi}{2\pi}$ (pure gauge)
- Hamiltonian, $\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$

$$H_\Phi = \frac{\hbar^2}{2mr^2}\left(\frac{1}{i}\partial_\phi - \frac{e\Phi}{2\pi\hbar c}\right)^2.$$

- Aharonov-Bohm effect: The expectation values dependend on the magnetic field.
- Surprises:
  (a) The magnetic field is vanishing along the path of the propagation
  (b) The physical effects of the magnetic flux can be eliminated by an aperiodic gauge transformation which lead out of the Hilbert space.

$$\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \frac{\hbar c}{e}\boldsymbol{\nabla}\chi(\boldsymbol{x}), \quad \psi(\boldsymbol{x}) \to e^{i\chi(\boldsymbol{x})}\psi(\boldsymbol{x}),$$

  - $A_\phi = 0$ and $\chi = \phi \frac{e}{\hbar c} \frac{\Phi}{2\pi} \to A_\phi = \frac{\Phi}{2\pi r}$, magnetic flux Φ.
  - $\mathcal{H}_\theta \to \mathcal{H}_{\theta + \frac{e}{\hbar c}\Phi}$, $H_\theta \to H_{\theta + \frac{e}{\hbar c}\Phi}$.
- H_Φ is the free Hamiltonian (no magnetic flux) in $\mathcal{H}_{\frac{e\Phi}{\hbar c}}$.
- Relation to quantum mechanics on the circle: $\Phi \quad \Leftrightarrow \quad \theta$

4. **Dynamical condition of the multi-valuedness of wave function:** The particle has to be excluded from a region and that makes the coordinate space multiply connected.

5. **Topological symmetry:** The fundamental group is a (classical) symmetry and its irreducible representations generate a new quantum number, Θ, e.g. rotor dynamics,

$$\pi_1(SO(3)) = Z_2, \quad \Theta = \begin{cases} 0 & \text{boson,} \\ \pi & \text{fermion.} \end{cases}$$

D. Projective representations

1. The vectors $|\psi\rangle$ and $e^{i\alpha}|\psi\rangle$ represent the same physical state.

2. The representations of symmetries are to be generalized to projective representations,

$$\begin{aligned} U(gg') &= U(g)U(g')e^{i\alpha(g,g')} \\ U(g_3g_2g_1) &= U(g_3)U(g_2g_1) = U(g_3g_2)U(g_1) \\ \alpha(g_3, g_2g_1) + \alpha(g_2, g_1) &= \alpha(g_3, g_2) + \alpha(g_3g_2, g_1). \end{aligned}$$

Question: Can the phase factor be eliminated by a "gauge transformation" in the group space?

$$U(g) \to U(g)e^{i\beta(g)}$$

Necessary conditions:

1. *Preliminary:*

   (a) Central charge: the coefficient of the identity in the commutator of the generators,

   $$[\tau, \tau'] = c\mathbb{1} + \cdots$$

   (b) Simple group: has generator which commute with all the other generators, eg. $i\mathbb{1}$ for $U(n)$

   (c) Semi-simple group: has no generator which commute with all the other generators, eg. $SU(n)$

2. *Local:* (in the vicinity of the identity)

(a) The group has no central charge.

(b) The central charge of a semi-simple group can be eliminated by the appropriate redefinition of the generators.

3. *Global:* The topology of the group must be simply connected. In case of a multiply connected topology the phase $\alpha(g, g')$ gives a representation of the fundamental group.

Rotations:

1. $SO(3)$ is semi-simple but doubly connected, $\pi_1(SO(3)) = Z_2 \Longrightarrow \exists$ projective representations.

2. The spin is pseudo-scalar and is left unchanged by space inversion $\Longrightarrow U(P) = z_P \mathbb{1}$.

3. Projective representation: $U^2(P) = \mathbb{1} \in Z_2 \Longrightarrow U^2(P) = \pm\mathbb{1}$, $z_P \in \{1, -1, i, -i\}$.

4. $U(R_{\boldsymbol{n}}(2\pi)) = \mathbb{1}e^{i\Theta}$

   (a) $d = 3$: $e^{i\Theta} \in \pi_1(SO(3)) = Z_2$, bosons ($\Theta = 0$) or fermions ($\Theta = \pi$)

   (b) $d = 2$: $\pi_1(SO(2)) = Z$, $-\pi < \Theta < \pi$, anyons

E. Relativistic spinors

Spin is a relativistic effect:

- Irreducible representations of the Poincare group are labeled by $P^\mu P_\mu$ and $\boldsymbol{S}^2$
- The Lorentz boost introduce projective representations of the Poincare group
- Lorentz group is represented on the universal covering space of $L_+^\uparrow$ which is $SL(2, C)$

Spinor representation of $SL(2, C)$: $\dim(Sl(2, C)) = 6$

- Parametrization:

$$A(a, \boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a}\boldsymbol{\sigma} = \begin{pmatrix} a + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a - ia_3 \end{pmatrix}, \quad a, \boldsymbol{a} \in C, \quad \det A(a, \boldsymbol{a}) = a^2 + \boldsymbol{a}^2 = 1$$

- Fundamental representation: $\xi \to A\xi$.
- Complex conjugate fundamental representation: $\eta \to A^*\eta$.

- van der Waerden conventions: ξ_a, $\eta^{\dot a}$, $a, \dot a \in \{1,2\}$,

$$\begin{aligned}
\xi_a &\to A_a{}^b\xi_b, \quad \eta_{\dot a} \to A^{*\dot b}_{\dot a}\psi_{\dot b} \\
g_{ab} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 = g_{\dot a\dot b} = -g^{ab} = -g^{\dot a\dot b} \\
\xi_a\chi^a &= \xi_a g^{ab}\chi_b = \xi_1\chi_2 - \xi_2\chi_1 \to (\xi_a\chi_b - \xi_b\chi_a)A_1{}^a A_2{}^b = (\xi_1\chi_2 - \xi_1\chi_2)\det A_a{}^b = \xi_a\chi^a.
\end{aligned}$$

- Adjoint representation:

$$\begin{aligned}
X &\to AXA^\dagger \\
X^{a\dot a}(x^\mu) &= x^0\mathbb{1} + \boldsymbol{x\sigma} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} = X^{*\dot a a}(x^\mu) \\
X_{a\dot a} &= g_{ab}g_{\dot a\dot b}X^{b\dot b}, \qquad g = i\sigma_2, \quad \boldsymbol{\sigma}^{\mathrm{tr}} = \boldsymbol{\sigma}^* = -\sigma_2\boldsymbol{\sigma}\sigma_2 = \sigma_2\boldsymbol{\sigma}\sigma_2^{\mathrm{tr}} \\
&= i\sigma_2(x^0\mathbb{1} + \boldsymbol{x\sigma})i\sigma_2^{\mathrm{tr}} = x^0\mathbb{1} - \boldsymbol{x\sigma}^{\mathrm{tr}}.
\end{aligned}$$

Lorentz group and $SL(2,C)$: $Dim(X) = 4$, X is a four vector x^μ.

1. length $\det X = x^2$ is preserved.

2. infinitesimal $SL(2,C)$ transformations are infinitesimal Lorentz transformation:

   Rotations: $R_{\boldsymbol n}(\alpha) = A_{\boldsymbol n}(\alpha) = e^{-\frac{i}{2}\alpha\boldsymbol{n\sigma}}$, $\alpha \sim 0$

$$X(0,\boldsymbol x) \to A_{\boldsymbol n}(\alpha)X(0,\boldsymbol x)A^\dagger_{\boldsymbol n}(\alpha) = X(0, \boldsymbol x + \alpha\boldsymbol n \times \boldsymbol x)$$

   Lorentz boost $\boldsymbol v = c\beta\boldsymbol n$, $\beta \sim 0$

$$\begin{aligned}
A_{\boldsymbol n}(i\beta) &= e^{\frac{\beta}{2}\boldsymbol{n\sigma}} = \mathbb{1}\cosh\frac{\beta}{2} + \boldsymbol{n\sigma}\sinh\frac{\beta}{2} \\
A_{\boldsymbol n}(i\beta)X(x^0,\boldsymbol x)A^\dagger_{\boldsymbol n}(i\beta) &\approx \left(\mathbb{1} + \frac{\beta}{2}\boldsymbol{n\sigma}\right)(x^0\mathbb{1} + \boldsymbol{x\sigma})\left(\mathbb{1} + \frac{\beta}{2}\boldsymbol{n\sigma}\right) \\
&\approx x^0\mathbb{1} + (\boldsymbol x + x^0\beta\boldsymbol n)\boldsymbol\sigma + \frac{\beta}{2}\{\boldsymbol{n\sigma}, \boldsymbol{x\sigma}\} \\
&= X(x^0 + \beta\boldsymbol{nx}, \boldsymbol x + x^0\beta\boldsymbol n)
\end{aligned}$$

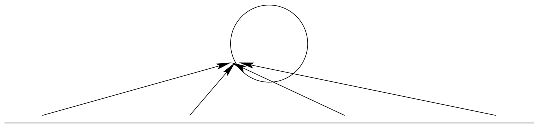
3. Mapping: $SL(2,c) \to L_+^\uparrow$: two-to-one (A and $-A$).

4. Topology: $L_+^\uparrow = SO(3) \otimes R^3$, $\pi_1(L_+^\uparrow) = \pi_1(SO(3)\otimes R^3) = Z_2$.

Universal covering space:

1. Γ and Γ_c are locally identical, Γ_c is globally simply connected, $\pi_1(\Gamma_c) = \mathbb{1}$

2. $R \to U(1)$ locally identical, $\pi_1(R) = \mathbb{1}$, $\pi_1(U(1)) = Z$.



3. $SU(2) \to SO(3)$ locally identical, $\pi_1(SU(2)) = \mathbb{1}$, $\pi_1(SO(3)) = Z_2$.

4. $SL(2,C) \to L_+^\uparrow$ locally identical, $\pi_1(SL(2,C)) = \mathbb{1}$, $\pi_1(L_+^\uparrow) = \pi_1(SO(3) \otimes R^3) = Z_2$.

