MULTIPARTITE ENTANGLEMENT IN DICKE STATES

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Abstract. We study the different aspects of multipartite entanglement in multiqu*d*it Dicke states for the case of arbitrary number of subsystems of arbitrary dimension. For this we calculate the Schmidt decomposition and the reduced states of Dicke states in a general, closed form, which makes possible to obtain some important general results, as well as to show some explicit illustrations. We prove genuine multipartite entanglement of non-positive partial transpose in the reduced (mixed) Dicke states. We calculate the measures of different types of partial correlation and entanglement for pure and reduced Dicke states. We also determine the matrix product state form of Dicke states.





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1. Introduction

In 1935, Erwin Schrödinger coined the term *entanglement* [1–3], emphasizing that

> *"[entanglement] is not one, but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."*

The principle of superposition, which is a characteristic of quantum systems, leads to so-called *non-classical or quantum correlations* between subsystems. *Entanglement* is the most important manifestation of these non-classical correlations [3, 4]. Classical correlations can be created by performing "local" quantum operations on the subsystems and classical communication among them. The latter can be realized through classical interactions, using the classical systems transmitted between them. Entanglement is the correlation which cannot be created by these methods, but require genuine quantum interaction. It is stronger than the correlation in classical systems, difficult to exploit, but an important resource for achieving the *"quantum advantage"* in, for example, quantum algorithms, quantum communication, quantum metrology and quantum cryptography, which are popular research fields today [5, 6]. On the other hand, a bipartite system can be entangled or separable, but this classification becomes much more complicated in the multipartite case, which is called *partial entanglement* [7–11].

Dicke states of a composite system of many qubits are strongly but not maximally entangled states, which play an important role in quantum networks, quantum game theory, or quantum metrology. These are important bosonic (symmetric) pure quantum states, originally appeared in quantum optics [12], and later, thanks to their simple yet interesting structure, became widely used examples and tools in the theory of multipartite entanglement. Multipartite entanglement criteria in the vicinity of Dicke states were formulated [13–16], also based on quantum metrology [17–20], which made possible the experimental detection of multipartite entanglement in photonic [21] or cold atomic systems [22–25]. Entanglement was also characterized in terms of different entanglement measures in pure Dicke states and even in the mixtures of symmetric Dicke states in the qubit [26, 27] and also the qu*d*it [28–32] cases. Recently the general theory of the entanglement of mixtures of qu*d*it Dicke states [33] was also worked out. Dicke states are also important examples in tomography [34]. Dicke states of small numbers of qubits were also prepared directly in quantum optical experiments [35–38], different methods of preparation were worked out also for quantum computers [39–42], and even the matrix product state form of qu*d*it Dicke states could be derived explicitly [43].

In this thesis we study the *multipartite entanglement* of Dicke states in the general case of *multi-qudit systems*. For this first we recall the important theoretical background. In Section 2 we describe *discrete finite quantum systems* by introducing *state spaces*, with an example, the *qubit*, in more detail. (For this we also review the *Hilbert space formalism* used in quantum mechanics for the finite-dimensional case in the Appendix.) In Section 3 we present the description of *multipartite discrete finite composite systems*, then recall the *correlation* and *entanglement* of these systems. Detecting entanglement in mixed states is a difficult task, the so called *separability problem*. We also recall here the important methods used for entanglement detection, the positive maps criterion and entanglement witnesses, with some examples. We also recall the relative entropy of correlation and entanglement to quantify correlation and entanglement. The subsequent sections contain our results. In Section 4 we introduce the *elementary symmetric* or *Dicke states*, and examine the properties of these states in the *qubit* (two-dimensional) and *qudit* (arbitrary-dimensional) cases. In Section 5 we detect entanglement and correlation in the pure and reduced (mixed) Dicke states. In Section 6 we calculate the measures of elementary correlation and entanglement properties of the pure and reduced (mixed) Dicke states. In Section 7 we calculate the relative entropy of k-producibility, k-partitionability and k-stretchability correlation of the pure and reduced (mixed) Dicke states. In Section 8 we derive the *matrix*

product state (MPS) form of the Dicke state vectors. Finally, in Section 9, we summarize our results.

Our new results are as follows.

- The (4.53), (4.54) *Schmidt decomposition of Dicke states* for the arbitrary dimensional, qu*d*it case (this result is already known for qubits in the literature).
- A *new form* (4.57b) *of the generalization of the Chu-Vandermonde identity*, together with an equivalent *telescopic summation rule* (4.59) *for the generalized hypergeometric distribution.*
- The *detection of NPT-GME entanglement in the reduced Dicke states* in Section 5.2. We published this in Ref. [44], and this was also justified later, following from the general theory of the entanglement of mixtures of qu*d*it Dicke states [33].
- The evaluation of the (6.2) and (6.4) *relative entropy of ξ-correlation of pure Dicke states and reduced Dicke states.*
- The evaluation of the (6.3) *relative entropy of $X|\overline{X}$-entanglement of pure Dicke states.*
- The evaluation of the *relative entropy of k-producibility, k-partitionability and k-stretchability correlation* of pure Dicke states (7.2) and of reduced Dicke states (7.4).
- The (8.15), (8.16) and (8.17) *MPS and right canonical MPS form of Dicke states* for general-dimensional, qu*d*it elementary subsystems (this result is already known for qubits in the literature). We derived this result independently, which was also reported later in [43].

2. Quantum systems

In this section, we review the description of quantum systems using the Hilbert space methods presented in Appendix A. At the end of this section, we demonstrate the description of quantum systems by the simplest possible example, the qubit.

2.1. State spaces.

In quantum mechanics, *measurable quantities* or *observables*, are represented by normal operators. The systems where the measurements can take finite numbers of values are called *discrete finite systems*, then the observables can be represented on a finite dimensional Hilbert space [45]. In this thesis we study such systems. If the dimension of the Hilbert space is 2, then the system is called a *qubit*, if the dimension is an arbitrary number d, then it is called a *qudit*. The possible results of the measurement of an observable are the eigenvalues a_i in the spectral decomposition (A.51) of the operator, and the post-measurement states of the system are the so-called *eigenstates* represented by the eigenvectors $|\alpha_i\rangle$ of the operator. According to the *Born rule*, the system is described by a *state vector* $|\psi\rangle \in \mathcal{H}, \||\psi\rangle\| = 1$, and measuring the observable A, the probability the outcome a_i is

$$p_i = |\langle\alpha_i|\psi\rangle|^2. \tag{2.1}$$

Based on this, the *expectation value* of the observable A in the state $|\psi\rangle$ is

$$\mathbb{E}_{|\psi\rangle}(A) = \sum_i p_i a_i = \sum_i |\langle\alpha_i|\psi\rangle|^2 a_i = \sum_i a_i \langle\psi|\alpha_i\rangle\langle\alpha_i|\psi\rangle = \langle\psi|\Big[\sum_i a_i |\alpha_i\rangle\langle\alpha_i|\Big]|\psi\rangle = \langle\psi|A|\psi\rangle. \tag{2.2}$$

Introducing the operator

$$\pi := |\psi\rangle\langle\psi|, \tag{2.3}$$

which is a rank-1 projector, called a *pure state*. Using the Hilbert-Schmidt inner product (A.41), the expectation value is

$$\mathbb{E}_{|\psi\rangle}(A) = \mathrm{Tr}(\pi A) = \mathrm{Tr}(\pi^\dagger A) = (\pi|A). \tag{2.4}$$

When we consider a statistical ensemble of systems described by different state vectors $|\psi_j\rangle$, of *weights* w_i (that is, $w_j \geq 0$ and $\sum_j w_j = 1$ by definition). In this case, the expectation value of the observable A is the weighted average of the expectation values in states $|\psi_j\rangle$,

$$\mathbb{E}_{\{|\psi_j\rangle,w_j\}_j}(A) = \sum_j w_j \mathbb{E}_{|\psi_j\rangle}(A) = \sum_j \mathrm{Tr}\big[|\psi_j\rangle\langle\psi_j|A\big] = \mathrm{Tr}\Big[\Big(\sum_j w_j|\psi_j\rangle\langle\psi_j|\Big)A\Big]. \tag{2.5}$$

Introducing the operator

$$\rho := \sum_j w_j|\psi_j\rangle\langle\psi_j|, \tag{2.6}$$

which is a positive semi-definite operator of unit trace, called a *mixed state*, the expectation value is

$$\mathbb{E}_{\{|\psi_j\rangle,w_j\}_j}(A) = \mathrm{Tr}(\rho A) = \mathrm{Tr}(\rho^\dagger A) = (\rho|A). \tag{2.7}$$

Based on these, we can define the *state space*, whose elements are the operators ρ introduced above, that is,

$$\mathcal{D}(\mathcal{H}) := \Big\{\rho \in \mathrm{Lin}(\mathcal{H}) \;\Big|\; \rho = \sum_j w_j|\psi_j\rangle\langle\psi_j|, |\psi_j\rangle \in \mathcal{H}, \||\psi_j\rangle\| = 1, \sum_j w_j = 1\Big\}. \tag{2.8}$$

The operators $\rho \in \mathcal{D}(\mathcal{H})$ are called *density operators*, and the expectation value of an arbitrary observable A in the system in state ρ is

$$\mathbb{E}_\rho(A) = (\rho|A) = \mathrm{Tr}(\rho A). \tag{2.9}$$

A special point of the state space is the so-called *white noise*

$$\rho = \frac{1}{d}I, \tag{2.10}$$

where $d = \dim(\mathcal{H})$. In this state, every measurement outcome of every (non-degenerate) observable has a probability of $\frac{1}{d}$. Note that the form of the density operator written in the definition (2.8) is not a spectral decomposition, but a *general convex decomposition*, since the vectors $\{|\psi_j\rangle\}$ are not orthogonal, only normalized. The special case of this convex decomposition is the *spectral decomposition*, in which the vectors are orthonormal. For the spectral decomposition of a density operator $\rho \in \mathcal{D}(\mathcal{H})$, we will use the notation

$$\rho = \sum_i \lambda_i|\varphi_i\rangle\langle\varphi_i|. \tag{2.11}$$

It can be shown that the definition (2.8) is equivalent to

$$\mathcal{D}(\mathcal{H}) = \big\{\rho \in \mathrm{Lin}(\mathcal{H}) \;\big|\; \rho^\dagger = \rho \geq 0, \mathrm{Tr}(\rho) = 1\big\}, \tag{2.12}$$

that is, the density operators are exactly the positive semi-definite operators with unit trace. The space of pure states can also be defined, as the set in $\mathrm{Lin}(\mathcal{H})$, containing the pure states π introduced above, that is,

$$\mathcal{P}(\mathcal{H}) := \big\{\pi \in \mathrm{Lin}(\mathcal{H}) \;\big|\; \pi = |\psi\rangle\langle\psi|, |\psi\rangle \in \mathcal{H}, \||\psi\rangle\| = 1\big\}. \tag{2.13}$$

It can be shown that this definition is equivalent to

$$\mathcal{P}(\mathcal{H}) = \big\{\pi \in \mathcal{D}(\mathcal{H}) \;\big|\; \pi^2 = \pi\big\}. \tag{2.14}$$

Based on (2.8) and (2.13), the state space is the *convex hull* of the pure states,

$$\mathcal{D}(\mathcal{H}) = \mathrm{Conv}\big(\mathcal{P}(\mathcal{H})\big), \tag{2.15}$$

meaning that a density operator $\rho \in \mathcal{D}(\mathcal{H})$ can be written as a *convex combination* of pure states, $\rho = \sum_j w_j \pi_j$, where the w_j-s are the weights introduced above [4, 46]. Furthermore, it can be shown that the pure states are the *extreme points* of the state space,

$$\mathcal{P}(\mathcal{H}) = \mathrm{Extr}\big(\mathcal{D}(\mathcal{H})\big), \tag{2.16}$$

meaning that any convex decomposition of a pure state by states is trivial, that is, if a pure state $\pi \in \mathcal{P}(\mathcal{H})$ is to be written as a convex combination of density operators $\rho_i \in \mathcal{D}(\mathcal{H})$, it can only be done in one way, as $\pi = 1 \cdot \pi + \sum_i 0 \cdot \rho_i$ [4, 46]. It can be shown [4] that the *dimension* of the state space is

$$\dim\big(\mathcal{D}(\mathcal{H})\big) = \dim(\mathcal{H})^2 - 1, \tag{2.17}$$

and the (real) dimension of the space of pure states is

$$\dim\big(\mathcal{P}(\mathcal{H})\big) = 2\dim(\mathcal{H}) - 2. \tag{2.18}$$

2.2. **Qubit.** For a qubit, $\dim(\mathcal{H}) = 2$, so the dimension of the state space is $\dim\big(\mathcal{D}(\mathcal{H})\big) = 3$, and the dimension of the space of pure states is $\dim\big(\mathcal{P}(\mathcal{H})\big) = 2$. Since the half-spin can be represented on a 2-dimensional Hilbert space, we choose the generators of the SU(2) group, the so-called *Pauli operators* as the basis besides the identity operator,

$$\{I, \sigma_1, \sigma_2, \sigma_3\}. \tag{2.19}$$

The Pauli operators are self-adjoint, $\sigma_a^\dagger = \sigma_a$, traceless $\mathrm{Tr}(\sigma_a) = 0$, their multiplication rule is $\sigma_a\sigma_b = \delta_{ab}I + i\sum_{c=1}^3 \varepsilon_{abc}\sigma_c$, where ε is the Levi-Civita symbol, the totally antisymmetric $3\times3\times3$ array. The commutation relation of the Pauli operators is $[\sigma_a, \sigma_b] = \sigma_a\sigma_b - \sigma_b\sigma_a = 2i\sum_{c=1}^3 \varepsilon_{abc}\sigma_c$, and the anti-commutation relation is $\{\sigma_a, \sigma_b\} = \sigma_a\sigma_b + \sigma_b\sigma_a = 2\delta_{ab}I$. We introduce the notation

$$\boldsymbol{v}\boldsymbol{\sigma} := v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3, \tag{2.20}$$

where $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{C}^3$. Using this, we have the

$$(\boldsymbol{v}\boldsymbol{\sigma}) \cdot (\boldsymbol{u}\boldsymbol{\sigma}) = \boldsymbol{v}\boldsymbol{u}I + i(\boldsymbol{v} \times \boldsymbol{u})\boldsymbol{\sigma} \tag{2.21}$$

form of the multiplication rule, being more useful in calculations. Let an arbitrary density operator $\rho \in \mathcal{D}$ be written in the form

$$\rho = \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}). \tag{2.22}$$

Then $\mathrm{Tr}(\rho) = 1$ holds immediately. Since $\rho^\dagger = \rho$, the elements of $\boldsymbol{r}$ are all real, so $\boldsymbol{r} \in \mathbb{R}^3$. These vectors $\boldsymbol{r}$ are called *Bloch vectors*. Thus, (2.22) is an *affine mapping*

$$\begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathcal{D} \\ \boldsymbol{r} & \longmapsto & \rho(\boldsymbol{r}) \end{array} \tag{2.23}$$

because it preserves the convex combination, that is, for weights w_i

$$\rho\Big(\sum_i w_i \boldsymbol{r}_i\Big) = \sum_i w_i \rho(\boldsymbol{r}_i). \tag{2.24}$$

This tells us that the geometry of the Bloch vectors in $\mathbb{R}^3$ as the geometry of the state space $\mathcal{D}$. According to (2.14), density operator ρ describes a pure state exactly when it is idempotent, that is, $\rho^2 = \rho$. In this case, using the form (2.22) of the density operators, we get that

$$\rho^2 = \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma})\frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}) = \frac{1}{4}\Big(I + \|\boldsymbol{r}\|^2 I + 2\boldsymbol{r}\boldsymbol{\sigma}\Big) = \rho = \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}), \tag{2.25}$$

which is satisfied if $\frac{1}{2}(1 + \|\boldsymbol{r}\|^2) = 1$, that is,

$$\|\boldsymbol{r}\| = 1. \tag{2.26}$$

Thus, the possible Bloch vectors $\boldsymbol{r}$ describing a pure state form a *spherical surface* of radius 1, called the *Bloch sphere*. Based on this, the space of pure qubit states is

$$\mathcal{P} = \left\{ \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}) \;\middle|\; \boldsymbol{r} \in \mathbb{R}^3, \|\boldsymbol{r}\|^2 = 1 \right\}. \tag{2.27}$$

Since, according to (2.15), the state space is the convex hull of the pure states, the possible Bloch vectors $\boldsymbol{r}$ describing the states define a *solid sphere* of radius 1, which is called the *Bloch ball*. Based on this, the qubit state space is

$$\mathcal{D} = \mathrm{Conv}(\mathcal{P}) = \left\{ \frac{1}{2}(I + \boldsymbol{r}\boldsymbol{\sigma}) \;\middle|\; \boldsymbol{r} \in \mathbb{R}^3, \|\boldsymbol{r}\|^2 \leq 1 \right\}. \tag{2.28}$$

A special point of the state space is the white noise, $\frac{1}{2}I = \rho(\mathbf{0})$, it corresponds to the center of the Bloch ball $\boldsymbol{r} = \mathbf{0}$.

To determine the eigenvalues and eigenvectors of the density operators of the form (2.22), it is enough to determine the eigenvalues and eigenstates of the operator $\boldsymbol{r}\boldsymbol{\sigma}$. Using (2.21) $(\boldsymbol{r}\boldsymbol{\sigma})^2 = \|\boldsymbol{r}\|^2 I$ and $\mathrm{Tr}(\boldsymbol{r}\boldsymbol{\sigma}) = 0$, the squares of the eigenvalues are $\|\boldsymbol{r}\|^2$ and their sum is 0. Based on this, the two eigenvalues are $\pm\|\boldsymbol{r}\|$. Thus, the spectral decomposition of the operator $\boldsymbol{r}\boldsymbol{\sigma}$ is

$$\boldsymbol{r}\boldsymbol{\sigma} = \|\boldsymbol{r}\| \cdot |\varphi_+\rangle\langle\varphi_+| - \|\boldsymbol{r}\| \cdot |\varphi_-\rangle\langle\varphi_-|. \tag{2.29}$$

Since the eigenvectors $|\varphi_+\rangle$ and $|\varphi_-\rangle$ form a basis on $\mathcal{H}$, the identity operator can be written in the form $I = |\varphi_+\rangle\langle\varphi_+| + |\varphi_-\rangle\langle\varphi_-|$. This together with the spectral decomposition (2.29) of the operator $\boldsymbol{r}\boldsymbol{\sigma}$ imply that the dyads formed by the eigenvectors, which are pure states, can be written as

$$|\varphi_\pm\rangle\langle\varphi_\pm| = \frac{1}{2}\left(I \pm \frac{\boldsymbol{r}\boldsymbol{\sigma}}{\|r\|}\right). \tag{2.30}$$

Because of the (2.22) form of the density operators, the eigenvalues of ρ are

$$\lambda_\pm = \frac{1}{2}(1 \pm \|\boldsymbol{r}\|). \tag{2.31}$$

Based on these, the spectral decomposition of the density operator ρ can be written in the form

$$\rho = \lambda_+ |\varphi_+\rangle\langle\varphi_+| + \lambda_- |\varphi_-\rangle\langle\varphi_-|. \tag{2.32}$$

From the point of view of the Bloch sphere, the Bloch vectors $\boldsymbol{r}_\pm$ corresponding to the pure states $|\varphi_\pm\rangle\langle\varphi_\pm|$ formed by the eigenvectors of ρ can be obtained, based on (2.30), by taking the Bloch vector $\boldsymbol{r}$ belonging to $\rho(\boldsymbol{r})$, extending it in both directions until it intersects the Bloch sphere, $\boldsymbol{r}_\pm = \pm\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}$. The eigenvalues indicate the ratio by which the Bloch vector of the density operator $\rho(\boldsymbol{r})$ divides the line drawn between the two pure states, as illustrated in Figure 1.

3. Quantum theory of composite systems

In this thesis, we study composite systems consisting of multiple subsystems. In this section we review the description of these systems, as well as the correlation and entanglement, together with the mathematical methods necessary for their quantification and detection.

3.1. Systems, subsystems and state spaces.

Let the composite system consist of $n \in \mathbb{N}$ *elementary subsystems* and the Hilbert spaces used to describe these be denoted by $\mathcal{H}_a$, where the index $a \in L := [n] := \{1, 2, 3, \ldots, n\}$ denotes a given elementary subsystem. The dimensions of these spaces are $\dim(\mathcal{H}_a) =: d_a \geq 2$. Let the set of vectors

$$\left\{ |\phi_{a,j}\rangle \in \mathcal{H}_a \;\middle|\; j \in [d_a] \right\} \tag{3.1}$$

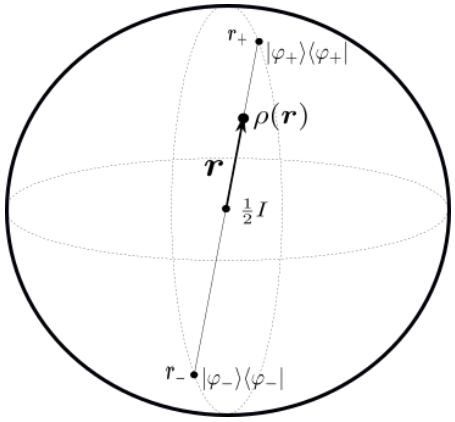


FIGURE 1. Determination of the pure states $|\varphi_\pm\rangle\langle\varphi_\pm|$.

be a basis in the Hilbert spaces of the elementary subsystems which are orthonormal, that is, for every $a \in L$

$$\langle\phi_{a,j}|\phi_{a,i}\rangle = \delta_{ji}. \tag{3.2}$$

The Hilbert space $\mathcal{H}_X$ of a general *subsystem* $X \subseteq L$ is the tensor product of the Hilbert spaces of the elementary subsystems in the subsystem, indexed by $a \in X$, that is,

$$\mathcal{H}_X = \bigotimes_{a\in X} \mathcal{H}_a. \tag{3.3}$$

Based on this, the dimension of $\mathcal{H}_X$ is the product of the dimensions of its elementary subsystems, that is, $d_X := \dim(\mathcal{H}_X) = \prod_{a\in X} d_a$. We can also find a basis in the Hilbert space $\mathcal{H}_X$ of this subsystem using the bases of the elementary subsystems in the following way

$$\left\{|\phi_{X,\boldsymbol{i}}\rangle = \bigotimes_{a\in X} |\phi_{a,i_a}\rangle \in \mathcal{H}_X\right\}. \tag{3.4}$$

Here $\boldsymbol{i} = (i_a)_{a\in X} \in \times_{a\in X}[d_a] =: \mathrm{I}_X$ is the *multiindex*, which indicates which indexed basis vector is taken in the given Hilbert space of index a. The state space $\mathcal{D}_X := \mathcal{D}(\mathcal{H}_X)$ can also be introduced on this Hilbert space and the space of pure states is $\mathcal{P}_X := \mathcal{P}(\mathcal{H}_X)$. If there is a system described by the state ρ_L, then the *reduced state* ρ_X of the subsystem $X \subseteq L$ is the one that gives the expected value of the observables $A_X \in \mathrm{Lin}(\mathcal{H}_X)$ of the subsystem. To determine this state ρ_X, let us take a subsystem $X \subseteq L$, with complement $\overline{X} = L \setminus X$, and let us have an operator $A_X \in \mathrm{Lin}(\mathcal{H}_X)$. An operator acting on the whole space $\mathcal{H}_L$ can be constructed from this, via the *embedding*

$$A_X \quad \longmapsto \quad A_L := A_X \otimes I_{\overline{X}}. \tag{3.5}$$

Let us have the state of the whole system $\rho_L \in \mathcal{D}_L$, which we can decompose as $\rho_L = \sum_k R_k \otimes S_k$, where $R_k \in \mathrm{Lin}(\mathcal{H}_X)$, $S_k \in \mathrm{Lin}(\mathcal{H}_{\overline{X}})$ are *arbitrary operators*. Based on (2.9), the expected value

of the observable A_L in this state is

$$\begin{aligned}\mathbb{E}_{\rho_L}(A_L) = (\rho_L|A_L) = \operatorname{Tr}\big[\rho_L(A_X\otimes I_{\overline{X}})\big] = \operatorname{Tr}\Big[\Big(\sum_k R_k\otimes S_k\Big)\otimes\big(A_X\otimes I_{\overline{X}}\big)\Big]\\ = \sum_k \operatorname{Tr}\Big[(R_kA_X)\otimes(S_kI_{\overline{X}})\Big] = \sum_k \operatorname{Tr}(R_kA_X)\operatorname{Tr}(S_k)\\ = \operatorname{Tr}\Big[\sum_k\big(R_k\operatorname{Tr}(S_k)\big)A_X\Big] = \operatorname{Tr}(\rho_XA_X) = (\rho_X|A_X) = \mathbb{E}_{\rho_X}(A_X),\end{aligned} \tag{3.6}$$

where by using the (A.40) partial trace map, we get that $\rho_X = \sum_k R_k\operatorname{Tr}(S_k) = \operatorname{Tr}_{\overline{X}}\big(\sum_k R_k\otimes S_k\big)$ is the *partial trace* of the original density operator ρ_L, that is,

$$\rho_X = \operatorname{Tr}_{\overline{X}}(\rho_L). \tag{3.7}$$

So we have found that the reduced state can be obtained by the partial trace map as in (3.7). We have actually seen that the Hilbert-Schmidt adjoint of the embedding (3.5) is the state reduction (3.7), as $(\rho_L|A_X\otimes I_{\overline{X}}) = (\operatorname{Tr}_{\overline{X}}(\rho_L)|A_X)$. We will also see later that the partial trace map is a quantum channel. Note, that using the (A.40) definition of the partial trace map, it is easy to see, that for arbitrary subsystems $Z\subseteq Y\subseteq X$,

$$\operatorname{Tr}_{Y\setminus Z}\circ\operatorname{Tr}_{X\setminus Y} = \operatorname{Tr}_{X\setminus Z} \tag{3.8}$$