Representations of the Lorentz group: $L = L_+^\uparrow \oplus L_+^\downarrow \oplus L_-^\uparrow \oplus L_-^\downarrow$

1. Representation of $L_+^\uparrow$:

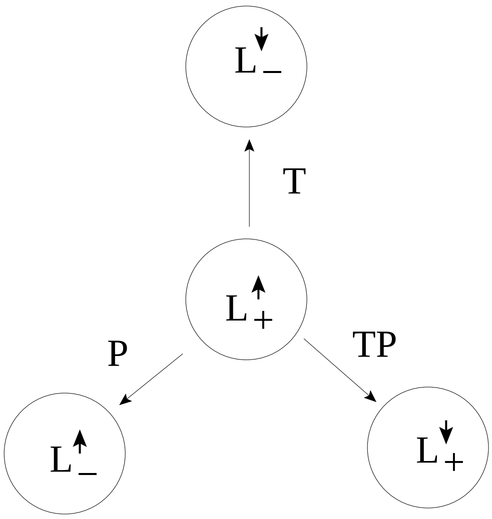
   (a) Simply connected $\pi_1(SL(2,C) \otimes R^3) = \mathbb{1}$

   (b) Doubly connected $\pi_1(SO(3) \otimes R^3) = \pi_1(L_+^\uparrow) = Z_2$

   $\Longrightarrow$projective representations (bosons, fermions)

2. Representation of discrete inversions P, T

3. Representation of the cosets $L_+^\downarrow = TPL_+^\uparrow$, $L_-^\uparrow = PL_+^\uparrow$, $L_-^\downarrow = TL_+^\uparrow$



Representation of discrete symmetries by bi-spinors:

1. P:

   (a) Nonrelativistic case: $P = z_P \mathbb{1}$

   (b) Relativistic case: $PL(\boldsymbol{v}) = L(-\boldsymbol{v})P \neq L(\boldsymbol{v})P \Longrightarrow [L(\boldsymbol{v}), P] \neq 0 \Longrightarrow P \neq z_P \mathbb{1}$

   (c) Irreducible representation of $L_+^\uparrow \cup L_-^\uparrow$: unitary, $s_z \to s_z$, exchange the irreps. of $L_+^\uparrow$

$$\psi = \begin{pmatrix} \xi^a \\ \eta_{\dot a} \end{pmatrix},$$

$$\begin{aligned} P\xi^a &= z_P\eta_a, \quad P\eta_a = z_P\xi^a, \\ z_P &= \pm 1,\ (P^2 = 1\!\!1) \quad \text{or} \quad \pm\, i\ (P^2 = -1\!\!1). \end{aligned}$$

2. T:

   (a) $TL(\boldsymbol{v}) = L(-\boldsymbol{v})T \neq L(\boldsymbol{v})T \Longrightarrow [L(\boldsymbol{v}), T] \neq 0 \Longrightarrow T \neq z_T 1\!\!1$

   (b) Irreducible representation of $L_+^\uparrow \cup L_-^\downarrow$: anti-unitary, $s_z \to -s_z$, keep the irreps. of $L_+^\uparrow$

$$\begin{aligned} T\xi^a &= z_T g_{ab}\xi^{*b}, \quad T\eta_a = z_T g^{ab}\eta_b^*, \\ T^2 &= U(R_{\boldsymbol{n}}(2\pi)) = -1\!\!1, \\ z_T &= \pm i. \end{aligned}$$

3. C: anti-unitary, $s_z \to -s_z$, exchange the irreps. of $L_+^\uparrow$

$$\begin{aligned} C\xi^a &= z_C g^{ab}\eta_a^*, \quad C\eta_a = -z_C g_{ab}\xi^{*b} \\ z_C &= \pm 1, \pm i. \end{aligned}$$

Usual convention of a PTC-symmetric quantum field theory: $z_P = -z_T = -z_C = i$.

F. Dirac equation

1. Covariant equation for $\psi = (\xi^a, \eta_{\dot a})$,

2. Involving $p^{a\dot a} = p^0 1\!\!1 + \boldsymbol{p}\boldsymbol{\sigma}$ and $p_{a\dot a} = p^0 1\!\!1 - \boldsymbol{p}\boldsymbol{\sigma}^{\mathrm{tr}}$.

$$\begin{aligned} p^{a\dot a}\eta_{\dot a} &= m\xi^a, \\ p_{a\dot a}\xi^a &= m\eta_{\dot a}, \end{aligned}$$

or

$$\begin{aligned} (p^0 + \boldsymbol{p}\boldsymbol{\sigma})\eta &= m\xi, \\ (p^0 - \boldsymbol{p}\boldsymbol{\sigma})\xi &= m\eta. \end{aligned}$$

Dynamical role of the mass: coupling the two spinors

Elimination of one spinor:

$$(p^2 - m^2)\xi = (p^2 - m^2)\eta = 0.$$

Covariant form: $p_\mu = i\partial_\mu$

$$
\begin{aligned}
0 &= (i\gamma^\mu_{ch}\partial_\mu - m)\psi \\
\gamma^0_{ch} &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_{ch} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma^0_{st} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \boldsymbol{\gamma}_{st} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \\
\gamma^\mu_{ch} &= U\gamma^\mu_{st}U^\dagger, \quad U = \frac{1}{\sqrt{2}}(1-\gamma^5\gamma^0) = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix},
\end{aligned}
$$

Scalar Dirac matrix:

$$
\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma^5_{ch} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}
$$

Minimal coupling: $p_\mu \to p_\mu - eA_\mu$, $\partial_\mu \to \partial_\mu + ieA_\mu$, $(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0$

Discrete symmetries:

- $P\psi = U_P\psi_P$, $f_P(t,\boldsymbol{x}) = f(t,-\boldsymbol{x})$, $A_{\mu P}(t,\boldsymbol{x}) = (\phi(t,\boldsymbol{x}),\boldsymbol{A}(t,\boldsymbol{x}))_P = (\phi(t,-\boldsymbol{x}),-\boldsymbol{A}(t,-\boldsymbol{x}))$,

  $U_P = i\gamma^0$, $U_P\gamma^0U_p^{-1} = \gamma^0$, $U_P\boldsymbol{\gamma}U_P^{-1} = -\boldsymbol{\gamma}$

  $(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0 \Longrightarrow U_P(i\partial\!\!\!/ - eA\!\!\!/ - m)U_P^{-1}U_P\psi = 0 \Longrightarrow (i\partial\!\!\!/ - eA\!\!\!/_P - m)\psi_P = 0$

- $T\psi = U_T\bar{\psi}_T$, $f_T(t,\boldsymbol{x}) = f(-t,\boldsymbol{x})$, $A_{\mu T}(t,\boldsymbol{x}) = (\phi(t,\boldsymbol{x}),\boldsymbol{A}(t,\boldsymbol{x}))_T = (\phi(-t,\boldsymbol{x}),-\boldsymbol{A}(-t,\boldsymbol{x}))$,

  $U_T = -i\gamma^1\gamma^3\gamma^0$, $U_T^{-1} = i\gamma^0\gamma^3\gamma^1$, $U_T\gamma^{\mathrm{tr}0}U_T^{-1} = \gamma^0$, $U_T\boldsymbol{\gamma}^{\mathrm{tr}}U_T^{-1} = -\boldsymbol{\gamma}$

  $\bar{\psi}(-i\partial\!\!\!/ - eA\!\!\!/ - m) = 0 \Longrightarrow U_T(-i\partial\!\!\!/^{\mathrm{tr}} - eA\!\!\!/^{\mathrm{tr}} - m)U_T^{-1}U_T\bar{\psi} = 0 \Longrightarrow (i\partial\!\!\!/ - eA\!\!\!/_T - m)\psi_T = 0$

- $C\psi = U_C\bar{\psi}$, $U_C = -i\gamma^2\gamma^0$, $U_C\gamma^\mu U_C^{-1} = -\gamma^{\mathrm{tr}\mu}$, $(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi_C = 0$.

Lorentz transformations:

- Fundamental representation:

$$
\xi^a \to A^a{}_b\xi^b, \quad \xi \to A\xi, \quad A(a,\boldsymbol{a}) = a\mathbb{1} + i\boldsymbol{a\sigma}
$$

- Complex conjugate representation:

$$
\begin{aligned}
A^*(a,\boldsymbol{a}) &= (a\mathbb{1} + i\boldsymbol{a\sigma})^* = a^*\mathbb{1} - i\boldsymbol{a}^*\boldsymbol{\sigma}^*, \quad \boldsymbol{\sigma}^* = -\sigma_2\boldsymbol{\sigma}\sigma_2 \\
&= \sigma_2 A(a^*,\boldsymbol{a}^*)\sigma_2 \\
\eta_{\dot{a}} &\to A_{\dot{a}}{}^{\dot{b}}\eta_{\dot{b}} = (g_{aa'}g^{bb'}A^{a'}{}_{b'})^*\eta_{\dot{b}}, \qquad g = i\sigma_2 \\
\eta &\to i\sigma_2 A^*(a,\boldsymbol{a})i\sigma_2^{\mathrm{tr}}\eta = i\sigma_2\sigma_2 A(a^*,\boldsymbol{a}^*)\sigma_2 i\sigma_2^{\mathrm{tr}}\eta = A(a^*,\boldsymbol{a}^*)\eta.
\end{aligned}
$$

- Bi-spinors: $A_{\boldsymbol{n}}(\alpha + i\beta)$, $\boldsymbol{u} = \alpha\boldsymbol{n}$, $\boldsymbol{v} = \beta\boldsymbol{n}$,

1. Lorentz covariance:

$$\begin{aligned}
A_{\boldsymbol{n}}(\alpha+i\beta) &= e^{-\frac{i}{2}(\alpha+i\beta)\boldsymbol{n}\boldsymbol{\sigma}} = e^{-\frac{i}{2}(\boldsymbol{u}\boldsymbol{\sigma}+i\boldsymbol{v}\boldsymbol{\sigma})} = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \\
v_j &= \omega_{0j} = -\omega_{j0}, \quad u_\ell = \frac{1}{2}\epsilon_{jk\ell}\omega_{jk}, \quad \omega_{jk} = \epsilon_{jk\ell}u_\ell, \quad \epsilon_{jk\ell}\epsilon_{jkm} = 2\delta_{\ell,m} \\
\omega_{\mu\nu} &= \begin{pmatrix} 0 & v_1 & v_2 & v_3 \\ -v_1 & 0 & u_3 & -u_2 \\ -v_2 & -u_3 & 0 & u_1 \\ -v_3 & u_2 & -u_1 & 0 \end{pmatrix}, \quad \sigma^{0j}_{ch} = i\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad \sigma^{jk}_{ch} = \epsilon^{jk\ell}\begin{pmatrix} \sigma_\ell & 0 \\ 0 & \sigma_\ell \end{pmatrix} \\
\psi &\to e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi
\end{aligned}$$

2. Representation independent form:

$$\begin{aligned}
\gamma^0_{ch} &= \begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}_{ch} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \\
\sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu]
\end{aligned}$$

3. Spin: generators of $SO(3)$ rotations:

$$\begin{aligned}
S_j &= \frac{1}{2}\epsilon_{jk\ell}\sigma_{k\ell} \\
(S_{ch})_j &= \frac{1}{2}\epsilon_{jk\ell}\epsilon^{k\ell m}\begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}
\end{aligned}$$

TABLE III: Transformation properties of bilinears.