Let us take an arbitrary vector $|\psi\rangle\in\mathcal{H}_L$ and the pure state $|\psi\rangle\langle\psi|\in\mathcal{P}_L$ formed from it. With the help of the (A.47) Schmidt decomposition, the vector $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{i=1}^r \sqrt{\eta_i}|\varphi_{X,i}\rangle\otimes|\varphi_{\overline{X},i}\rangle \tag{3.9}$$

for an arbitrary bipartition of the system consisting of subsystems X and $\overline{X}$, the Schmidt rank of the vector $|\psi\rangle$ is $r\le d_X, d_{\overline{X}}$. The reduced states of $|\psi\rangle\langle\psi|$ for the subsystems X and $\overline{X}$ can be calculated easily from the Schmidt decomposition, since the Schmidt bases are orthonormal,

$$\rho_X = \operatorname{Tr}_{\overline{X}}(|\psi\rangle\langle\psi|) = \sum_{i=1}^r \eta_i|\varphi_{X,i}\rangle\langle\varphi_{X,i}|, \tag{3.10a}$$

$$\rho_{\overline{X}} = \operatorname{Tr}_X(|\psi\rangle\langle\psi|) = \sum_{i=1}^r \eta_i|\varphi_{\overline{X},i}\rangle\langle\varphi_{\overline{X},i}|, \tag{3.10b}$$

so the nonzero part of the spectra of the two reduced states are the same, the squares of the Schmidt coefficients.

Note that if the Schmidt rank of the vector $|\psi\rangle$ is 1, $r=1$, then the vector is the elementary tensor $|\psi\rangle = |\varphi_X\rangle\otimes|\varphi_{\overline{X}}\rangle$, which, as we will later see, means that the pure state $|\psi\rangle\langle\psi|$ is uncorrelated and separable.

3.2. **Matrix product state.** The Schmidt decomposition (3.9) is a powerful tool, because it contains only one summation and the Schmidt vectors form orthonormal sets. Unfortunately there is no such powerful tensor decomposition method for dividing the system into more than two parts. Here we present a possible method, called the *matrix product state (MPS) form*, which plays a very important role in the treatment of quantum mechanical many-body systems, both in the numerical and the theoretical description [47].

Let us assume that for an arbitrary vector $|\psi\rangle\in\mathcal{H}_L$, the Schmidt decomposition is known for all bipartitions $X=\{1,2,\dots,k\}$ and $\overline{X}=\{k+1,k+2,\dots,n\}$ in the form

$$|\psi\rangle = \sum_{\alpha_k}\sqrt{\eta^{(k)}_{\alpha_k}}|\varphi^X_{\alpha_k}\rangle\otimes|\varphi^{\overline{X}}_{\alpha_k}\rangle, \tag{3.11}$$

where the index k denotes the size of the subsystem X, $k = |X|$. In these calculations, we use the notation $|\phi_i\rangle = |i\rangle$ for the basis vectors. For simplicity, let us consider a composite system consisting of $n = 4$ elementary subsystems,

$$|\psi\rangle = \sum_{i_1,i_2,i_3,i_4} \psi_{i_1,i_2,i_3,i_4} |i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |i_4\rangle. \tag{3.12}$$

In this case, the vector $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_{i_1} |i_1\rangle \otimes |\tau_{i_1}^{\{2,3,4\}}\rangle, \tag{3.13}$$

where $|\tau_{i_1}^{\{2,3,4\}}\rangle \in \mathcal{H}_{\{2,3,4\}}$. Since the Schmidt vectors $\{|\varphi_{\alpha_k}^X\rangle\}$ form a basis in the space $\mathcal{H}_X$, we can write the Schmidt basis expansion of the vector $|\tau_{i_1}^{\{2,3,4\}}\rangle$ as

$$|\tau_{i_1}^{\{2,3,4\}}\rangle = \sum_{\alpha_1} |\varphi_{\alpha_1}^{\{2,3,4\}}\rangle\langle\varphi_{\alpha_1}^{\{2,3,4\}}|\tau_{i_1}^{\{2,3,4\}}\rangle, \tag{3.14}$$

where we can relabel the scalar product as $\langle\varphi_{\alpha_1}^{\{2,3,4\}}|\tau_{i_1}^{\{2,3,4\}}\rangle =: \Gamma_{\alpha_1}^{(1)i_1}\sqrt{\eta_{\alpha_1}^{(1)}}$, that is, the corresponding Schmidt coefficient, which is nonzero in this case, is factored out. Then the form of the vector $|\psi\rangle$ becomes

$$|\psi\rangle = \sum_{i_1}\sum_{\alpha_1} \Gamma_{\alpha_1}^{(1)i_1}\sqrt{\eta_{\alpha_1}^{(1)}}|i_1\rangle \otimes |\varphi_{\alpha_1}^{\{2,3,4\}}\rangle. \tag{3.15}$$

The vector $|\varphi_{\alpha_1}^{\{2,3,4\}}\rangle$ can be written in the form $|\varphi_{\alpha_1}^{\{2,3,4\}}\rangle = \sum_{i_2} |i_2\rangle \otimes |\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle$ similarly to step (3.13), where $|\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle \in \mathcal{H}_{\{3,4\}}$. In the same way as in step (3.14), we can write the Schmidt basis expansion of the vector $|\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle$ with the basis $\{|\varphi_{\alpha_2}^{\{3,4\}}\rangle\}$ as $|\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle = \sum_{\alpha_2} |\varphi_{\alpha_2}^{\{3,4\}}\rangle\langle\varphi_{\alpha_2}^{\{3,4\}}|\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle$, where we relabel the scalar product as $\langle\varphi_{\alpha_2}^{\{3,4\}}|\tau_{\alpha_1,i_2}^{\{3,4\}}\rangle =: \Gamma_{\alpha_1,\alpha_2}^{(2)i_2}\sqrt{\eta_{\alpha_2}^{(2)}}$. Substituting this back into (3.15), we get

$$|\psi\rangle = \sum_{i_1,i_2}\sum_{\alpha_1,\alpha_2} \Gamma_{\alpha_1}^{(1)i_1}\sqrt{\eta_{\alpha_1}^{(1)}}\Gamma_{\alpha_1,\alpha_2}^{(2)i_2}\sqrt{\eta_{\alpha_2}^{(2)}}|i_1\rangle \otimes |i_2\rangle \otimes |\varphi_{\alpha_2}^{\{3,4\}}\rangle. \tag{3.16}$$

Similarly to the previous steps, we can decompose the vector $|\varphi_{\alpha_2}^{\{3,4\}}\rangle$, and then we get that

$$|\psi\rangle = \sum_{i_1,i_2,i_3}\sum_{\alpha_1,\alpha_2,\alpha_3} \Gamma_{\alpha_1}^{(1)i_1}\sqrt{\eta_{\alpha_1}^{(1)}}\Gamma_{\alpha_1,\alpha_2}^{(2)i_2}\sqrt{\eta_{\alpha_2}^{(2)}}\Gamma_{\alpha_2,\alpha_3}^{(3)i_3}\sqrt{\eta_{\alpha_3}^{(3)}}|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |\varphi_{\alpha_3}^{\{4\}}\rangle. \tag{3.17}$$

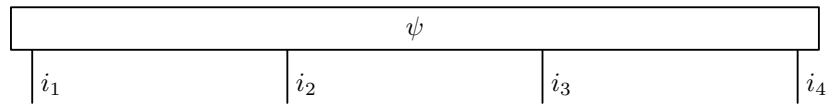
Finally, let us expand the vector $|\varphi_{\alpha_3}^{\{4\}}\rangle$ as $|\varphi_{\alpha_3}^{\{4\}}\rangle = \sum_{i_4} |i_4\rangle\langle i_4|\varphi_{\alpha_3}^{\{4\}}\rangle$, and rename the resulting scalar product as $\Gamma_{\alpha_3}^{(4)i_4}$. Substituting this back into (3.17), we obtain the MPS form of the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{i_1,i_2,i_3,i_4}\sum_{\alpha_1,\alpha_2,\alpha_3} \Gamma_{\alpha_1}^{(1)i_1}\sqrt{\eta_{\alpha_1}^{(1)}}\Gamma_{\alpha_1,\alpha_2}^{(2)i_2}\sqrt{\eta_{\alpha_2}^{(2)}}\Gamma_{\alpha_2,\alpha_3}^{(3)i_3}\sqrt{\eta_{\alpha_3}^{(3)}}\Gamma_{\alpha_3}^{(4)i_4}|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |i_4\rangle, \tag{3.18}$$

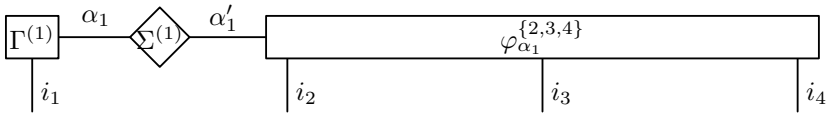
where the indices i_1, i_2, i_3, i_4 are the so-called *physical indices*, and the indices $\alpha_1, \alpha_2, \alpha_3$ are the so-called *virtual indices*. Note that in the summation over the virtual indices α_k, we did not write out the upper limit of the summation, but based on (3.9), for a given X and $\overline{X}$, it will be $b_k \leq \min(d_X, d_{\overline{X}})$, which is called the *bond dimension*. If we introduce the

$$\Sigma_{\alpha_k,\alpha_k'}^{(k)} := \sqrt{\eta_{\alpha_k}^{(k)}}\delta_{\alpha_k,\alpha_k'} \tag{3.19}$$

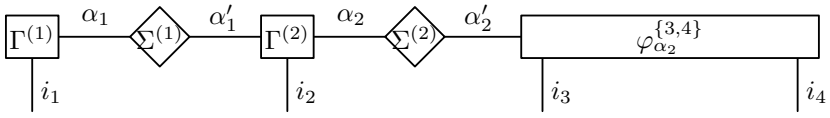
(3.12)



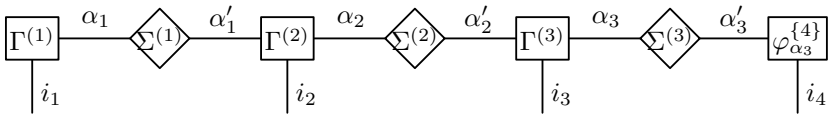
(3.15)



(3.16)



(3.17)



(3.18)

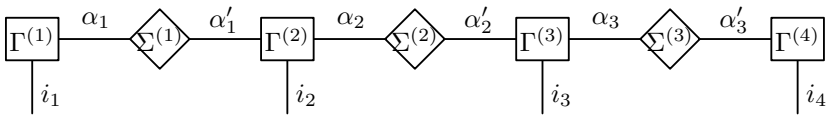


FIGURE 2. Demonstration of the steps of deriving the MPS form of a vector $|\psi\rangle$ consisting of 4 elementary subsystems.

matrices, the MPS takes the form

$$|\psi\rangle = \sum_{i_1,i_2,i_3,i_4} \Gamma^{(1)i_1}\Sigma^{(1)}\Gamma^{(2)i_2}\Sigma^{(2)}\Gamma^{(3)i_3}\Sigma^{(3)}\Gamma^{(4)i_4}|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |i_4\rangle \tag{3.20}$$

with the matrices $\Sigma^{(k)}$ and $\Gamma^{(k),i_k}$. Using that the vectors $|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |i_4\rangle$ form a basis in the space $\mathcal{H}_L$, and the vector $|\psi\rangle$ is written in this basis in the form $\sum_{i_1,i_2,i_3,i_4} \psi_{i_1,i_2,i_3,i_4}|i_1\rangle \otimes |i_2\rangle \otimes |i_3\rangle \otimes |i_4\rangle$, then the expansion coefficients of the vector $|\psi\rangle$ are obtained in the *matrix product state* form,

$$\psi_{i_1,i_2,i_3,i_4} = \Gamma^{(1)i_1}\Sigma^{(1)}\Gamma^{(2)i_2}\Sigma^{(2)}\Gamma^{(3)i_3}\Sigma^{(3)}\Gamma^{(4)i_4}. \tag{3.21}$$

In general, the representation (3.20) or (3.21) is called the *matrix product state form* of $|\psi\rangle$. The steps of deriving the MPS are demonstrated in Figure 2.

The $\Sigma^{(k)}$ matrices are often not treated separately. By introducing the matrices $A^{(1)i_1} := \Gamma^{(1)i_1}$ and $A^{(k)i_k} := \Sigma^{(k-1)}\Gamma^{(k)i_k}$, $k > 1$, we obtain the *left canonical MPS* form of the vector $|\psi\rangle$ as

$$\psi_{i_1,i_2,i_3,i_4} = A^{(1)i_1}A^{(2)i_2}A^{(3)i_3}A^{(4)i_4}. \tag{3.22}$$

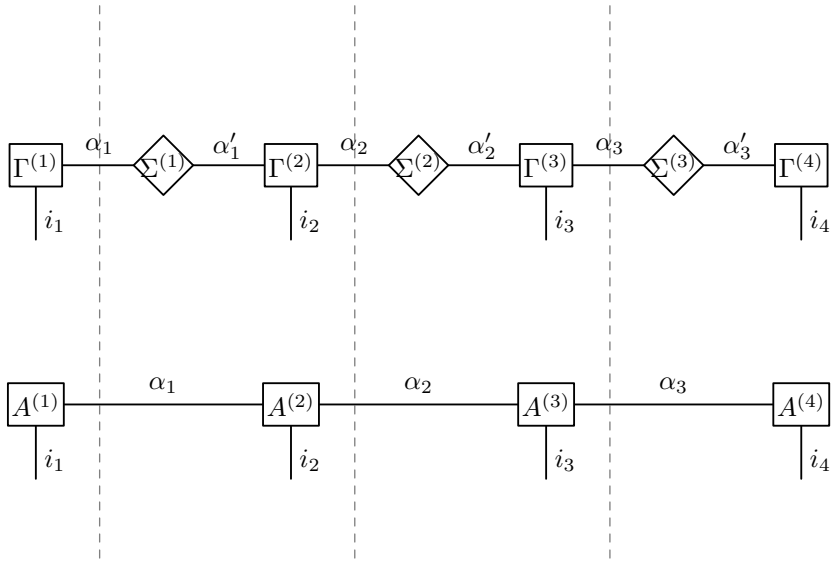


FIGURE 3. Demonstration of the left canonical MPS

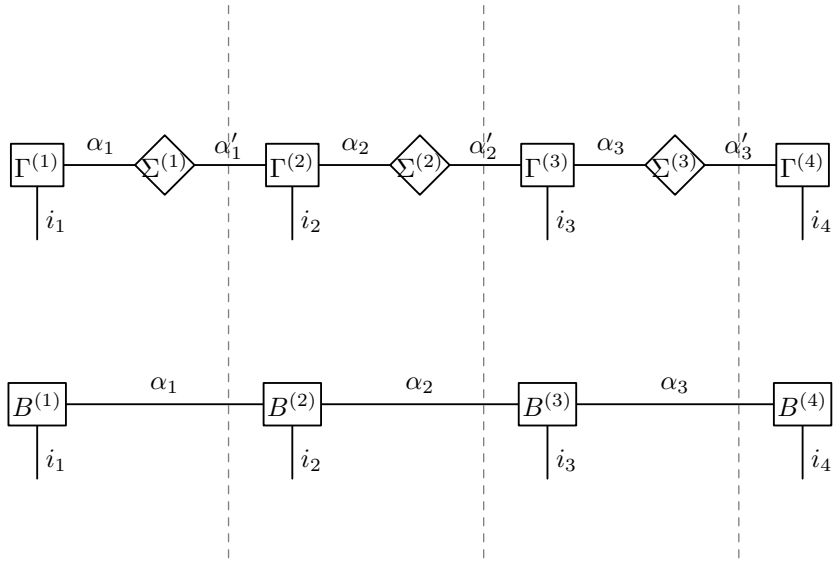


FIGURE 4. Demonstration of the right canonical MPS

The left canonical MPS is demonstrated in Figure 3.

By introducing the matrices $B^{(n)i_n} := \Gamma^{(n)i_n}$ and $B^{(k)i_k} := \Gamma^{(k)i_k}\Sigma^{(k)}$, $k < n$, we obtain the *right canonical MPS* form of the vector $|\psi\rangle$ as

$$\psi_{i_1,i_2,i_3,i_4} = B^{(1)i_1}B^{(2)i_2}B^{(3)i_3}B^{(4)i_4}. \tag{3.23}$$

The right canonical MPS is demonstrated in Figure 4.

The method presented here produces one possible MPS form, but generally, the form of the matrices in the MPS is not unique. This is called *gauge invariance*, that is, the vector $|\psi\rangle$

is invariant with respect to the *gauge transformation* $\Gamma^{(k)} \mapsto (G^{(k-1)})^{-1}\Gamma^{(k)}G^{(k)}$ and $\Sigma^{(k)} \mapsto (G^{(k)})^{-1}\Sigma^{(k)}G^{(k)}$, where $G^{(k)} \in \mathrm{GL}(b_k)$, that is, $b_k \times b_k$ invertible matrices.

The MPS form is useful because numerical experiences show that if the interactions in a spin chain are short ranged (each subsystem only interacts with its not far neighbors) and the system is not critical (far from a phase transition), then the correlations in the ground state will be short ranged (they decay exponentially with distance) and the MPS form gives a very good approximation of the state by limiting the bond dimensions [48].

3.3. **Quantum channels.** The change of quantum systems, for example a discrete step in a time evolution, is described by superoperators acting on quantum states (see Appendix A.6), called *quantum channels* [4,6]. The channels have to preserve the properties of quantum states, even if they act only on a subsystem. Using the (2.12) definition of state spaces, we can easily see that quantum channels are exactly the

$$\Phi : \mathcal{D}(\mathcal{H}) \quad \longrightarrow \quad \mathcal{D}(\mathcal{K}). \tag{3.24}$$

stochastic (completely positive, trace-preserving, CPTP) superoperators.

We call a quantum channel *deterministic*, if it maps pure states into pure states

$$\Phi : \mathcal{P}(\mathcal{H}) \quad \longrightarrow \quad \mathcal{P}(\mathcal{K}). \tag{3.25}$$

These are not necessarily reversible, for example the (constant) map preparing a pure state $\Phi_{|\alpha\rangle\langle\alpha|}(|\psi\rangle\langle\psi|) = |\alpha\rangle\langle\alpha|$. The discrete time evolution of closed systems are deterministic and *reversible*, these are

$$\Phi_U(\rho) = \mathrm{Ad}_U(\rho) = U\rho U^\dagger, \tag{3.26}$$

where $U \in \mathrm{U}(\mathcal{H})$ is a unitary operator. This is infinitesimal generator of the *von Neumann equation*, which is the *Schrödinger equation* for density operators. The quantum channel corresponding to adding an ancillary system, described by the state $\omega \in \mathcal{D}(\mathcal{K})$, to the system is

$$\begin{aligned} \Phi_{\omega,\mathcal{K}} : \mathcal{D}(\mathcal{H}) \quad &\longrightarrow \quad \mathcal{D}(\mathcal{H}\otimes\mathcal{K}) \\ \rho \quad &\longmapsto \quad \rho\otimes\omega. \end{aligned} \tag{3.27}$$

As we have seen it before in (3.7), the quantum channel corresponding to state reduction is partial trace

$$\begin{aligned} \Phi_{\mathrm{red},\mathcal{K}} : \mathcal{D}(\mathcal{H}\otimes\mathcal{K}) \quad &\longrightarrow \quad \mathcal{D}(\mathcal{H}) \\ \rho \quad &\longmapsto \quad \mathrm{Tr}_{\mathcal{K}}(\rho) \end{aligned} \tag{3.28}$$

(note that we have seen in Appendix A.6 that the trace is bistochastic, so it is a quantum channel). The quantum channel corresponding to mixing with with a state $\omega \in \mathcal{D}(\mathcal{H})$ with weight w is

$$\begin{aligned} \Phi_{\omega,w} : \mathcal{D}(\mathcal{H}) \quad &\longrightarrow \quad \mathcal{D}(\mathcal{H}) \\ \rho \quad &\longmapsto \quad (1-w)\rho + w\omega. \end{aligned} \tag{3.29}$$

For every quantum channel Φ, there exists a $\mathcal{D}(\mathcal{K})$, $\omega \in \mathcal{D}(\mathcal{K})$ and a $U \in \mathrm{U}(\mathcal{H}\otimes\mathcal{K})$, such that for every $\rho \in \mathcal{D}(\mathcal{H})$

$$\Phi(\rho) = \mathrm{Tr}_{\mathcal{K}}\big(U(\rho\otimes\omega)U^\dagger\big), \tag{3.30}$$

or in other words

$$\Phi = \Phi_{\mathrm{red},\mathcal{K}} \circ \Phi_U \circ \Phi_{\omega,\mathcal{K}}, \tag{3.31}$$

meaning that every quantum channel or change of systems can be interpreted as an interaction with an ancillary system, which is later neglected. The (3.31) form is called the *environmental representation* of the quantum channel Φ.

In Appendix A.6 we have seen that there exist non completely-positive superoperators, for example the (A.33) transpose. Such maps are nonphysical in the sense that they can not arise in the time evolution of quantum systems. However these are still important mathematical tools in the detection of entanglement, as we will see in Section 3.8. The mere existence of such non CP superoperators is deeply related to the *entanglement* in quantum systems.

3.4. **Qubit channels.** Let us consider the one qubit channels [4], which are also called *binary channels*. These channels map qubit states to qubit states. According to (2.28), these maps are transforming Bloch vectors into Bloch vectors. So we only have to consider the maps

$$\boldsymbol{r} \quad \longrightarrow \quad \boldsymbol{r'} = \boldsymbol{T}\boldsymbol{r} + \boldsymbol{u}, \tag{3.32}$$

where $\boldsymbol{T}$ is a real 3×3 matrix and $\boldsymbol{u} = (u_1, u_2, u_3)$ is the *translation vector*, which is displacement of the white noise. Since we can diagonalize the matrix $\boldsymbol{T}$ with the help of orthogonal transformations $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$, we can write the (3.32) as

$$\boldsymbol{r'} = \boldsymbol{O}_1 \boldsymbol{D} \boldsymbol{O}_2^{\mathrm{T}} \boldsymbol{r} + \boldsymbol{u}, \tag{3.33}$$

where $\boldsymbol{D}$ is a diagonal matrix with entries η_1, η_2, η_3. If we collect these entries into a vector $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$, we get the *distortion vector* of the channel. The first term in (3.33) transforms the Bloch ball into an ellipsoid, but when we are not interested in the orientation of this ellipsoid, we neglect the orthogonal transformations $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$. So we consider the canonical form of binary channels acting on Bloch vectors as

$$\boldsymbol{r'} = \boldsymbol{D}\boldsymbol{r} + \boldsymbol{u}. \tag{3.34}$$

These maps are positive if and only if the ellipsoid $\boldsymbol{D}\boldsymbol{r}$ lies within the Bloch ball. These maps are unital (noise preserving) if $\boldsymbol{u} = 0$. According tho (A.64) Choi's theorem, a binary map is completely positive if and only if the Choi operator of the map is positive. With the help of the translation and the distortion vector, the matrix of the Choi operator reads as

$$C_\Phi = \frac{1}{2} \begin{pmatrix} 1+\eta_3+u_3 & 0 & u_1+iu_2 & \eta_1+\eta_2 \\ 0 & 1-\eta_3+u_3 & \eta_1-\eta_2 & u_1+iu_2 \\ u_1-iu_2 & \eta_1-\eta_2 & 1-\eta_3-u_3 & 0 \\ \eta_1+\eta_2 & u_1-iu_2 & 0 & 1+\eta_3-u_3 \end{pmatrix}. \tag{3.35}$$

In the case of unital channels ($\boldsymbol{u} = \boldsymbol{0}$), the eigenvalues of the Choi operator are

$$\lambda_{0,3} = \frac{1}{2}[1+\eta_3 \pm (\eta_1+\eta_2)], \qquad \lambda_{1,2} = \frac{1}{2}[1-\eta_3 \pm (\eta_1-\eta_2)]. \tag{3.36}$$

These eigenvectors are positive and therefore the map is completely positive if the *Fujiwara-Algoet conditions*

$$(1 \pm \eta_3)^2 \geq (\eta_1 \pm' \eta_2)^2 \tag{3.37}$$

hold. It turns out that this means that the $\boldsymbol{\eta}$ vector lies within a tetrahedron in $\mathbb{R}^3$.