Bilinear	Special Lorentz tr.	Space inv.	Time inv.	Charge conj.
$S = \bar{\psi}\psi$	S	S	S	S
$P = \bar{\psi}\gamma^5\psi$	P	$-P$	P	P
$V^\mu = \bar{\psi}\gamma^\mu\psi = (V^0, \boldsymbol{V})$	$\omega^\mu{}_\nu V^\nu$	$(V^0, -\boldsymbol{V})$	$(-V^0, \boldsymbol{V})$	V^μ
$A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$	$\omega^\mu{}_\nu V^\nu$	$(-V^0, \boldsymbol{V})$	$(-V^0, \boldsymbol{V})$	A^μ
$T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi = T^{\mu\nu}(\boldsymbol{u}, \boldsymbol{v})$	$\omega^\mu{}_\nu\omega^{\mu'}{}_{\nu'}T^{\nu\nu'}$	$T^{\mu\nu}(-\boldsymbol{u}, \boldsymbol{v})$	$T^{\mu\nu}(-\boldsymbol{u}, \boldsymbol{v})$	$T^{\mu\nu}(\boldsymbol{u}, \boldsymbol{v})$

G. Free particles

$$\psi^{(+)}(x) = e^{-ipx}u_{\boldsymbol{p}}, \quad \psi^{(-)}_p(x) = e^{ipx}v_{\boldsymbol{p}}, \qquad 0 = (\not{p} - m)u_{\boldsymbol{p}} = (\not{p} + m)v_{\boldsymbol{p}}$$

$p^0 = \omega_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2} \geq 0$, $p^2 = m^2c^2$

$\boldsymbol{p} \neq 0$: $p_0^\mu = (mc, \mathbf{0})$,

$$
\begin{aligned}
0 &= (\gamma^0 - 1)u_{\mathbf{0}} = (\gamma^0 + 1)v_{\mathbf{0}} \\
u_{\mathbf{0}}^{(1)} &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi^{(1)} \\ 0 \end{pmatrix}, \quad u_{\mathbf{0}}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi^{(2)} \\ 0 \end{pmatrix}, \\
v_{\mathbf{0}}^{(1)} &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi^{(1)} \end{pmatrix}, \quad v_{\mathbf{0}}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi^{(2)} \end{pmatrix}.
\end{aligned}
$$

$\boldsymbol{p} \neq 0$: $p^\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$, $(\not{p} \pm m)(\not{p} \mp m) = p^2 - m^2$, $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$

$$
\begin{aligned}
u_{\boldsymbol{p}}^{(\alpha)} &= \frac{\not{p} + m}{\sqrt{2m(m + \omega_p)}} u_{\mathbf{0}}^{(\alpha)} = \begin{pmatrix} \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}} \phi^{(\alpha)} \\ \frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} \phi^{(\alpha)} \end{pmatrix} \\
v_{\boldsymbol{p}}^{(\alpha)} &= \frac{-\not{p} + m}{\sqrt{2m(m + \omega_p)}} v_{\mathbf{0}}^{(\alpha)} = \begin{pmatrix} \frac{\boldsymbol{\sigma p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} \chi^{(\alpha)} \\ \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}} \chi^{(\alpha)} \end{pmatrix}
\end{aligned}
$$

Normalization: $\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}$

$$
\begin{aligned}
\bar{u}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} &= \frac{u_{\mathbf{0}}^{(\alpha)\dagger} (\not{p}^\dagger + m) \gamma^0 (\not{p} + m) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} = \frac{u_{\mathbf{0}}^{(\alpha)\dagger} \gamma^{02} (\not{p}^\dagger + m) \gamma^0 (\not{p} + m) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} \\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)} (\not{p} + m)(\not{p} + m) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} = \frac{\bar{u}_{\mathbf{0}}^{(\alpha)} (m^2 + \not{p}^2 + 2m\not{p}) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} \\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)} [m^2 + p^2 + 2m(\omega_p \gamma^0 - \boldsymbol{p}\boldsymbol{\gamma})] u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} = \frac{\bar{u}_{\mathbf{0}}^{(\alpha)} (m^2 + \omega_p^2 - \boldsymbol{p}^2 + 2m\omega_p) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} \\
&= \frac{\bar{u}_{\mathbf{0}}^{(\alpha)} (m^2 + m^2 + \boldsymbol{p}^2 - \boldsymbol{p}^2 + 2m\omega_p) u_{\mathbf{0}}^{(\beta)}}{2m(m + \omega_p)} = 1 \\
\bar{u}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} &= -\bar{v}_{\boldsymbol{p}}^{(\alpha)} v_{\boldsymbol{p}}^{(\beta)} = \delta_{\alpha,\beta}, \qquad \bar{u}_{\boldsymbol{p}}^{(\alpha)} v_{\boldsymbol{p}}^{(\beta)} = \bar{v}_{\boldsymbol{p}}^{(\alpha)} u_{\boldsymbol{p}}^{(\beta)} = 0
\end{aligned}
$$

Projection to positive and negative energy: momentum-dependence as for scalar particles

$$
\begin{aligned}
P_+(\boldsymbol{p}) &= \sum_{\alpha=1}^{2} u_{\boldsymbol{p}}^{(\alpha)} \otimes \bar{u}_{\boldsymbol{p}}^{(\alpha)} = \frac{\not{p} + m}{\sqrt{2m(m + \omega_{\boldsymbol{p}})}} \frac{1 + \gamma^0}{2} \frac{\not{p} + m}{\sqrt{2m(m + \omega_{\boldsymbol{p}})}} = \frac{\not{p} + m}{2m} \\
P_-(\boldsymbol{p}) &= -\sum_{\alpha=1}^{2} v_{\boldsymbol{p}}^{(\alpha)} \otimes \bar{v}_{\boldsymbol{p}}^{(\alpha)} = \frac{\not{p} - m}{\sqrt{2m(m + \omega_{\boldsymbol{p}})}} \frac{1 - \gamma^0}{2} \frac{\not{p} - m}{\sqrt{2m(m + \omega_{\boldsymbol{p}})}} = \frac{m - \not{p}}{2m}
\end{aligned}
$$

$$(\not{p} - m)u_{\boldsymbol{p}} = (\not{p} + m)v_{\boldsymbol{p}} = 0$$

Bi-spinor: (2 spin) $\times$ ($\pm$ energy)

H. Helicity, chirality, Weyl and Majorana fermions, twistors

Bi-spinor: 8 real (4 complex) dimensional space: $8 = 4 + 4$

- **Helicity:**

1. Rotation invariant spin projection: projection of the total angular momentum on the momentum,

$$\begin{aligned} h_{\boldsymbol{p}} &= \frac{\boldsymbol{J}\boldsymbol{p}}{|\boldsymbol{p}|}, \quad \boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S} \\ &= \frac{\boldsymbol{S}\boldsymbol{p}}{|\boldsymbol{p}|}, \end{aligned}$$

Dirac fermion: $h = \pm\frac{\hbar}{2}$.

2. Conservation: $S_j = \frac{1}{2}\epsilon_{jk\ell}\sigma_{k\ell}$,

$$\begin{aligned} [\boldsymbol{\alpha}\boldsymbol{p} + \beta m, \boldsymbol{p}\boldsymbol{S}] &= \left[\boldsymbol{p}\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} + m\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}, \boldsymbol{p}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right] \\ &= p^j p^k \left[\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\right] + mp^k \left[\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}, \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}\right] \\ &= p^j p^k [\sigma_j, \sigma_k] \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix} + mp^k [1\!\!1_2, \sigma_k] \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix} = 0 \end{aligned}$$

3. Spatial rotation: invariant

4. Lorentz boost: $\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\frac{v^2}{c^2}}}$,

   (a) $L_{-\lambda\boldsymbol{v}}$, $\lambda > 1$ such that $\boldsymbol{v}$ changes sign

   (b) $\boldsymbol{p} = p\boldsymbol{n}$, $\boldsymbol{S}\boldsymbol{n} = \frac{1}{2}n_j\epsilon_{jk\ell}\sigma^{k\ell} = \frac{1}{2}n_j\epsilon_{jk\ell}\frac{i}{2}[\gamma^k, \gamma^\ell] = \frac{i}{2}n_j\epsilon_{jk\ell}\gamma^k\gamma^\ell = \frac{i}{2}\boldsymbol{\gamma}(\boldsymbol{n} \times \boldsymbol{\gamma})$

   (c) Three-vectors which are orthogonal to the boost velocity remain invariant

   (d) $\Longrightarrow \boldsymbol{S}\boldsymbol{n}$ invariant

   (e) $\boldsymbol{v}$ changes sign $\Longrightarrow h$ changes sign $\Longrightarrow$helicity is non-Lorentz invariant

   (f) A massive fermion can not be composed exclusively from a given helicity components.

- **Chirality:**

1. Definition: Eigenvalue of $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^5_{ch} = \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}$, $\gamma^5\psi = \chi\psi$, $\chi = \pm 1$

2. Projectors:

$$P_+ = R = \frac{1}{2}(1\!\!1 + \gamma^5), \quad P_- = L = \frac{1}{2}(1\!\!1 - \gamma^5),$$

3. Lorentz invariant, $\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$, $\gamma^5_{ch}\xi = \xi$, $\gamma^5_{ch}\eta = -\eta$

4. Conserved by massless particles only,

$$\begin{aligned}(p^0 + \boldsymbol{p\sigma})\eta &= m\xi, \\ (p^0 - \boldsymbol{p\sigma})\xi &= m\eta.\end{aligned}$$

5. Massless particles:

   (a) Chiral symmetry:

$$\xi \to e^{i\alpha}\xi, \quad \eta \to e^{-i\alpha}\eta, \quad \psi \to e^{i\alpha\gamma^5}\psi$$

broken by the axial anomaly at the UV cutoff (suppression of high energy modes), $\pi^0 \to \gamma + \gamma$

FIG. 1: J. Ambjorn, J. Greensite, C. Peterson, Nucl. Phys. **B221** (1983) 381

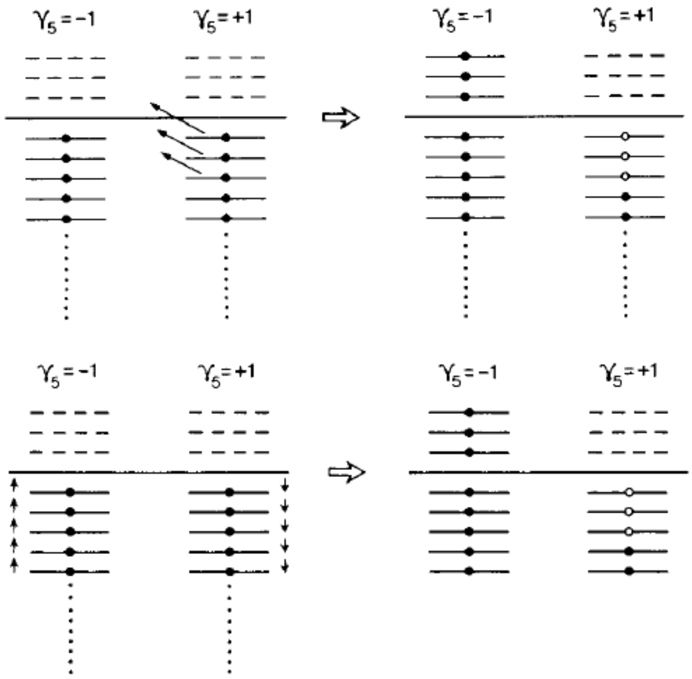


Fig. 1. (a) Explicit excitations of particles in the negative energy sea with $\gamma_5 = +1$ into positive energy states with $\gamma_5 = -1$. This transition is forbidden by a chirally invariant lagrangian. (Full and open circles denote particles and 'holes', respectively.) (b) Simultaneous shift of all energy levels in the Dirac sea due to pair creation in an external electromagnetic field.

   (b) helicity = chirality

$$\begin{aligned}\not{p}\psi &= (\omega_p\gamma^0 - \boldsymbol{p\gamma})\psi = 0, \quad \boldsymbol{p\gamma}\psi = \omega_p\gamma^0\psi \\ \boldsymbol{p}\gamma^0\boldsymbol{\gamma}\psi &= \boldsymbol{p}\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\psi = \boldsymbol{p}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}\psi = \boldsymbol{pS}\begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}\psi = |\boldsymbol{p}|h\begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}\psi \\ &= |\boldsymbol{p}|h\chi\psi = \omega_p\psi = |\boldsymbol{p}|\psi \quad \to \quad h\chi = 1, \; h,\chi = \pm 1, \quad \to \quad h = \chi\end{aligned}$$

(c) R: $\chi = 1$, $h = 1$, $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$, right hand rule

6. Weyl fermions: ξ and η respect no space inversion.

   (a) Lagrangian:

$$L_\eta = \eta^\dagger (i\partial_0 - i\boldsymbol{\nabla}\boldsymbol{\sigma})\eta, \quad L_\xi = \xi^\dagger (i\partial_0 + i\boldsymbol{\nabla}\boldsymbol{\sigma})\xi.$$

   (b) "Neutrino equations": $(p^0 + \boldsymbol{p}\boldsymbol{\sigma})\eta = (p^0 - \boldsymbol{p}\boldsymbol{\sigma})\xi = 0$

   (c) ν, seen in the mirror does not exist in Nature.

TABLE IV: Invariance properties of the splitting of the bi-spinor space.

Invariance	Helicity	Chirality	Majorana fermion
$L_+^\uparrow$	$\times$	✓	✓
Time evolution of a free particle	✓	$m \neq 0$: $\times$; $m = 0$: ✓	✓

- **Majorana fermion:**

1. Definition: The basis transformation

$$U_M = \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & i(\sigma_2 - 1) \\ i(1-\sigma_2) & 1+\sigma_2 \end{pmatrix}$$

   makes

   (a) $\gamma_M^\mu = U\gamma_{ch}^\mu U^\dagger$ imaginary,

$$\gamma_M^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma_M^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_M^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma_M^3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix},$$

   (b) $(i\gamma^\mu\partial_\mu - m)\psi = 0$ real

   (c) phase of ψ preserved

2. Majorana-Weyl fermion:

   (a) They have the same dimensionality, Majorana $\sim$ Weyl ?

   (b) Problem: the realness of a Dirac fermion, $\psi^* = \psi$, is not a Lorentz invariant condition.

   (c) Can a Majorana fermion be build up from a Weyl spinor?

   Yes:

$$\psi_{Mch} = \begin{pmatrix} \xi \\ -i\sigma_2\xi^* \end{pmatrix}$$

$$\begin{aligned}
\psi_M &= \frac{1}{2}\begin{pmatrix} 1+\sigma_2 & i(\sigma_2-1) \\ i(1-\sigma_2) & 1+\sigma_2 \end{pmatrix}\begin{pmatrix} \xi \\ -i\sigma_2\xi^* \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (1+\sigma_2)\xi+(\sigma_2-1)\sigma_2\xi^* \\ i(1-\sigma_2)\xi-i(1+\sigma_2)\sigma_2\xi^* \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (1+\sigma_2)\xi+(1-\sigma_2)\xi^* \\ i(1-\sigma_2)\xi-i(1+\sigma_2)\xi^* \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (1+\sigma_2)\xi+[(1+\sigma_2)\xi]^* \\ i(1-\sigma_2)\xi+[i(1-\sigma_2)\xi]^* \end{pmatrix} = \psi_M^*
\end{aligned}$$

(d) Mas generation?

i. Lagrangian:

$$\begin{aligned}
L_M &= \frac{1}{2}\bar\psi_M(i\partial\!\!\!/ - m)\psi_M \\
&= \frac{1}{2}[\xi^\dagger i(\partial_0 - \boldsymbol{\nabla\sigma})\xi - m\xi^{\mathrm{tr}} i\sigma_2\xi] + c.c.
\end{aligned}$$

ii. E.O.M.:

$$(\partial_0 - \boldsymbol{\nabla\sigma})\xi - m\sigma_2\xi^* = 0.$$

iii. Are the neutrinos massive?

- **Twistors:** space-time coordinates from spinors?

1. Observables are bosonic tensor operator with integer spin.
   - Observed coordinates belong to $\ell = 1$.
   - Can the elementary, microscopic representation be simpler, say $\ell = \frac{1}{2}$?
2. Idea:
   - Adjoint representation

$$x^\mu \leftrightarrow X^{a\dot a} = x^0 \mathbb{1} + \boldsymbol{x\sigma} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \leftrightarrow \nu^a\tilde\nu^{\dot a}$$

   - $d = 4 \Longrightarrow X^\dagger = X \Longrightarrow \tilde\nu = \nu^*$
   - Twistor coordinates: $X^{a\dot a}(x) = \nu^a\nu^{*\dot a}$.
3. Problem:
   - However $\nu \to e^{i\alpha}\nu$ is a symmetry $\Longrightarrow d \to d-1 = 3$ can be represented.
   - $\mathrm{Rank}(\nu^a\nu^{*\dot a}) = 1 \Longrightarrow \det X = x^\mu x_\mu = 0.$

     $\Longrightarrow$Twistors are available only for light-like vectors.
   - General vectors might be represented by a bi-twistor, $(\nu^a, \tau_b) \Longrightarrow$simplicity is lost

I. Non-relativistic limit

$$\frac{i}{c}\partial_t\psi = \left[\boldsymbol{\alpha}(-i\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}) + \beta mc + \frac{e}{c}A_0\right]\psi = \left[\boldsymbol{\alpha}\boldsymbol{\pi} + \beta mc + \frac{e}{c}A_0\right]\psi, \quad \boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$$

Pauli's equation: Standard representation, $\psi = (\phi, \chi)$,

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \\
i\partial_t\phi &= c\boldsymbol{\sigma}\boldsymbol{\pi}\chi + (eA_0 + mc^2)\phi \\
i\partial_t\chi &= c\boldsymbol{\sigma}\boldsymbol{\pi}\phi + (eA_0 - mc^2)\chi
\end{aligned}$$

Separation of the rest mass: $\Phi = e^{imc^2t}\phi$ and $X = e^{imc^2t}\chi$

$$\begin{aligned}
i\partial_t\Phi &= e^{imc^2t}(i\partial_t - mc^2)\phi = \boldsymbol{\sigma}\boldsymbol{\pi}X + eA_0\Phi \\
i\partial_t X &= e^{imc^2t}(i\partial_t - mc^2)\chi = c\boldsymbol{\sigma}\boldsymbol{\pi}\Phi + (eA_0 - 2mc^2)X \\
X &= \frac{\boldsymbol{\sigma}\boldsymbol{\pi}}{2mc}\Phi \quad \leftarrow \quad \partial_t,\ eA_0 \ll mc^2 \\
i\partial_t\Phi &= \left[\frac{(\boldsymbol{\sigma}\boldsymbol{\pi})^2}{2m} + eA_0\right]\Phi,
\end{aligned}$$

Summary: Use Schrödinger's equations with $\boldsymbol{p} \to \boldsymbol{\sigma}\boldsymbol{\pi}$,

$$i\partial_t\Phi = \left[\frac{(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2}{2m} - \frac{e}{2mc}\boldsymbol{\sigma}\boldsymbol{B} + eA_0\right]\Phi.$$

Problems:

1. Irrealistic velocity:

   (a) Representations of the time evolution:

   i. Schrödinger: $i\partial_t|\psi(t)\rangle_S = H|\psi(t)\rangle_S$, $i\partial_t A_S = 0$, $|\psi(t)\rangle_S = e^{-i(t-t_i)H}|\psi(t_i)\rangle_S$

   ii. Heisenberg: $|\psi(t)\rangle_H = e^{i(t-t_i)H}|\psi(t)\rangle_S = |\psi(t_i)\rangle_S$,

$$\begin{aligned}
\langle\psi(t)|A|\psi(t)\rangle_H &= \underbrace{\langle\psi(t_i)|e^{i(t-t_i)H}}_{\langle\psi(t)|_S}\underbrace{A}_{A_S}\underbrace{e^{-i(t-t_i)H}|\psi(t_i)\rangle_S}_{|\psi(t)\rangle_S} \\
&= \underbrace{\langle\psi(t_i)|}_{\langle\psi|_H}\underbrace{e^{i(t-t_i)H}Ae^{-i(t-t_i)H}}_{A_H(t)}\underbrace{|\psi(t_i)\rangle_S}_{|\psi\rangle_H} \\
A_H(t) &= e^{i(t-t_i)H}A_S e^{-i(t-t_i)H} \\
i\partial_t A_H(t) &= [A_H, H], \quad i\partial_t|\psi(t)\rangle_H = 0
\end{aligned}$$

   (b) Dirac equation:

$$\frac{i}{c}\partial_t\boldsymbol{x} = [\boldsymbol{x}, \boldsymbol{\alpha}\boldsymbol{p} + \beta mc^2] = i\boldsymbol{\alpha},$$

   $\alpha_j\psi = \pm\psi$, the particle moves with the speed of light.

(c) Zitterbewegung: interference between positive and negative energy solutions of the Dirac equation.

(d) Heisenberg uncertainty principle: good knowledge of $\boldsymbol{x}$ $\Longrightarrow$large momentum fluctuations.