3.5. **Elementary correlation and entanglement properties.** We consider the correlation and entanglement of multipartite composite systems with respect to different *partitions* [9,10,49]. A *partition* [50] of the system L is a set $\xi = \{X_1, X_2, \ldots, X_{|\xi|}\}$ of subsystems $X_i \in L$ whose elements are non-empty (that is, $\forall X \in \xi : X \neq \emptyset$), *disjoint* (that is, $X \neq X' \in \xi : X \cap X' = \emptyset$) and cover the entire system (that is, $\bigcup_{X\in\xi} X = L$). The set of all partitions of L is denoted by $\Pi(L)$. In the following, we use the $P_\mathrm{I} := \Pi(L)$ notation as L is fixed in this work. For arbitrary partitions $\upsilon, \xi \in P_\mathrm{I}$, we say that υ is *finer* than ξ, or ξ is *coarser* than υ, if υ can be obtained by cutting the elements of ξ. This is denoted by $\upsilon \preceq \xi$, and

$$\upsilon \preceq \xi \quad \Leftrightarrow \quad \forall Y \in \upsilon \;\; \exists X \in \xi : Y \subseteq X. \tag{3.38}$$

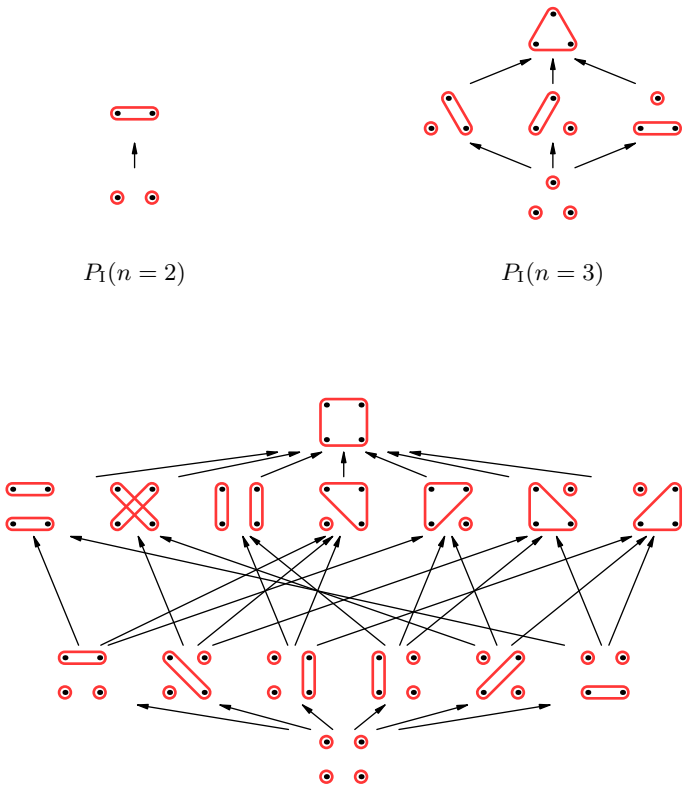


FIGURE 5. The P_{I} set of partitions of composite systems consisting of n elementary subsystems for $n = 2, 3, 4$. The arrows point from the finer partition to the coarser one.

This relation is a *partial order*, since for all $\xi, \upsilon, \zeta \in P_{\mathrm{I}}$, the relation is *reflexive* (that is, $\xi \preceq \xi$), *antisymmetric* (that is, $\xi \preceq \upsilon, \upsilon \preceq \xi \Leftrightarrow \xi = \upsilon$) and *transitive* (that is, $\xi \preceq \upsilon, \upsilon \preceq \zeta \Rightarrow \xi \preceq \zeta$). Furthermore, with respect to this partial order, the set P_{I} forms a *lattice*, that is, for all partitions $\xi, \upsilon \in P_{\mathrm{I}}$, there exists a *least upper bound*, $\xi \vee \upsilon$, and a *greatest lower bound*, $\xi \wedge \upsilon$. The *finest partition*, whose elements are the elementary subsystems, is denoted by $\bot = \{\{i\}, i \in L\} \in P_{\mathrm{I}}$. The *coarsest partition*, or *trivial partition*, whose only element is the whole system, is denoted by $\top = \{L\} \in P_{\mathrm{I}}$. All partitions of composite systems consisting of 2,3 and 4 elementary subsystems are shown in Figure 5.

For a given partition $\xi \in P_{\mathrm{I}}$, the *covariance* of the operators A_X, $X \in \xi$ is

$$\mathrm{Cov}_\xi\big(\rho, \{A_X\}_{X\in\xi}\big) = \mathbb{E}_\rho\Big(\bigotimes_{X\in\xi} A_X\Big) - \prod_{X\in\xi} \mathbb{E}_{\rho_X}(A_X). \tag{3.39}$$

Using (2.9), we get

(3.40)
$$\mathrm{Cov}_\xi\big(\rho,\{A_X\}_{X\in\xi}\big)=\Big(\rho\Big|\bigotimes_{X\in\xi}A_X\Big)-\prod_{X\in\xi}(\rho_X|A_X)=\mathrm{Tr}\Big(\rho\bigotimes_{X\in\xi}A_X\Big)-\prod_{X\in\xi}\mathrm{Tr}(\rho_XA_X)=$$
$$=\mathrm{Tr}\Big(\rho\bigotimes_{X\in\xi}A_X\Big)-\mathrm{Tr}\Big(\bigotimes_{X\in\xi}(\rho_XA_X)\Big),$$

where $\rho_X=\mathrm{Tr}_{\overline{X}}(\rho)$. Introducing the operators

$$\Gamma_\xi:=\rho-\bigotimes_{X\in\xi}\rho_X, \tag{3.41a}$$

$$A_\xi:=\bigotimes_{X\in\xi}A_X, \tag{3.41b}$$

we get that

$$\mathrm{Cov}_\xi\big(\rho,\{A_X\}_{X\in\xi}\big)=\mathrm{Tr}(\Gamma_\xi A_\xi)=(\Gamma_\xi|A_\xi). \tag{3.42}$$

If $\mathrm{Cov}_\xi\big(\rho,\{A_X\}_{X\in\xi}\big)=0$, then the *observables* $\{A_X\}_{X\in\xi}$ *are uncorrelated.*

If

$$\mathrm{Cov}_\xi\Big(\rho,\{A_X\}_{X\in\xi}\Big)=0\qquad\forall\{A_X\}_{X\in\xi}, \tag{3.43}$$

then *the state* ρ *is* ξ*-uncorrelated.* Based on (3.42) and using the definition (3.41a) of Γ_ξ, we get that a state ρ is ξ-uncorrelated if and only if it can be written in a product form, that is,

$$\rho=\bigotimes_{X\in\xi}\rho_X. \tag{3.44}$$

Based on this, the *space of* ξ*-uncorrelated states* is

$$\mathcal{D}_{\xi\text{-unc}}=\Big\{\bigotimes_{X\in\xi}\rho_X\;\Big|\;\rho_X\in\mathcal{D}_X\Big\}. \tag{3.45}$$

The pure ξ-uncorrelated states are

$$\mathcal{P}_{\xi\text{-unc}}=\mathcal{D}_{\xi\text{-unc}}\cap\mathcal{P}_L. \tag{3.46}$$

Note that these are the products of pure states, since $\mathrm{rk}\big(\bigotimes_{X\in\xi}\rho_X\big)=\prod_{X\in\xi}\mathrm{rk}(\rho_X)$. These are exactly the states

$$|\psi_L\rangle\langle\psi_L|=\bigotimes_{X\in\xi}|\psi_X\rangle\langle\psi_X|, \tag{3.47a}$$

which are *projections to the product vectors*,

$$|\psi_L\rangle=\bigotimes_{X\in\xi}|\psi_X\rangle. \tag{3.47b}$$

The elements of the remainder $\mathcal{D}_L\setminus\mathcal{D}_{\xi\text{-unc}}$ of the state space are the ξ*-correlated states*, which cannot be prepared without classical communication.

Classical probability distributions can always be obtained as a statistical mixture of product distributions, but this is not true for quantum states. This is called *entanglement. Classical statistical mixture* refers to classical communication representing classical interaction. A state is *entangled* if it cannot be created by classical mixing, but requires genuine quantum interaction. For this, we introduce the *space of* ξ*-separable states* as the convex hull of the space of ξ-uncorrelated states,

$$\mathcal{D}_{\xi\text{-sep}}=\mathrm{Conv}\big(\mathcal{D}_{\xi\text{-unc}}\big). \tag{3.48}$$

The pure ξ-separable states are uncorrelated, and these are the extremal points of the ξ-separable states

$$\mathcal{P}_{\xi\text{-sep}} = \mathcal{D}_{\xi\text{-sep}} \cap \mathcal{P} = \mathcal{P}_{\xi\text{-unc}} = \operatorname{Extr}\bigl(\mathcal{D}_{\xi\text{-sep}}\bigr), \tag{3.49}$$

which are of the form (3.47). The elements of the remainder $\mathcal{D}_L \setminus \mathcal{D}_{\xi\text{-sep}}$ of the state space are the *ξ-entangled states*, which cannot be prepared by local operations and classical communication.

It is important to see that both correlation and entanglement are *independent of the local basis choice*, so correlation and entanglement are invariant under *local* unitary transformations. They are, however, *dependent of the tensor product structure*, because this defines the subsystems, and we speak of entanglement and correlation with respect to the subsystems. So correlation and entanglement are *not* invariant under *global* unitary transformations.

It can be shown that states uncorrelated or separable with respect to a finer partition will also be uncorrelated or separable with respect to a coarser partition, that is, for arbitrary partitions $\upsilon, \xi \in P_{\mathrm{I}}$ we have the *inclusion hierarchy* [49]

$$\upsilon \preceq \xi \quad \Longleftrightarrow \quad \mathcal{D}_{\upsilon\text{-unc}} \subseteq \mathcal{D}_{\xi\text{-unc}}, \tag{3.50a}$$

$$\upsilon \preceq \xi \quad \Longleftrightarrow \quad \mathcal{D}_{\upsilon\text{-sep}} \subseteq \mathcal{D}_{\xi\text{-sep}}. \tag{3.50b}$$

The space of states uncorrelated or separable with respect to the top element (trivial partition) is equal to the state space of the system, $\mathcal{D}_{\top\text{-unc}} = \mathcal{D}_{\top\text{-sep}} = \mathcal{D}_L$, while with respect to the bottom element (finest partition), $\mathrm{D}_{\bot\text{-unc}}$ is the space of *fully uncorrelated states* and $\mathrm{D}_{\bot\text{-sep}}$ is the space of *fully separable states*.

3.6. **Correlation and entanglement properties.** There exist states that are convex mixtures of ξ-uncorrelated states for different partitions ξ but cannot be written as convex mixtures of ξ-uncorrelated states for the same partition ξ. Such states should be considered as separable, although not ξ-separable for any partition ξ. To handle these states, we introduce the different *correlation and entanglement properties*, considering multiple partitions at the same time [9, 10, 49]. The *elementary correlation and entanglement properties*, considering one single partition at the time, shown in the previous section, are special cases of these.