2. Higher order time derivatives $\Longrightarrow$Ostrogadsky's instabilities, $\ddot{\boldsymbol{x}} = \frac{\boldsymbol{F}}{m}$ is Newton's great luck

   $\frac{d^jx}{dt^j} = x^{(j)}$, $L = L(x, x^{(1)}, \ldots, x^{(n)}, t)$:

   EOM:

$$\begin{aligned}\delta S[x] &= \delta \int dt L(x) = \sum_{j=0}^{n} \int dt \delta x^{(j)} \frac{\delta L}{\delta x^{(j)}} = \int dt \delta x \left(\sum_j (-1)^j \frac{d^j}{dt^j} \frac{\delta L}{\delta x^{(j)}} \right) \\ 0 &= \sum_j (-1)^j \frac{d^j}{dt^j} \frac{\delta L}{\delta x^{(j)}}\end{aligned}$$

   Energy conservation: $t \to t + \xi$, $\delta x = -\xi \dot{x}$,

$$\begin{aligned}S[\xi] &= -\sum_{j=0}^{n} \int dt \frac{d^j \xi \dot{x}}{dt^j} \frac{\delta L}{\delta x^{(j)}} = -\sum_{j=0}^{n} \int dt \left[\xi \frac{\delta L}{\delta x^{(j)}} x^{(j+1)} + \sum_{k=1}^{j} \binom{j}{k} \frac{\delta L}{\delta x^{(j)}} \xi^{(k)} x^{(j-k+1)} \right] \\ \frac{d}{dt} L &= \partial_t L + \sum_{j=0}^{n} \frac{\delta L}{\delta x^{(j)}} x^{(j+1)} \\ S[\xi] &= \int dt \xi \left(\partial_t L - \frac{d}{dt} L \right) - \sum_{j=1}^{n} \sum_{k=1}^{j} \binom{j}{k} \int dt \frac{\delta L}{\delta x^{(j)}} \xi^{(k)} x^{(j-k+1)}\end{aligned}$$

   EOM for ξ:

$$\begin{aligned}0 &= \partial_t L - \frac{d}{dt} L - \sum_{j=1}^{n} \sum_{k=1}^{j} \binom{j}{k} (-1)^k \frac{d^k}{dt^k} \left[\frac{\delta L}{\delta x^{(j)}} x^{(j-k+1)} \right] = \partial_t L + \dot{H} \\ n = 1: \quad H_1 &= \frac{\delta L}{\delta \dot{x}} \dot{x} - L \\ n \geq 2: \quad H &= H_1 + \sum_{j=2}^{n} \sum_{k=1}^{j} \binom{j}{k} (-1)^k \frac{d^{k-1}}{dt^{k-1}} \left(\frac{\delta L}{\delta x^{(j)}} x^{(j-k+1)} \right) \quad \text{non} - \text{definite !!!}\end{aligned}$$

   Instead cluster expansion:

$$S = \sum_{n \geq 1} \frac{1}{n!} \int dt_1 \cdots dt_n L_n(t_1, x(t_1), \dot{x}(t_1), \cdots, t_n, x(t_n), \dot{x}(t_n))$$

   non-definite energy !!!

Foldy-Wouthuysen transformation:

1. Basis transformation $|\psi\rangle \to S|\psi\rangle$ to decouple particles and anti-particles

   Klein-Gordon equation:

$$S_{\boldsymbol{p}} = \frac{m + \omega_p - \sigma_1(m - \omega_p)}{2\sqrt{m\omega_p}} = \cosh \kappa + \sigma_1 \sinh \kappa = e^{\kappa \sigma_1}$$

Dirac equation:

$$\begin{aligned}
S &= e^{\frac{\gamma p}{|p|}\theta} = \mathbb{1}\cos\theta + \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta, \qquad ((\boldsymbol{\gamma p})^2 = -\boldsymbol{p}^2) \\
S(\boldsymbol{\alpha p} + \beta m)S^{-1} &= \left(\mathbb{1}\cos\theta + \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)(\boldsymbol{\alpha p} + \beta m)\left(\mathbb{1}\cos\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right) \\
p^j p^k\{\gamma^j, \alpha^k\} &= p^j p^k(\gamma^j\gamma^0\gamma^k + \gamma^0\gamma^k\gamma^j) = p^j p^k \gamma^0[\gamma^j, \gamma^k] = 0 \\
S(\boldsymbol{\alpha p} + \beta m)S^{-1} &= (\boldsymbol{\alpha p} + \beta m)\left(\mathbb{1}\cos\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)^2 \\
&= (\boldsymbol{\alpha p} + \beta m)\left(\mathbb{1}(\cos^2\theta - \sin^2\theta) - 2\frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\cos\theta\sin\theta\right) \\
&= (\boldsymbol{\alpha p} + \beta m)\left(\mathbb{1}\cos 2\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin 2\theta\right) \\
(\boldsymbol{\alpha p})(\boldsymbol{\gamma p}) &= p^j p^k \alpha^k \gamma^j = p^j p^k \gamma^0 \gamma^j \gamma^k = \frac{1}{2} p^j p^k \gamma^0 \{\gamma^j, \gamma^k\} = -\beta \boldsymbol{p}^2 \\
S(\boldsymbol{\alpha p} + \beta m)S^{-1} &= \boldsymbol{\alpha p}\underbrace{\left(\cos 2\theta - \frac{m}{|\boldsymbol{p}|}\sin 2\theta\right)}_{0} + \beta(m\cos 2\theta + |\boldsymbol{p}|\sin 2\theta), \\
\tan 2\theta &= \frac{|\boldsymbol{p}|}{m}, \quad \sin 2\theta = \frac{|\boldsymbol{p}|}{\omega_p}, \quad \cos 2\theta = \frac{m}{\omega_p}, \quad H_{FW} = \beta\omega_p.
\end{aligned}$$

2. Projectors:

$$\begin{aligned}
P_\pm(\boldsymbol{p}) &= \frac{m \pm \not{p}}{2m} \\
SP_\pm(\boldsymbol{p})S^{-1} &= \left(\mathbb{1}\cos\theta + \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\frac{m \pm (\beta\omega_p - \boldsymbol{\gamma p})}{2m}\left(\mathbb{1}\cos\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right) \\
&= \frac{m \mp \boldsymbol{\gamma p}}{2m} \pm \frac{\beta\omega_p}{2m}\left(\mathbb{1}\cos\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)^2 \\
&= \frac{m \mp \boldsymbol{\gamma p}}{2m} \pm \frac{\beta\omega_p}{2m}\left(\mathbb{1}\frac{m}{\omega_p} - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\frac{|\boldsymbol{p}|}{\omega_p}\right) \qquad \leftarrow \qquad \cos^2\theta - \sin^2\theta = \frac{m}{\omega_p} \\
&= \frac{\mathbb{1} \pm \beta}{2}
\end{aligned}$$

3. Coordinate operator: $\boldsymbol{x} = i\boldsymbol{\nabla}_p$

$$\begin{aligned}
\boldsymbol{x}_{FW} &= S\boldsymbol{x}S^{-1} \\
&= \left(\mathbb{1}\cos\theta + \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right)\boldsymbol{x}\left(\mathbb{1}\cos\theta - \frac{\boldsymbol{\gamma p}}{|\boldsymbol{p}|}\sin\theta\right) = \boldsymbol{x} + i\boldsymbol{a} \\
\boldsymbol{a} &= -\frac{1}{2\omega_p}\left[\left(1 - \frac{m}{\omega_p}\right)\boldsymbol{p}\frac{\boldsymbol{\gamma p}}{\boldsymbol{p}^2} + \boldsymbol{\gamma} - (\mathbb{1} + \boldsymbol{\gamma}\otimes\boldsymbol{\gamma})\boldsymbol{p}\frac{\omega_p - m}{\boldsymbol{p}^2}\right] \\
&= -\frac{1}{2mc}\left[\boldsymbol{\gamma} - (\mathbb{1} + \boldsymbol{\gamma}\otimes\boldsymbol{\gamma})\frac{\boldsymbol{p}}{2mc} + \mathcal{O}\left(\frac{\boldsymbol{p}^2}{m^2c^2}\right)\right]
\end{aligned}$$

4. Velocity operator:

$$\frac{1}{c}\partial_t \boldsymbol{x} = -i[\boldsymbol{x}, \beta\omega_p] = \beta\boldsymbol{\nabla}_p\omega_p = \beta\boldsymbol{v}_{group}$$

J. Spherically symmetric potential

1. **Hamiltonian**

$$H = \boldsymbol{\alpha p} + \beta m + U(r), \quad \{\alpha_j, \alpha_k\} = 2\delta_{j,k}, \quad \beta^2 = \mathbb{1}, \quad \{\boldsymbol{\alpha}, \beta\} = 0$$

2. **Commuting observables:**

(a) *Total angular momentum:*

i. Angular momentum:

$$\begin{aligned}
[L_j, H] &= [L_j, \boldsymbol{\alpha p} + \beta m + U(r)] = [L_j, \boldsymbol{\alpha p}] \\
&= \alpha_m \epsilon_{jk\ell}[x_k p_\ell, p_m] = \alpha_m \epsilon_{jk\ell}[x_k, p_m] p_\ell = i\alpha_k \epsilon_{jk\ell} p_\ell = i(\boldsymbol{\alpha} \times \boldsymbol{p})_j
\end{aligned}$$

ii. Spin:

$$\begin{aligned}
S_j &= \frac{1}{2}\epsilon_{jk\ell}\sigma_{k\ell} = \frac{1}{2}\epsilon_{jk\ell}\frac{i}{2}[\gamma^k, \gamma^\ell] = \frac{i}{2}\epsilon_{jk\ell}\gamma^k\gamma^\ell = \frac{i}{2}\epsilon_{jk\ell}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_\ell \\ -\sigma_\ell & 0 \end{pmatrix} \\
&= -\frac{i}{2}\epsilon_{jk\ell}\begin{pmatrix} \sigma_k\sigma_\ell & 0 \\ 0 & \sigma_k\sigma_\ell \end{pmatrix} = -\frac{i}{2}\epsilon_{jk\ell} i\epsilon_{k\ell m}\begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix} = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \\
\beta &= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \\
[S_j, H] &= \frac{1}{2}\left[\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, p_m \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, m \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\right] \\
&= \frac{p_m}{2}\left(\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}\begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}\right) \\
&= \frac{p_m}{2}\begin{pmatrix} 0 & [\sigma_j, \sigma_m] \\ [\sigma_j, \sigma_m] & 0 \end{pmatrix} = ip_m\epsilon_{jm\ell}\begin{pmatrix} 0 & \sigma_\ell \\ \sigma_\ell & 0 \end{pmatrix} = -i(\boldsymbol{\alpha} \times \boldsymbol{p})_j
\end{aligned}$$

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \quad [\boldsymbol{J}, H] = 0$$

(b) *Spatial inversion:* $\psi_P(t, \boldsymbol{x}) = U_P\psi(t, -\boldsymbol{x}) = i\gamma^0 P\psi(t, \boldsymbol{x})$

$$\begin{aligned}
[U_p, H] &= i[P\beta, \boldsymbol{\alpha p} + \beta m + U(r)] = i[P\beta, \boldsymbol{\alpha p}] \\
&= i\left[P\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \begin{pmatrix} 0 & \boldsymbol{p\sigma} \\ \boldsymbol{p\sigma} & 0 \end{pmatrix}\right] = iP\left[\begin{pmatrix} 0 & \boldsymbol{p\sigma} \\ -\boldsymbol{p\sigma} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\boldsymbol{p\sigma} \\ \boldsymbol{p\sigma} & 0 \end{pmatrix}\right] = 0
\end{aligned}$$

(c) *Maximal set of commuting observables:* J^2, J_z, P

$$
\begin{aligned}
H\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= E\begin{pmatrix}\phi\\ \chi\end{pmatrix}\\
J^2\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= J(J+1)\begin{pmatrix}\phi\\ \chi\end{pmatrix}\\
J_z\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= M\begin{pmatrix}\phi\\ \chi\end{pmatrix}\\
P\begin{pmatrix}\phi\\ \chi\end{pmatrix} &= (-1)^\ell\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \quad J=\ell+\frac{\sigma}{2}, \quad \sigma=\pm 1
\end{aligned}
$$