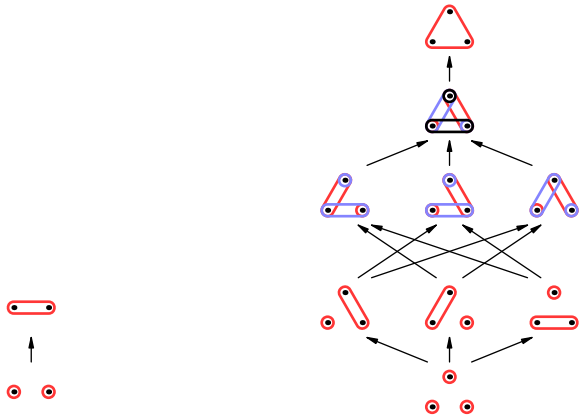
Because of the inclusion hierarchy (3.50a), when we consider ξ-uncorrelated states, we automatically consider υ-uncorrelated states for all $\upsilon \preceq \xi$. So for the labeling of the properties, we use nonempty down-sets of partitions

$$\boldsymbol{\xi} = \{\xi_1, \xi_2, \ldots, \xi_{|\boldsymbol{\xi}|}\} \subseteq P_{\mathrm{II}}, \tag{3.51}$$

called *partition ideals*. These sets are closed downwards with respect to the (3.38) $\preceq$ partial order on partitions, meaning that if an arbitrary $\xi \in \boldsymbol{\xi}$, then for every $\upsilon \preceq \xi$ we have $\upsilon \in \boldsymbol{\xi}$. The set of all nonempty partition ideals for a system L is denoted by $P_{\mathrm{II}} := \mathcal{O}_{\downarrow}(P_{\mathrm{I}}) \setminus \{\emptyset\}$. We also have a partial ordering for partition ideals. We say that a partition ideal $\boldsymbol{\upsilon}$ is finer if it is a subsystem of the coarser partition ideal $\boldsymbol{\xi}$

$$\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \quad \Leftrightarrow \quad \boldsymbol{\upsilon} \subseteq \boldsymbol{\xi}. \tag{3.52}$$

The set P_{II} also forms a lattice with the partial ordering (3.52), where the least upper bound $\vee$ is simply the union $\cup$ and the greatest lower bound $\wedge$ is simply the intersection $\cap$ of the partition ideals. For an arbitrary set of partitions $R \subseteq P_{\mathrm{I}}$, we have its *down-closure*, denoted by $\downarrow R = \{\upsilon \in P_{\mathrm{I}} \mid \exists \xi \in R : \upsilon \preceq \xi\}$, containing all the partitions which are finer than a partition in R. The down-closure always leads to an ideal. The $\boldsymbol{\xi} = \downarrow \{\xi\}$ down-closure of one-element sets is called a *principal ideal*. The finest partition ideal contains only the finest partition $\downarrow \{\bot\} = \{\bot\}$ and the coarsest partition ideal contains all the partitions in $\downarrow \{\top\} = P_{\mathrm{I}}$. We can see the lattice of the partition ideals for 2 and 3 elementary subsystems in Figure 6



$P_{\mathrm{II}}(n=2)$ $P_{\mathrm{II}}(n=3)$

FIGURE 6. The P_{II} set of partition ideals of composite systems consisting of n elementary subsystems for $n = 2, 3$. The arrows point from the finer partition ideal to the coarser one.

Based on the reasoning above, the state ρ is $\boldsymbol{\xi}$-uncorrelated, if it is ξ-uncorrelated for at least one partition $\xi \in \boldsymbol{\xi}$. Therefore the *set of $\boldsymbol{\xi}$-uncorrelated states* is

$$\mathcal{D}_{\boldsymbol{\xi}\text{-unc}} = \bigcup_{\xi\in\boldsymbol{\xi}} \mathcal{D}_{\xi\text{-unc}}. \tag{3.53}$$

These are the states that can be written as the mixture of ξ-uncorrelated states for possibly different partitions $\xi \in \boldsymbol{\xi}$. The elements of the remainder $\mathcal{D}_L \setminus \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}$ state space are the *$\boldsymbol{\xi}$-correlated states*. It is easy to see that for principal ideals we get back the elementary correlation properties,

$$\mathcal{D}_{\downarrow\{\xi\}\text{-unc}} = \mathcal{D}_{\xi\text{-unc}}. \tag{3.54}$$

Similarly to (3.48) the *set of $\boldsymbol{\xi}$-separable states* is the convex hull of the $\boldsymbol{\xi}$-uncorrelated states,

$$\mathcal{D}_{\boldsymbol{\xi}\text{-sep}} = \mathrm{Conv}(\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}). \tag{3.55}$$

The elements of the remainder $\mathcal{D}_L \setminus \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}$ of the state space are the *$\boldsymbol{\xi}$-entangled states*. It is easy to see that the pure $\boldsymbol{\xi}$-separable states are exactly the pure $\boldsymbol{\xi}$-uncorrelated states, just like in (3.49),

$$\mathcal{P}_{\boldsymbol{\xi}\text{-sep}} = \mathcal{D}_{\boldsymbol{\xi}\text{-sep}} \cap \mathcal{P}_L = \mathcal{P}_{\boldsymbol{\xi}\text{-unc}} = \mathcal{D}_{\boldsymbol{\xi}\text{-unc}} \cap \mathcal{P}_L. \tag{3.56}$$

It is easy to see that for principal ideals we get back the elementary entanglement properties,

$$\mathcal{D}_{\downarrow\{\xi\}\text{-sep}} = \mathcal{D}_{\xi\text{-sep}}. \tag{3.57}$$

From (3.52) one can see that for arbitrary partition ideals $\boldsymbol{\xi}, \boldsymbol{\upsilon} \in P_{\mathrm{II}}$ we also have the *inclusion hierarchy*,

$$\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \quad\Longleftrightarrow\quad \mathcal{D}_{\boldsymbol{\upsilon}\text{-unc}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-unc}}, \tag{3.58a}$$

$$\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \quad\Longleftrightarrow\quad \mathcal{D}_{\boldsymbol{\upsilon}\text{-sep}} \subseteq \mathcal{D}_{\boldsymbol{\xi}\text{-sep}}. \tag{3.58b}$$

Because of the correspondence (3.54) and (3.57), we have that the states of uncorrelated or separable states with respect to the top element is equal to the state space of the system, $\mathcal{D}_{\downarrow\{\top\}\text{-unc}} = \mathcal{D}_{\top\text{-unc}} = \mathcal{D}_{\downarrow\{\top\}\text{-sep}} = \mathcal{D}_{\top\text{-sep}} = \mathcal{D}_L$, while with respect to the bottom element

$\mathrm{D}_{\downarrow\{\perp\}\text{-unc}} = \mathrm{D}_{\perp\text{-unc}}$ is the space of *fully uncorrelated states* and $\mathrm{D}_{\downarrow\{\perp\}\text{-sep}} = \mathrm{D}_{\perp\text{-sep}}$ is the space of *fully separable states*.

Bipartitions (that is, partitions consisting of two subsystems, $\xi = X|\overline{X}$, labeling elementary entanglement properties) play an important role when studying entanglement. We call a state *bipartition separable* if it is separable with respect to an arbitrary bipartition. We call a state *partition separable* if it is separable with respect to a nontrivial partition $\xi \neq \top$. Because of (3.50b), if a state is partition-separable then it is also bipartition-separable. A state is *biseparable* if it is the convex mixture of the (bi)partition separable states, that are exactly the $\boldsymbol{\xi}_{\text{bisep}} = (P_{\text{I}} \setminus \{\top\}) =\downarrow \{X|\overline{X} \mid X \subset L, X \neq \emptyset\}$-separable states. A state shows *genuine multipartite entanglement (GME)* if it is not biseparable. These properties play an important role in multipartite entanglement theory and are subject of many ongoing research.

3.7. **Partitionability, producibility and stretchability of correlation and entanglement.** The structure of the multipartite correlation and entanglement properties become complicated when we increase the number of elementary subsystems. For $n = 3$ there are 5 partitions and 9 partition ideals and for $n = 4$ there are 15 partitions and 346 partition ideals. (And there are 203 partitions for $n = 6$ [51].) Although there are too many properties (partition ideals) to handle, but there are some which are more important than the others [10, 11, 49], for example, when we are only interested in the size or the number of the correlated or entangled subsystems. These properties are labeled by the *k-partitionable, k-producible and k-stretchable* partitions. For this, let us introduce the *height*, *width* and *rank* of a partition as

$$h(\xi) := |\xi|, \tag{3.59a}$$

$$w(\xi) := \max_{X \in \xi} |X|, \tag{3.59b}$$

$$r(\xi) := w(\xi) - h(\xi). \tag{3.59c}$$

The k-partitionable partitions are the partitions consisting of at least k subsystems, the k-producible partitions are the partitions consisting of subsystems of size at most k and the k-stretchable partitions are the partitions of rank at most k,

$$\boldsymbol{\xi}_{k\text{-part}} := \{\xi \in P_{\text{I}} \mid h(\xi) \geq k\} \in P_{\text{II}}, \tag{3.60a}$$

$$\boldsymbol{\xi}_{k\text{-prod}} := \{\xi \in P_{\text{I}} \mid w(\xi) \leq k\} \in P_{\text{II}}, \tag{3.60b}$$

$$\boldsymbol{\xi}_{k\text{-str}} := \{\xi \in P_{\text{I}} \mid r(\xi) \leq k\} \in P_{\text{II}}. \tag{3.60c}$$

It is easy to verify that these sets are indeed partition ideals. This form chains (totally ordered sets) with respect to the refinement (3.52),

$$\boldsymbol{\xi}_{k\text{-part}} \preceq \boldsymbol{\xi}_{l\text{-part}} \quad \Longleftrightarrow \quad k \geq l, \tag{3.61a}$$

$$\boldsymbol{\xi}_{k\text{-prod}} \preceq \boldsymbol{\xi}_{l\text{-prod}} \quad \Longleftrightarrow \quad k \leq l, \tag{3.61b}$$

$$\boldsymbol{\xi}_{k\text{-str}} \preceq \boldsymbol{\xi}_{l\text{-str}} \quad \Longleftrightarrow \quad k \leq l, \tag{3.61c}$$

which is a much simpler structure than that of the general properties. We can see the k-partitionable and k-producible partitions for 2,3 and 4 elementary subsystems in Figure 7.

Since the (3.60) sets of k-producible, k-partitionable and k-stretchable partitions are partition ideals, they define the *k-producibility, k-partitionability and k-stretchability properties* and the *spaces of k-producibly, k-partitionably and k-stretchably uncorrelated states*, using (3.53), are

$$\mathcal{D}_{k\text{-part unc}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-part}}\text{-unc}} = \left\{ \bigotimes_{X \in \xi} \rho_X \;\middle|\; \xi \in \boldsymbol{\xi}_{k\text{-part}} \right\}, \tag{3.62a}$$

$$\mathcal{D}_{k\text{-prod unc}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-prod}}\text{-unc}} = \left\{ \bigotimes_{X \in \xi} \rho_X \;\middle|\; \xi \in \boldsymbol{\xi}_{k\text{-prod}} \right\}, \tag{3.62b}$$

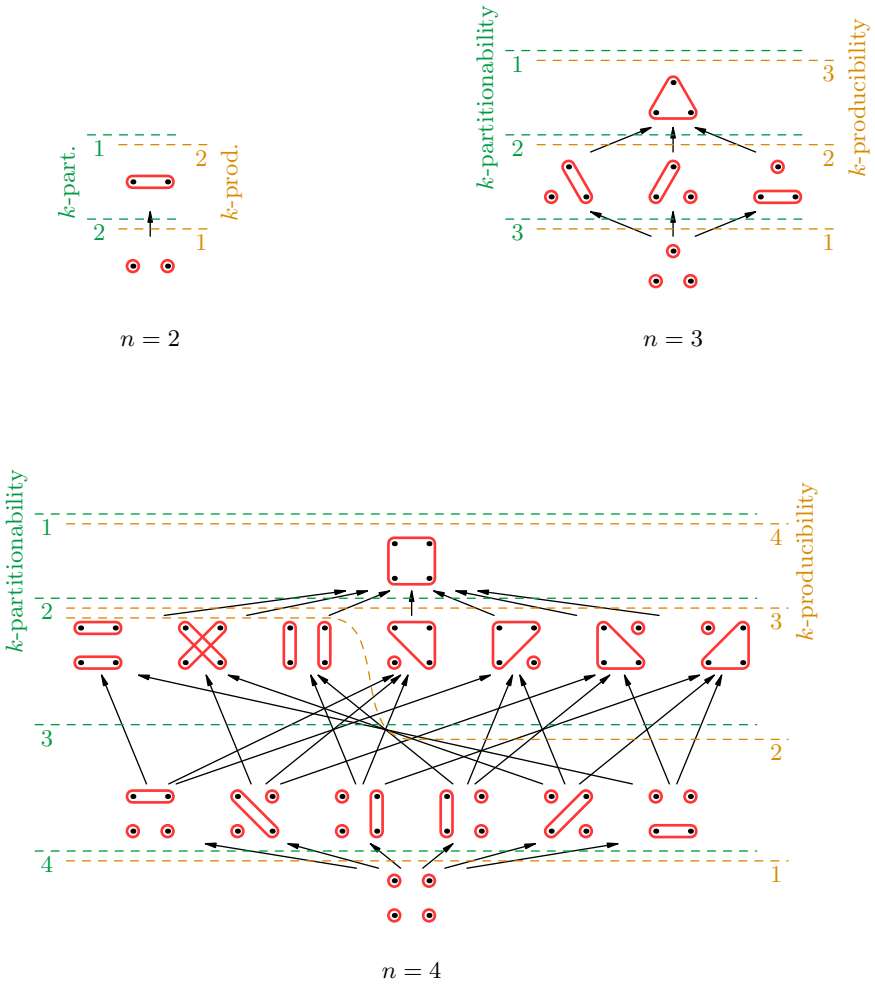


FIGURE 7. The k-partitionable and k-producible partitions in the P_{I} set of partitions of composite systems consisting of n elementary subsystems for $n = 2, 3, 4$. The arrows point from the finer partition to the coarser one.