3. **Spherical harmonics:**

$$
\begin{aligned}
\mathcal{Y}_{\ell J}^M &= \begin{cases}\begin{pmatrix}\sqrt{\frac{J+M}{2J}}Y_{M-\frac12}^{J-\frac12}\\ \sqrt{\frac{J-M}{2J}}Y_{M+\frac12}^{J-\frac12}\end{pmatrix} & \sigma=+1, \quad J=\ell+\frac12\\ \begin{pmatrix}\sqrt{\frac{J-M+1}{2(J+1)}}Y_{M-\frac12}^{J+\frac12}\\ -\sqrt{\frac{J+M+1}{2(J+1)}}Y_{M+\frac12}^{J+\frac12}\end{pmatrix} & \sigma=-1, \quad J=\ell-\frac12, \quad \ell>0\end{cases}\\
\psi_{\sigma J}^M &= \frac{1}{r}\begin{pmatrix}F(r)\mathcal{Y}_{J+\frac{\sigma}{2}J}^M\\ iG(r)\mathcal{Y}_{J-\frac{\sigma}{2}J}^M\end{pmatrix}
\end{aligned}
$$

$\sigma_r=\boldsymbol{\sigma}\frac{\boldsymbol{r}}{r}$, $\boldsymbol{p}\boldsymbol{\sigma}$ scalar with negative space inversion parity: $\Delta J=0$, $\Delta\ell=\pm1$

$$
\sigma_r\mathcal{Y}_{J+\frac{\sigma}{2}J}^M=-\mathcal{Y}_{J-\frac{\sigma}{2}J}^M
$$

4. **Polar coordinates:** $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_\phi)$, $\boldsymbol{x}\to(r,\phi,\theta)$, $\boldsymbol{\nabla}=\boldsymbol{e}_r\nabla_r+\boldsymbol{e}_\theta\nabla_\theta+\boldsymbol{e}_\phi\nabla_\phi$, $\alpha_r=\boldsymbol{e}_r\boldsymbol{\alpha}$

$$
\begin{aligned}
\boldsymbol{\nabla} &= \boldsymbol{e}_r(\boldsymbol{e}_r\boldsymbol{\nabla})-\boldsymbol{e}_r\times(\boldsymbol{e}_r\times\boldsymbol{\nabla})=\boldsymbol{e}_r\nabla_r-\boldsymbol{e}_r\times(\boldsymbol{e}_\phi\nabla_\theta-\boldsymbol{e}_\theta\nabla_\phi)\\
&= \boldsymbol{e}_r\nabla_r-\frac{i}{r}\boldsymbol{e}_r\times\boldsymbol{L}\\
\boldsymbol{\alpha p} &= \alpha_r\frac{1}{i}\partial_r-\frac{1}{r}\boldsymbol{\alpha}(\boldsymbol{e}_r\times\boldsymbol{L})\\
\sigma_a\sigma_b &= \delta_{ab}\mathbb{1}+i\epsilon_{abc}\sigma_c, \quad (\boldsymbol{a\alpha})(\boldsymbol{b\alpha})=\boldsymbol{ab}+2i\boldsymbol{S}(\boldsymbol{a}\times\boldsymbol{b}), \quad \boldsymbol{a}=\boldsymbol{e}_r, \quad \boldsymbol{b}=2\boldsymbol{S}\\
\alpha_r(\boldsymbol{\alpha L}) &= \boldsymbol{e}_r\boldsymbol{L}+2i\boldsymbol{S}(\boldsymbol{e}_r\times\boldsymbol{L})=2i\boldsymbol{S}(\boldsymbol{e}_r\times\boldsymbol{L}) \quad \leftarrow\times\begin{pmatrix}0 & \mathbb{1}_2\\ \mathbb{1}_2 & 0\end{pmatrix}, \quad \boldsymbol{\alpha}\leftrightarrow 2\boldsymbol{S}\\
2\alpha_r(\boldsymbol{SL}) &= i\boldsymbol{\alpha}(\boldsymbol{e}_r\times\boldsymbol{L})\\
\boldsymbol{\alpha p} &= \alpha_r\frac{1}{i}\partial_r+\frac{2i}{r}\alpha_r(\boldsymbol{SL})=\frac{1}{i}\alpha_r\left(\partial_r+\frac{1}{r}-\frac{\beta}{r}K\right)\\
\beta K &= 1+2\boldsymbol{SL}=1+\boldsymbol{J}^2-\boldsymbol{S}^2-\boldsymbol{L}^2\\
&= 1+\begin{pmatrix}J(J+1)-\frac34-(J+\frac{\sigma}{2})(J+1+\frac{\sigma}{2})\\ J(J+1)-\frac34-(J-\frac{\sigma}{2})(J+1-\frac{\sigma}{2})\end{pmatrix}=1+\begin{pmatrix}-\frac44-\sigma J-\frac{\sigma}{2}\\ -\frac44+\sigma J+\frac{\sigma}{2}\end{pmatrix}
\end{aligned}
$$

$$= \sigma \begin{pmatrix} -\frac{1}{2} - J \\ \frac{1}{2} + J \end{pmatrix} = -\sigma\beta\left(\frac{1}{2} + J\right)$$

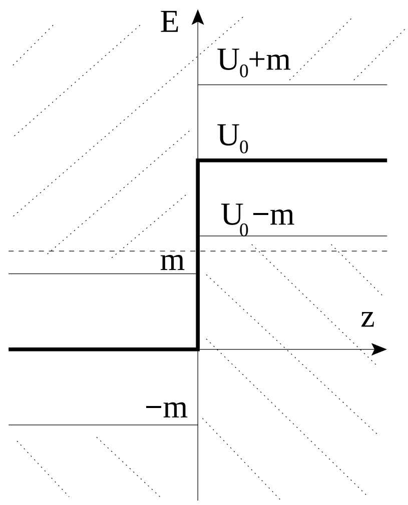
$$\boldsymbol{\alpha p}\frac{1}{r}f = \frac{1}{i}\alpha_r\left(\partial_r + \frac{1}{r} - \frac{\beta}{r}K\right)\frac{1}{r}f = \frac{1}{ir}\alpha_r\left(\partial_r - \frac{\beta}{r}K\right)f$$

5. **Stationary state:**

$$\begin{aligned}
H\psi^M_{\sigma J} &= \left[\frac{1}{i}\alpha_r\left(\partial_r + \frac{1}{r} - \frac{\beta}{r}K\right) + \beta m + U(r)\right]\frac{1}{r}\begin{pmatrix} F(r)\mathcal{Y}^M_{J+\frac{\sigma}{2}J} \\ iG(r)\mathcal{Y}^M_{J-\frac{\sigma}{2}J}\end{pmatrix} \\
E\begin{pmatrix} F(r)\mathcal{Y}^M_{J+\frac{\sigma}{2}J} \\ iG(r)\mathcal{Y}^M_{J-\frac{\sigma}{2}J}\end{pmatrix} &= \left[\frac{1}{i}\alpha_r\left(\partial_r - \frac{\beta}{r}K\right) + \beta m + U(r)\right]\begin{pmatrix} F(r)\mathcal{Y}^M_{J+\frac{\sigma}{2}J} \\ iG(r)\mathcal{Y}^M_{J-\frac{\sigma}{2}J}\end{pmatrix} \\
E\begin{pmatrix} F \\ iG \end{pmatrix} &= \left[-\frac{1}{i}\begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}\left(\partial_r + \frac{\sigma}{r}\beta\left(\frac{1}{2} + J\right)\right) + \beta m + U\right]\begin{pmatrix} F \\ iG \end{pmatrix} \\
(E - m - U)F &= \left[-\partial_r + \frac{\sigma}{r}\left(\frac{1}{2} + J\right)\right]G \\
(E + m - U)G &= \left[\partial_r + \frac{\sigma}{r}\left(\frac{1}{2} + J\right)\right]F
\end{aligned}$$

IV. KLEIN PARADOX

1+1 dimensions, $U(z) = U_0\Theta(z)$, $U_0 > 2m$, $m < E < U_0 - m$



1. **Dirac equation:**

$$\begin{aligned}
0 &= (i\partial\!\!\!/ - eA\!\!\!/ - m)\psi, \quad \psi(t,z) = \chi(z)e^{-itE}, \quad eA_0 = U, \quad \partial\!\!\!/ = \partial^\mu\partial_\mu \\
0 &= [\gamma^0(E - U(z)) + i\gamma^z\nabla_z - m]\chi(z) = 0
\end{aligned}$$

2. **Plane wave:**

$$
\begin{aligned}
\gamma^0 &= \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \\
u_{\boldsymbol{p}}^{(\alpha)} &= \frac{\not{p} + m}{\sqrt{2m(m+\omega_p)}} u_{\boldsymbol{0}}^{(\alpha)} = \frac{\begin{pmatrix} \omega_p + m & -\boldsymbol{p}\boldsymbol{\sigma} \\ \boldsymbol{p}\boldsymbol{\sigma} & -\omega_p + m \end{pmatrix}}{\sqrt{2m(m+\omega_p)}} \begin{pmatrix} \phi^{(\alpha)} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}} \phi^{(\alpha)} \\ \frac{\boldsymbol{\sigma}\boldsymbol{p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} \phi^{(\alpha)} \end{pmatrix} \\
v_{\boldsymbol{p}}^{(\alpha)} &= \frac{-\not{p} + m}{\sqrt{2m(m+\omega_p)}} v_{\boldsymbol{0}}^{(\alpha)} = \frac{\begin{pmatrix} -\omega_p + m & \boldsymbol{p}\boldsymbol{\sigma} \\ -\boldsymbol{p}\boldsymbol{\sigma} & \omega_p + m \end{pmatrix}}{\sqrt{2m(m+\omega_p)}} \begin{pmatrix} 0 \\ \chi^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{\boldsymbol{\sigma}\boldsymbol{p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} \chi^{(\alpha)} \\ \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}} \chi^{(\alpha)} \end{pmatrix}
\end{aligned}
$$

3. **Wave function:**

$$
\begin{aligned}
\chi(z) &= \Theta(-z)[\chi_i(z) + \chi_r(z)] + \Theta(z)\chi_t(z) \\
\chi_i(z) &= e^{ipz} \sqrt{\frac{2m}{m+\omega_{\boldsymbol{p}}}} \begin{pmatrix} \sqrt{\frac{m+\omega_{\boldsymbol{p}}}{2m}} \phi^{(1)} \\ \frac{\boldsymbol{\sigma}\boldsymbol{p}}{\sqrt{2m(m+\omega_{\boldsymbol{p}})}} \phi^{(1)} \end{pmatrix} = e^{ipz} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{m+E} \\ 0 \end{pmatrix}, \quad E^2 = m^2 + p^2 \\
\chi_r(z) &= be^{-ipz} \begin{pmatrix} \phi^{(1)} \\ -\frac{\boldsymbol{\sigma}\boldsymbol{p}}{m+\omega_{\boldsymbol{p}}} \phi^{(1)} \end{pmatrix} + b'e^{-ipz} \begin{pmatrix} \phi^{(2)} \\ -\frac{\boldsymbol{\sigma}\boldsymbol{p}}{m+\omega_{\boldsymbol{p}}} \phi^{(2)} \end{pmatrix} = be^{-ipz} \begin{pmatrix} 1 \\ 0 \\ -\frac{p}{m+E} \\ 0 \end{pmatrix} + b'e^{-ipz} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{p}{m+E} \end{pmatrix} \\
\chi_t(z) &= de^{ip'z} \begin{pmatrix} \phi^{(1)} \\ \frac{\boldsymbol{\sigma}\boldsymbol{p}'}{m+\omega_{\boldsymbol{p}'}} \phi^{(1)} \end{pmatrix} + d'e^{ip'z} \begin{pmatrix} \phi^{(2)} \\ \frac{\boldsymbol{\sigma}\boldsymbol{p}'}{m+\omega_{\boldsymbol{p}'}} \phi^{(2)} \end{pmatrix} = de^{ip'z} \begin{pmatrix} 1 \\ 0 \\ \frac{p'}{m+E-U_0} \\ 0 \end{pmatrix} + d'e^{ip'z} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p'}{m+E-U_0} \end{pmatrix} \\
&(E - U_0)^2 = m^2 + p'^2
\end{aligned}
$$

Spin independent potential (no spin flip): $b' = d' = 0$.

4. **Matching conditions:**

$$
\begin{aligned}
\int_{-\epsilon}^{\epsilon} dz \nabla_z \chi(z) &= i \int_{-\epsilon}^{\epsilon} dz \gamma^z [m - \gamma^0(E - U(z))]\chi(z) \\
\text{Disc}\chi(0) &= i\gamma^z \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dz [m - \gamma^0(E - U(z))]\chi(z) = 0 \\
\chi_1(0^-) &= \chi_1(0^+) \quad \to \quad 1 + b = d \\
\chi_3(0^-) &= \chi_3(0^+) \quad \to \quad (1-b)\frac{p}{m+E} = d\frac{p'}{m+E-U_0}, \quad 1 - b = d\xi, \quad \xi = \frac{p'}{p} \underbrace{\frac{m+E}{m+E-U_0}}_{\text{spin}}
\end{aligned}
$$

$$b = \frac{1-\xi}{1+\xi}, \quad d = \frac{2}{1+\xi}$$

5. **Reflection and transmission coefficients:**

$$
\begin{aligned}
j^z &= \bar{\psi}\gamma^z\psi = \psi^\dagger\gamma^0\gamma^z\psi = \chi^\dagger \begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} \chi = \chi^\dagger \begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix} \chi, \quad R = -\frac{j_r}{j_i}, \quad T = \frac{j_t}{j_i} \\
j_i &= \left(1, 0, \frac{p}{m+E}, 0\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{m+E} \\ 0 \end{pmatrix} = 2\frac{p}{m+E} \\
j_r &= |b|^2 \left(1, 0, -\frac{p}{m+E}, 0\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -\frac{p}{m+E} \\ 0 \end{pmatrix} = -2|b|^2\frac{p}{m+E} \\
j_t &= |d|^2 \left(1, 0, \frac{p'}{m+E-U_0}, 0\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p'}{m+E-U_0} \\ 0 \end{pmatrix} = 2|d|^2\frac{p'}{m+E-U_0} \\
R &= |b|^2, \quad T = |d|^2\xi = |d|^2\frac{p'}{p}\frac{m+E-U_0}{m+E} \\
R+T &= |b|^2 + |d|^2\xi = \left(\frac{1-\xi}{1+\xi}\right)^2 + \left(\frac{2}{1+\xi}\right)^2 \xi = \frac{(1-\xi)^2+4\xi}{(1+\xi)^2} = 1
\end{aligned}
$$

6. **Klein paradox:** $m < E < U_0 - m \Longrightarrow \xi = \frac{p'}{p}\frac{m+E}{m+E-U_0} < 0 \Longrightarrow T < 0$ and $R > 1$ (spin effect)

Appendix A: Renormalization group

A typical result in QED, the interaction energy of two static charges:

$$U(r) \neq \frac{e_1 e_2}{r} \quad ???$$

Q: Why is it difficult to interpret the result of any calculation in quantum field theory?

A: Because it is given in terms of the parameters of the Lagrangian which are non-physical quantities

Q: Why are the parameters of the Lagrangian non-physical?

A: Because physical quantities depend on their scale of observation and we do not know the scale of observation of these parameters

Q: Is there such a problem in quantum mechanics?

A: No because quantum mechanical problems involve few particles and cover a small scale window

TABLE V: Main chapters in quantum field theory and renormalization group

Quantum field theory	Renormalization group	Time period
renormalized theories	multiplicativ RG	1954-1970
bare theories	blocking	1971-1984
	functional RG	1984-2006
open theories	quantum RG	2006-

1. Multiplicative RG

1. **UV divergences in perturbation expansion:**

$$E_n^{(2)} = \sum_{m\neq n}\frac{|\langle\psi_m^{(0)}|H_1|\psi_n^{(0)}\rangle|^2}{E_n^{(0)}-E_m^{(0)}} < \infty \quad ?$$
$$E_{\boldsymbol p}^{(2)} = \int d^3q\frac{|\langle\psi_{\boldsymbol q}^{(0)}|H_1|\psi_{\boldsymbol p}^{(0)}\rangle|^2}{E_{\boldsymbol p}^{(0)}-E_{\boldsymbol q}^{(0)}} \approx \int_0^\infty dqq^2\frac{|\langle\psi_q^{(0)}|H_1|\psi_{|\boldsymbol p|}^{(0)}\rangle|^2}{E_{|\boldsymbol p|}^{(0)}-q} = \infty$$

2. **Regularization:** restriction to known physics, $|\boldsymbol p| < \Lambda$, below the cutoff representing our ignorance

$$E_{\boldsymbol p}^{(2)}(\Lambda) = \int d^3q\Theta(\Lambda-|\boldsymbol q|)\frac{|\langle\psi_{\boldsymbol q}^{(0)}|H_1|\psi_{\boldsymbol p}^{(0)}\rangle|^2}{E_{\boldsymbol p}^{(0)}-E_{\boldsymbol q}^{(0)}}$$

   Spatial resolution: $\ell_{min} = \frac{1}{\Lambda}$

3. **Renormalization:** How to sweep the parameter of our ignorance under the rug?

   - *Constants in the action are NOT physical parameters:* $S[x;\alpha_B]$, $S[\phi;\alpha_B]$, $\alpha_B = \{g_B,\ldots,m_B,\ldots\}$

     Physical parameters

$$P = F(\alpha_B)$$

     are complicated functions of the bare parameters

   - *Compensation of the change of the cutoff:* $\alpha_B \to \alpha_B(\Lambda)$?

   - *Renormalization conditions:* QED for electrons: $S[\psi,\bar\psi,A_\mu;e_B,m_B]$

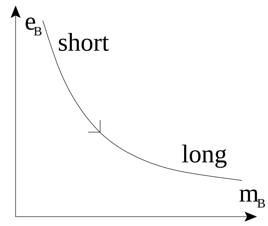
     – Naive solution:

$$P_1 = F_1(e_B,m_B), \quad P_2 = F_2(e_B,m_B) \quad\to\quad e_B,m_B$$

- Regulated solution:

$$P_1 = F_1(e_B, m_B, \Lambda), \quad P_2 = F_2(e_B, m_B, \Lambda) \quad \to \quad e_B(\Lambda), m_B(\Lambda)$$

- Renormalized trajectory: theories with different UV resolution but the same IR physics



- Interpretation: $\alpha_B(\Lambda)$ are running parameters, they parametrize the physics at scale Λ

  No constants in physics, only (slowly?) running parameters

- Theories with N parameters have $N-1$ free parameters only

• *Renormalized theory:* Removal of the cutoff $\Lambda \to \infty$, $\ell_{min} \to 0$

• *Renormalizable theory:* The finite scale physics converges as $\Lambda \to \infty$, $\ell_{min} \to 0$

$$F(e_B(\Lambda), m_B(\Lambda), \Lambda) \to P$$

• *Perturbative renormalizability by power counting:*

- Perturbation series for an observable $\langle O \rangle$:

$$\begin{aligned} \langle O \rangle &= \sum_{n=0}^{\infty} g^n I_n \\ I_n &= \int_{|\boldsymbol{p}_1|,\ldots,|\boldsymbol{p}_k|<\Lambda} d^4p_1 \cdots d^4p_k \frac{N(p_1,\ldots,p_k)}{D(p_1,\ldots,p_k)} \end{aligned}$$

- Wick rotation: $p^0 \to -ip^0$, $x^0 \to -ix^0$, $x^2 = x^{02} - \boldsymbol{x}^2 \to -(x^{02} + \boldsymbol{x}^2)$

$$I_n \to \int_{|p_1|,\ldots,|p_k|<\Lambda} d^4p_1 \cdots d^4p_k \frac{N(p_1,\ldots,p_k)}{D(p_1,\ldots,p_k)}$$

- Primitive degree of divergence: $\hbar = c = 1$, energy dimension of X: $[X]$.

$$\omega(I_n) = [I_n] = 4k + [N(p_1,\ldots,p_k)] - [D(p_1,\ldots,p_k)],$$

- UV divergence:

$$\int_{m_0}^{\Lambda} dp p^{\omega(I_n)-1} = \begin{cases} \left(\frac{\Lambda}{m_0}\right)^{\omega(I_n)} & \omega(I_n) \neq 0 \\ \ln\frac{\Lambda}{m_0} & \omega(I_n) = 0 \end{cases}$$

- Power counting:

$$[O] = n[g] + [I_n], \quad \omega(I_n) = [O] - n[g]$$

(a) $[g] < 0$: strong power divergences with increasing orders; non-renormalizable theory

(b) $[g] = 0$: the same divergence structure; renormalizable theory

(c) $[g] > 0$ finite number of UV divergent orders; super-renormalizable theory

– Example:

$$\begin{aligned}
S &= \int d^dx \left[\frac{1}{2}(\partial\phi)^2 - g_n\phi^n\right] \\
0 = [S] &= -d + 2 + 2[\phi], \quad [\phi] = \frac{d}{2} - 1 \to 1 \quad (d \to 4) \\
&= -d + [g_n] + n[\phi], \quad [g_n] = d - n\left(\frac{d}{2} - 1\right) \to 4 - n \geq 0
\end{aligned}$$