$$\mathcal{D}_{k\text{-str unc}} := \mathcal{D}_{\boldsymbol{\xi}_{k\text{-str}}\text{-unc}} = \left\{ \bigotimes_{X\in\xi} \rho_X \;\middle|\; \xi \in \boldsymbol{\xi}_{k\text{-str}} \right\}. \tag{3.62c}$$

The *spaces of k-producibly, k-partitionably and k-stretchably separable states* are the convex hull of the k-producibly, k-partitionably and k-stretchably uncorrelated states, just like in (3.55),

$$\mathcal{D}_{k\text{-part sep}} := \mathrm{Conv}(\mathcal{D}_{k\text{-part unc}}), \tag{3.63a}$$

$$\mathcal{D}_{k\text{-prod sep}} := \mathrm{Conv}(\mathcal{D}_{k\text{-prod unc}}), \tag{3.63b}$$

$$\mathcal{D}_{k\text{-str sep}} := \mathrm{Conv}(\mathcal{D}_{k\text{-str unc}}). \tag{3.63c}$$

For example, a state is k-producibly separable if and only if it can be mixed by the use of entanglement restricted inside k-partite subsystems. Note, that $\mathcal{D}_{n\text{-part sep}} = \mathcal{D}_{1\text{-prod sep}} = \mathcal{D}_{-(n-1)\text{-str sep}}$ are the spaces of fully separable states, $\mathcal{D}_{2\text{-part sep}} = \mathcal{D}_{(n-1)\text{-prod sep}} = \mathcal{D}_{(n-3)\text{-str sep}}$ are the spaces of biseparable states, $\mathcal{D}_{1\text{-part sep}} = \mathcal{D}_{n\text{-prod sep}} = \mathcal{D}_{(n-1)\text{-str sep}} = \mathcal{D}_L$ is the whole state space. Note that for other values of k, there are no such coincidences among these properties. The

k-partitionably separable states are also called k-separable states [52, 53] and the k-producibly separable states are also called k-producible states [53] in the literature. This would lead to confusion as we consider the whole picture of correlation and entanglement.

From (3.58) and (3.61) one can see, that the sets of k-partitionable, k-producible and k-stretchable partitions also have the *nested inclusion hierarchy*

$$k \geq l \iff \mathcal{D}_{k\text{-part unc}} \subseteq \mathcal{D}_{l\text{-part unc}}, \tag{3.64a}$$

$$k \leq l \iff \mathcal{D}_{k\text{-prod unc}} \subseteq \mathcal{D}_{l\text{-prod unc}}, \tag{3.64b}$$

$$k \leq l \iff \mathcal{D}_{k\text{-str unc}} \subseteq \mathcal{D}_{l\text{-str unc}}, \tag{3.64c}$$

$$k \geq l \iff \mathcal{D}_{k\text{-part sep}} \subseteq \mathcal{D}_{l\text{-part sep}}, \tag{3.64d}$$

$$k \leq l \iff \mathcal{D}_{k\text{-prod sep}} \subseteq \mathcal{D}_{l\text{-prod sep}}, \tag{3.64e}$$

$$k \leq l \iff \mathcal{D}_{k\text{-str sep}} \subseteq \mathcal{D}_{l\text{-str sep}}. \tag{3.64f}$$

For example k-producibly separable states are also $(k+1)$-producibly separable.

Note that partitionability, producibility and stretchability are particular cases of *permutation invariant partial correlation and entanglement properties* and also of *one- parameter partial correlation and entanglement properties*. The theory of these are also worked out in details [10, 11], which we will not use in the whole generality.

3.8. **Detecting correlation and entanglement.** Having a multipartite quantum state, it is easy to decide if the state is *uncorrelated or not* with respect to a given partial correlation property. First of all, the state spaces of a given partial correlation property are just the unions of state spaces of given elementary partial correlation properties (3.53), since we do not have the convex hull construction here. Deciding the productness, on the other hand, is simple as we just have to check if the product of the marginals gives back the original state.

Having a multipartite quantum state, it would be important to be able to decide if that state is *separable or not* with respect to a given partial separability property. As we have seen it before, pure entangled states are exactly the pure correlated states, so we can easily detect entanglement in pure states, by detecting correlation. Detecting entanglement in mixed states is much harder, involving an optimization task, which is infeasible for not too large system sizes. Fortunately, at least some necessary but not sufficient criteria for separability exist [3, 54, 55]. In this section we review some of these criteria for later use. We only consider here only bipartition separability and entanglement, although some of these criteria can easily be generalized for arbitrary partitions.

Positive but not completely positive superoperators play a key role in entanglement detection, since a state $\rho \in \mathcal{D}(L)$ is separable with respect to the bipartition $X|\overline{X}$ if and only if for all positive superoperator $\Phi : \mathrm{Lin}(\mathcal{H}_{\overline{X}}) \to \mathrm{Lin}(\mathcal{K}_{\overline{X}})$, the image of the state is positive [56],

$$\rho \in \mathcal{D}_{X|\overline{X}\text{-sep}} \iff (\mathcal{I}_{\mathcal{H}_X} \otimes \Phi)\rho \geq 0. \tag{3.65}$$

This means that if we can find a positive but not completely positive superoperator which acting on a subsystem maps the state to a not positive operator, the state is entangled.

For a given positive, but not completely-positive Φ superoperator (3.65) gives a necessary but not sufficient condition of separability. For example, as we have seen it before, the (A.33) transpose is a positive but not completely positive (A.76) superoperator, thus it can be used for entanglement detection. Writing (3.65) in terms of transpose, we get the *Peres-Horodecki criterion* or *Positive partial transpose (PPT) criterion* [57],

$$\rho \in \mathcal{D}_{X|\overline{X}\text{-sep}} \implies \rho^{\mathrm{T}_X}, \rho^{\mathrm{T}_{\overline{X}}} \geq 0, \tag{3.66}$$

where we introduced the *partial transpose* map, as the transpose acting on a subsystem,

$$\begin{aligned} \mathrm{T}_X : \mathrm{Lin}(\mathcal{H}_X) \otimes \mathrm{Lin}(\mathcal{H}_{\overline{X}}) \quad &\longrightarrow \quad \mathrm{Lin}(\mathcal{H}_X) \otimes \mathrm{Lin}(\mathcal{H}_{\overline{X}}), \\ A_X \otimes B_{\overline{X}} \quad &\longmapsto \quad (A_X \otimes B_{\overline{X}})^{\mathrm{T}_X} = A_X^{\mathrm{T}} \otimes B_{\overline{X}}. \end{aligned} \tag{3.67}$$

(Note that $(\rho^{\mathrm{T}_X})^{\mathrm{T}} = \rho^{\mathrm{T}_{\overline{X}}}$, so the two part in the right hand side of (3.66) are equivalent. Note that although the transpose of an operator is basis dependent, the positivity is basis independent, so the (3.66) criterion is basis independent.) Therefore, the (3.66) *PPT criterion* states that every $X|\overline{X}$-separable state has a positive partial transpose operator with respect to the subspaces X or $\overline{X}$. This means that if the partial transpose operator of a state is non-positive (NPT), then the state is entangled. Note that the (3.66) PPT criterion of separability in general is necessary but not sufficient, that is, there are entangled states of positive partial transpose, called *PPT entangled states*. It is known that PPT entangled states are undistillable, but the reverse is not known, and subject of ongoing research. Note, however, that the criterion is necessary and sufficient (meaning that the partial transpose of every entangled state is non-positive) only in the special cases of two-qubit ($d_X = d_{\overline{X}} = 2$) systems or qubit-qutrit ($d_X = 2$, $d_{\overline{X}} = 3$) systems [56].

Another powerful method for detecting entanglement is the use of *entanglement witnesses*. An observable W is an entanglement witness, if

$$\forall \sigma \in \mathcal{D}_{X|\overline{X}\text{-sep}} : \quad \mathrm{Tr}(W\sigma) \geq 0 \tag{3.68a}$$

$$\exists \rho \in \mathcal{D}_L \setminus \mathcal{D}_{X|\overline{X}\text{-sep}} : \quad \mathrm{Tr}(W\rho) < 0, \tag{3.68b}$$

that is, the observable W signals the entanglement of a given state ρ by taking negative expectation value. Moreover, there exist entanglement witnesses W for every entangled ρ, such that $\mathrm{Tr}(W\rho) < 0$. This is called the *completeness of witnesses*. Using the completeness of witnesses, we can say that a state ρ is entangled if and only if there exist an entanglement witness W, such that $\mathrm{Tr}(W\rho) < 0$.

Entanglement witnesses are deeply connected to the (3.65) criterion with positive but not completely positive maps [56]. To see that, let us suppose that we have an entangled state ρ. According to (3.65), there exists a positive but not completely positive superoperator Φ, such that $(\mathcal{I} \otimes \Phi)(\rho) \not\geq 0$. This means that the operator $(\mathcal{I} \otimes \Phi)(\rho)$ has a negative eigenvalue λ_- with the corresponding eigenvector $|\varphi_-\rangle$. In that case, the operator $W = (\mathcal{I} \otimes \Phi)^\dagger(|\varphi_-\rangle\langle\varphi_-|)$ is an entanglement witness detecting ρ ($(\mathcal{I} \otimes \Phi)^\dagger$ denotes the adjoint map of $(\mathcal{I} \otimes \Phi)$, satisfying $\mathrm{Tr}((\mathcal{I} \otimes \Phi)^\dagger(A)B) = \mathrm{Tr}(A(\mathcal{I} \otimes \Phi)(B))$, which can be derived from the (A.41) definition of the Hilbert-Schmidt inner product). The (3.68a) part of the (3.68) definition follows from the (3.65) criterion. To see that the (3.68b) part of the definition is satisfied, let us calculate $\mathrm{Tr}(W\rho)$,

$$\begin{aligned} \mathrm{Tr}(W\rho) &= \mathrm{Tr}\Big((\mathcal{I} \otimes \Phi)^\dagger(|\varphi_-\rangle\langle\varphi_-|)\rho\Big) = \mathrm{Tr}\Big(|\varphi_-\rangle\langle\varphi_-|(\mathcal{I} \otimes \Phi)(\rho)\Big) = \langle\varphi_-|(\mathcal{I} \otimes \Phi)(\rho)|\varphi_-\rangle \\ &= \lambda_- \leq 0. \end{aligned} \tag{3.69}$$

Formulating entanglement witnesses for pure states is somewhat straightforward. For every pure entangled state $|\psi\rangle\langle\psi|$, the

$$W := \alpha_{|\psi\rangle} I - |\psi\rangle\langle\psi|, \qquad \alpha_{|\psi\rangle} = \max_{\sigma \in \mathcal{D}_{X|\overline{X}\text{-sep}}} \mathrm{Tr}(\sigma|\psi\rangle\langle\psi|) = \max_{|\phi\rangle = |a\rangle \otimes |b\rangle} |\langle\psi|\phi\rangle|^2 = \eta_1 \tag{3.70}$$

operator is an entanglement witness detecting entanglement in $|\psi\rangle\langle\psi|$. When evaluating the maximum, we used that $\mathrm{Tr}(\sigma|\psi\rangle\langle\psi|)$ is a linear function of σ, which takes the maximum on a convex set at the extremal points, which are the pure uncorrelated (product) states, and this maximum can be directly calculated as square of the $\sqrt{\eta_1}$ highest Schmidt coefficient of $|\psi\rangle$. Of course, the point is not to detect the entanglement of the pure state $|\psi\rangle\langle\psi|$, which is straightforward because it is just correlation (non-productness), but to detect entanglement of

mixed states in the vicinity of the pure entangled $|\psi\rangle\langle\psi|$ state [54]. Note that by choosing an $\alpha_{|\psi\rangle} < \alpha' < 1$ in the formula (3.70), then the witness will not be optimal (it detects less entangled states), but still detects the entangled state $|\psi\rangle\langle\psi|$. The witness constructed in (3.70) detects $X|\overline{X}$ entanglement for a given bipartition. To detect GME states, we may have the

$$W := \alpha_{|\psi\rangle} I - |\psi\rangle\langle\psi|,$$

$$\alpha_{|\psi\rangle} = \max_{\sigma\in\mathcal{D}_{\text{2-part sep}}} \mathrm{Tr}(\sigma|\psi\rangle\langle\psi|) = \max_{X|\overline{X}\in P_{\mathrm{I}}} \max_{|\phi\rangle=|a\rangle_X\otimes|b\rangle_{\overline{X}}} |\langle\psi|\phi\rangle|^2 = \max_{X|\overline{X}\in P_{\mathrm{I}}} \eta_1^{X|\overline{X}}, \tag{3.71}$$

where $\eta_1^{X|\overline{X}}$ is the square of the maximal Schmidt coefficient with respect to the $X|\overline{X}$ bipartition.