- *Multiplicative RG:* Momentum-dependent UV divergences $\Longrightarrow \phi(x) = \sqrt{Z}\phi_B(x)$
- *Problem of the perturbation expansion:*
  - $\alpha_B(\Lambda) \to \infty$
  - Asymptotic series, minimal error in order $\frac{1}{g_B} \to 0$
  - **New small parameter is needed**
- *Renormalized perturbation expansion:*
  - Renormalization conditions for physical parameters:

$$\alpha_n = F_n(\alpha_B(\Lambda), \Lambda)$$

  - Counterterms

    ϕ^4 scalar theory:

$$\begin{aligned}
L_B &= \frac{Z}{2}(\partial\phi_B)^2 - \frac{m_B^2}{2}\phi_B^2 - \frac{g_B}{4!}\phi_B^4 \\
&= \underbrace{\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4}_{L_R} + \underbrace{\frac{\delta Z}{2}(\partial\phi)^2 - \frac{\delta m^2}{2}\phi^2 - \frac{\delta g}{4!}\phi^4}_{L_{CT}}
\end{aligned}$$

    In general: $\alpha_B = \alpha + \Delta\alpha$

$$S_B[\phi_B; \alpha_B] = S_R[\phi; \alpha] + S_{CT}[\phi; \Delta\alpha]$$

  - Reordering of the perturbation series for $\Delta\alpha = O(\Lambda^n)$, $n \geq 1$ or $\Delta\alpha = O(\ln\Lambda)$

$$\sum_n b_n(\Lambda) g_B^n = \sum_n b_n(\Lambda)[g + \Delta g(\Lambda)]^n = \sum_n r_n g^n$$

  - Problems:
    * asymptotic convergent series
    * no formal expression giving $\sum_n r_n g^n$ after expansion
  - Nonperturbative theories are defined only at the bare level

2. Blocking

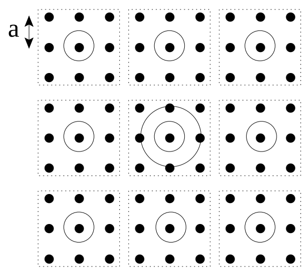
1. **Critical phenomenas:**

- *Phase transitions:*
  - Macroscopic averages (IR) are singular functions of the microscopic parameters (UV)
  - No singularity in fundamental (UV) physics laws
  - Where do critical temperature, density, pressure, etc. come from?
- *Critical phenonenon:*
  - Diverging correlation length (usually second order phase transition)
  - No perturbative approach
  - Surprising simplifications
    * critical exponents: thermodynamical singularities $(T - T_C)^{-\nu}$
    * universality: independence of the critical exponents from chemical composition, lattice structure, etc.
    * B. Widom notes that all these follow from the homogeneity of thermodynamical potentials

$$F(\lambda z) = \lambda^p F(z)$$

2. **Blocking in space-time:**

- *Block spin:* (Kadanoff) Ising model, $a \to sa$, $s = 2$



majority rule:

$$\sigma'_{j'} = F_{j'}[\sigma] = \text{sign}\left(\sum_{j \in j'} \sigma_j\right)$$

- *Blocked partition function:*

$$Z = \sum_{\{\sigma_j = \pm 1\}} e^{-H[\sigma_j]}$$

$$= \sum_{\{\sigma_j=\pm1\}} \underbrace{\sum_{\{\sigma'_{j'}=\pm1\}} \prod_{j'} \delta_{\sigma'_{j'},F_{j'}[\sigma]}}_{1} e^{-H[\sigma_j]}$$

$$= \sum_{\{\sigma'_{j'}=\pm1\}} \underbrace{\sum_{\{\sigma_j=\pm1\}} \prod_{j'} \delta_{\sigma'_{j'},F_{j'}[\sigma]} e^{-H[\sigma_j]}}_{e^{-H'[\sigma'_{j'}]}}$$

$$= \sum_{\{\sigma'_{j'}=\pm1\}} e^{-H'[\sigma'_{j'}]}$$

- *Blocking transformation of bare parameters:*

$$g'_n = B_n(g)$$

- *Physical interpretation:* $B_n(g) - g_n$ represents the contribution of modes within (a, sa) length scale interval

3. **Vicinity of the critical point:**

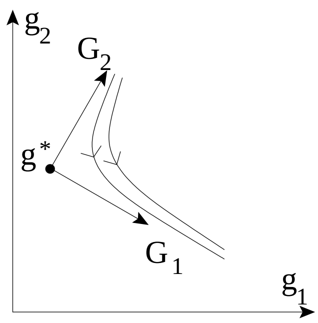
- *Blocking:*
  - lattice spacing: $a \to a' = sa$
  - coordinates: $x \to x' = \frac{x}{s}$
  - correlation length $\lambda = a\xi$, $\xi' = \frac{\xi}{s}$
  - free energy $F = a^d f$, $f' = s^d f$ in d-dimension
- *Scale invariance:* $\xi = \xi' = \infty$ at the critical point, $g^* = B(s, g^*)$
- *Around the critical point:*
  - $B(s, g)$ is analytic in g since a blocking sums up the physics in the finite interval $a < \ell < sa$
  - linearization in $\Delta g = g - g^*$:

$$\Delta g'_n = \sum_m M_{nm}(s)\Delta g_m$$

  - assuming the diagonalizability of M

$$MG_n = \lambda_n G_n$$

    * G_n: scaling operator
    * $\lambda > 1$ relevant
    * $\lambda < 1$ irrelevant
    * $\lambda = 1$ marginal



- All relevant and marginal scaling operators are needed in the Hamiltonian
- RG equation:

$$B(s_2, B(s_1, g)) = B(s_1 s_2, g)$$

- Linearized RG equation:

$$\lambda_n(s_2)\lambda_n(s_1) = \lambda_n(s_1 s_2)$$

- Continuous solution:

$$\lambda_n(s) = s^{\nu_n}$$

- Critical exponent of the H_n scaling operators

$$G_n \to s^{\nu_n} G_n$$

- Origin of singularity:

$$L_{macr} = s^N a, \quad N = \frac{\ln \frac{L_{macr}}{a}}{\ln s}$$

  $N \to \infty$ repetition of a regular blocking becomes singular

- Homogeneity: assume that all relevant operator is present in H,

$$f' = s^d f(G') \quad \to \quad f(s^{\nu_n} G_n) = s^d f(G_n)$$

- universality: independence of the critical exponent on the irrelevant parameters

4. **Statisitical Physics and Quantum Field Theory:**

- Critical phenomenon is SP: $\lambda \to \infty$, a fixed, $\frac{\lambda}{a} = \xi \to \infty$
- Renormalization in QFT: $\lambda = \lambda_C$ fixed, $a = \frac{1}{\Lambda} \to 0$, $\frac{\lambda_C}{a} \to \infty$
- Two equivalent classifications:

  renormalizable: relevant and marginal

  nonrenormalizable: irrelevant

5. **Functional RG:**

   - Physical small parameters: coupling constants, $\frac{m}{\Lambda}$, λ^ν small but finite
   - Infinitesimal paramater: infinitesimal blocking $\frac{\Delta\Lambda}{\Lambda} \to 0$ at the end of the calculation
   - Generator functionals for the Green functions

$$W[j;\Lambda] = \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n j((x_1) \cdots j(x_n) \langle 0|T[\phi(x_1) \cdots \phi(x_n)]|0\rangle$$

   - Exact differential equation for the Λ-dependence:

$$\frac{S_\Lambda[\phi]}{d\Lambda} = -\frac{\hbar}{2} \mathrm{Tr} \ln \frac{\delta^2 S_\Lambda[\phi]}{\delta\phi\delta\phi}$$

3. Quantum RG

1. **Cutoff independence:** Λ is not a fixed parameter
2. **Renormalization:** $\Lambda \to \Lambda - \Delta\Lambda$ generates mixed quantum state
3. **Environment:** Bare, cutoff theory has an environment, the particles modes in $\Lambda < E < \Lambda - \Delta\Lambda$
4. **Open QFT:** QFT is UV divergent $\Longrightarrow$need of regulator $\Longrightarrow$QFT is open
5. **CTP:** Open quantum systems:
   - bra and ket fluctuations are correlated in mixed states

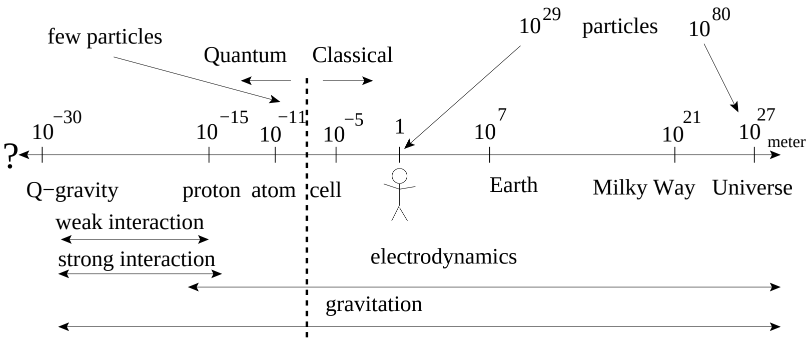
$$\langle \boldsymbol{x}_+|\rho|\boldsymbol{x}_-\rangle = \rho(\boldsymbol{x}_+, \boldsymbol{x}_-) = \psi_1(\boldsymbol{x}_+)\psi_1^*(\boldsymbol{x}_-) + \psi_2(\boldsymbol{x}_+)\psi_2^*(\boldsymbol{x}_-) + \cdots$$

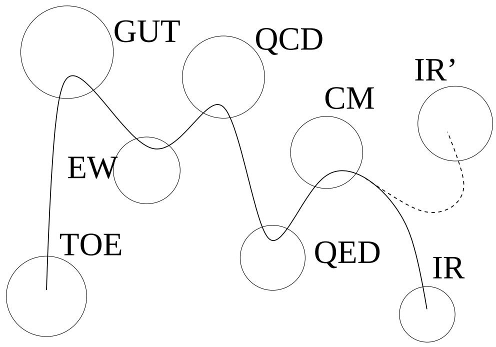
   - reduplication of degrees of freedom
     - QM: $\boldsymbol{x} \to (\boldsymbol{x}_+, \boldsymbol{x}_-)$
     - QFT: $\phi(x) \to (\phi_+(x), \phi_-(x))$
   - UV: new renormalizable coupling constants
   - IR: decoherence, classical limit
6. **Our world:**
   - *Decoupling theorem:* heavy particle M, light particle $m \ll M$

     Elimination of the heavy particle generates effective interaction terms gO in the Lagrangian with

$$g = \begin{cases} \mathcal{O}\left(\left(\frac{M}{m}\right)^n\right), & n > 0 \quad \text{super renormalizable} \\ \mathcal{O}\left(\left(\frac{m}{M}\right)^n\right), & n > 0 \quad \text{non} - \text{renormalizable} \\ \mathcal{O}\left(\ln\frac{M}{m}\right), & n > 0 \quad \text{renormalizable} \end{cases}$$

- *Effective theories:*
  - UV cutoff-dindependence $\Longrightarrow$approximative fixed points
  - the massive particles of higher energy theories deflect the renormalized trajectory from the lower energy (approximate) fixed points
- *Global renormalization group*





Oral exam questions:

1. Particles and anti-particles in special relativity

2. First order formalism of the Klein-Gorond equation

3. Mixing of partice and anti-particle components of the Klein-Gordon equation

4. Relativistic spinors

5. Free particles of the Dirac equation

6. Helicity, chirality, Weyl and Majorana fermions