Now let us demonstrate these entanglement detection methods with two famous three-qubit states, the W-state and the GHZ-state,

$$|\mathrm{W}\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) = \sqrt{\frac{2}{3}}|0\rangle\otimes\frac{|01\rangle+|10\rangle}{\sqrt{2}} + \frac{1}{\sqrt{3}}|1\rangle\otimes|00\rangle, \tag{3.72a}$$

$$|\mathrm{GHZ}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{\sqrt{2}}|0\rangle\otimes|00\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes|11\rangle. \tag{3.72b}$$

It is easy to see that the right hand sides of (3.72a) and (3.72b) are the Schmidt decomposition of the states. By looking at these decompositions, we can see that these states have ranks more than one, so the W and GHZ states are entangled. Let us form the 2-partite marginals of these sates,

$$\mathrm{Tr}_3\big(|\mathrm{W}\rangle\langle\mathrm{W}|\big) = \frac{1}{3}\big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|\big) \tag{3.73a}$$

$$\mathrm{Tr}_3\big(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|\big) = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big). \tag{3.73b}$$

To detect entanglement in these states, let us use the (3.66) PPT criterion. The partial transpose operators of these reduced states are

$$\mathrm{Tr}_3\big(|\mathrm{W}\rangle\langle\mathrm{W}|\big)^{\mathrm{T}_2} = \frac{1}{3}\big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |00\rangle\langle 11| + |11\rangle\langle 00|\big), \tag{3.74a}$$

$$\mathrm{Tr}_3\big(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|\big)^{\mathrm{T}_2} = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big). \tag{3.74b}$$

To see that the (3.74a) $\mathrm{Tr}_3\big(|\mathrm{W}\rangle\langle\mathrm{W}|\big)^{\mathrm{T}_2}$ operator is not positive, let us multiply it with the vector $|\psi\rangle = -|00\rangle + |11\rangle$ from both sides,

$$\langle\psi|\,\mathrm{Tr}_3\big(|\mathrm{W}\rangle\langle\mathrm{W}|\big)^{\mathrm{T}_2}|\psi\rangle = (-\langle 00| + \langle 11|)\left(-\frac{1}{3}|00\rangle - \frac{1}{3}|11\rangle + \frac{1}{3}|00\rangle\right) = -\frac{1}{3}. \tag{3.75}$$

This means, that the (3.73a) 2-partite reduced state of the W state is NPT, which means that it is entangled. It is easy to see that the (3.74b) partial transpose operator of the (3.73b) 2-partite reduced state of the GHZ state is positive, and since $\mathrm{Tr}_3\big(|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|\big)$ is a two-qubit state, this means that it is separable. Note that from (3.73b), one can also see directly that it is the convex combination of product states, therefore separable.

It is easy to get the (3.70) witnesses detecting entanglement in the vicinity of the W and GHZ states [54]. For this, we should look at the (3.72) Schmidt decompositions of the states and find the largest Schmidt coefficients. These witnesses are

$$W_{\mathrm{W}} = \frac{2}{3}I - |\mathrm{W}\rangle\langle\mathrm{W}|, \tag{3.76a}$$

$$W_{\mathrm{GHZ}} = \frac{1}{2}I - |\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|. \tag{3.76b}$$

Another, quantitative way of detecting correlation/entanglement is defining a *faithful correlation/entanglement measure*, which gives zero if and only if the state is uncorrelated/separable, and gives a positive value if the state is correlated/entangled. Moreover a measure can help us to compare the strength of correlation/entanglement, since these measures give higher values for stronger correlation of entanglement.

3.9. **Measures of correlation and entanglement.** In quantification correlation and entanglement, we use some the von Neumann entropy and Umegaki relative entropy, see Appendix B.

Let us have the *relative entropy of ξ-correlation*, which can also be called the *geometric measure of ξ-correlation* [9, 58] as

$$C_\xi(\rho) := \min_{\sigma\in\mathcal{D}_{\xi\text{-unc}}} D(\rho\|\sigma) = \sum_{X\in\xi} S(\rho_X) - S(\rho), \tag{3.77}$$

where the quantity on the right side is called the ξ-mutual information, and the equality is due to the fact that the minimization can be carried out, and the location of this minimum is $\sigma = \bigotimes_{X\in\xi} \mathrm{Tr}_{\overline{X}}(\rho)$ [58]. It is important that the relative entropy of ξ-correlation cannot increase for local operations (quantum channels) $\Phi_\xi := \bigotimes_{X\in\xi} \Phi_X$,

$$C_\xi(\Phi_\xi(\rho)) \leq C_\xi(\rho), \tag{3.78}$$

which is easy to show from the definition, since the ξ-local operations map the set of ξ-uncorrelated states $\mathcal{D}_{\xi\text{-unc}}$ to itself. Decreasing monotonicity for local operations is the defining property of correlation measures. The relative entropy of ξ-correlation for different partitions is partially ordered with respect to the refinement, which is called *multipartite monotonicity* [9, 49],

$$\upsilon \preceq \xi \quad\Longrightarrow\quad C_\upsilon \geq C_\xi, \tag{3.79}$$

that is, the correlation with respect to a finer partition is larger. This is based on the definition (3.77) of the relative entropy of ξ-correlation, and on the inclusion hierarchy (3.50a). (Due to the subadditivity of the von Neumann entropy (B.9), this is also visible from the writing of the relative entropy of correlation by the von Neumann entropies, which is the right-hand side of (3.77).)

Similarly, we can introduce the *relative entropy of ξ-entanglement*, which is also called *geometric measure of ξ-entanglement* [58, 59] as

$$E_\xi(\rho) := \min_{\sigma\in\mathcal{D}_{\xi\text{-sep}}} D(\rho\|\sigma). \tag{3.80}$$

Unlike correlation, the minimization cannot be given in a closed form here. It is important that if Λ_ξ are ξ-local operations with classical communication, then the relative entropy of ξ-entanglement cannot increase with respect to it,

$$E_\xi(\Lambda_\xi(\rho)) \leq E_\xi(\rho), \tag{3.81}$$

which is easy to show from the definition, since the ξ-local operations with classical communication map the set of ξ-separable states $\mathcal{D}_{\xi\text{-sep}}$ to itself. Decreasing monotonicity for local operations and classical communication is the defining property of entanglement measures. The relative entropy of ξ-entanglement for different partitions is partially ordered with respect to the refinement, which is called *multipartite monotonicity* [49],

$$\upsilon \preceq \xi \quad\Longrightarrow\quad E_\upsilon \geq E_\xi, \tag{3.82}$$

that is, the entanglement with respect to a finer partition is larger. This is based on the definition (3.80) of the relative entropy of ξ-entanglement, and on the inclusion hierarchy (3.50b).

Entanglement is always less than or equal to the total correlation, as a part of the total correlation is entanglement, namely the part that cannot be created by classical communication. This is hard to grasp operationally, but easy to see for the geometric measures (3.77) and (3.80),

$$E_\xi \leq C_\xi, \tag{3.83}$$

since the minimization is taken on a larger set leads to a smaller minimum.

For an arbitrary $\xi = X|\overline{X}$ bipartition, the correlation of pure states can be calculated as

$$\begin{aligned} C_{X|\overline{X}}(|\psi\rangle\langle\psi|) &= S\big(\mathrm{Tr}_X(|\psi\rangle\langle\psi|)\big) + S\big(\mathrm{Tr}_{\overline{X}}(|\psi\rangle\langle\psi|)\big) - S\big(|\psi\rangle\langle\psi|\big) \\ &= 2S\big(\mathrm{Tr}_{\overline{X}}(|\psi\rangle\langle\psi|)\big) = 2\Big(-\sum \eta_i \ln(\eta_i)\Big), \end{aligned} \tag{3.84}$$

where we used the definition (3.77) of the relative entropy of correlation, the Schmidt canonical form (3.9) of the vector $|\psi\rangle$, and the properties of the von Neumann entropy (B.1). A similar formula also arises for entanglement, but its derivation is not simple [59],

$$E_{X|\overline{X}}(|\psi\rangle\langle\psi|) = S\big(\mathrm{Tr}_{\overline{X}}(|\psi\rangle\langle\psi|)\big) = -\sum \eta_i \ln(\eta_i), \tag{3.85}$$

so

$$2E_{X|\overline{X}}(|\psi\rangle\langle\psi|) = C_{X|\overline{X}}(|\psi\rangle\langle\psi|). \tag{3.86}$$

Unfortunately, there is no similar result for a division into more than two parts, because the minimization in the relative entropy of entanglement (3.80), unlike the relative entropy of correlation (3.77), could not be given in a closed form.

Similarly to the (3.77) and (3.80) measures for partitions, we can define correlation and entanglement measures for (3.51) partition ideals [11]. Using the definition (3.53), the *relative entropy of $\boldsymbol{\xi}$-correlation*, which is also called geometric measure of $\boldsymbol{\xi}$-correlation is the minimum of the relative entropy of ξ-correlation for partitions ξ in the partition ideal $\boldsymbol{\xi}$,

$$C_{\boldsymbol{\xi}}(\rho) := \min_{\sigma\in\mathcal{D}_{\boldsymbol{\xi}\text{-unc}}} D(\rho\|\sigma) = \min_{\xi\in\boldsymbol{\xi}}\Big(\min_{\sigma\in\mathcal{D}_{\xi\text{-unc}}} D(\rho\|\sigma)\Big) = \min_{\xi\in\boldsymbol{\xi}} C_\xi(\rho). \tag{3.87}$$

Using the definition (3.55), the *relative entropy of $\boldsymbol{\xi}$-entanglement*, which is also called geometric measure of $\boldsymbol{\xi}$-entanglement is

$$E_{\boldsymbol{\xi}}(\rho) := \min_{\sigma\in\mathcal{D}_{\boldsymbol{\xi}\text{-sep}}} D(\rho\|\sigma). \tag{3.88}$$

Using (3.58), (3.79) and (3.82), we can see that for these multipartite measures *multipartite monotonicity* [11] holds,

$$\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \quad\Longrightarrow\quad C_{\boldsymbol{\upsilon}} \geq C_{\boldsymbol{\xi}}, \tag{3.89a}$$

$$\boldsymbol{\upsilon} \preceq \boldsymbol{\xi} \quad\Longrightarrow\quad E_{\boldsymbol{\upsilon}} \geq E_{\boldsymbol{\xi}}. \tag{3.89b}$$

Since the sets of k-partitionable (3.60a), k-producible (3.60b) and k-stretchable (3.60c) partitions are also partition ideals, we can easily define the *relative entropy of k-partitionable correlation*, the *relative entropy of k-producible correlation* and the *relative entropy of k-stretchable correlation* using (3.87) as

$$C_{k\text{-part}}(\rho) := C_{\boldsymbol{\xi}_{k\text{-part}}}(\rho) = \min_{\xi:h(\xi)\geq k} C_\xi(\rho) = \min_{\xi:h(\xi)\geq k}\Big(\sum_{X\in\xi} S(\rho_X) - S(\rho)\Big), \tag{3.90a}$$

$$C_{k\text{-prod}}(\rho) := C_{\boldsymbol{\xi}_{k\text{-prod}}}(\rho) = \min_{\xi:w(\xi)\leq k} C_\xi(\rho) = \min_{\xi:w(\xi)\leq k}\Big(\sum_{X\in\xi} S(\rho_X) - S(\rho)\Big), \tag{3.90b}$$

$$C_{k\text{-str}}(\rho) := C_{\boldsymbol{\xi}_{k\text{-str}}}(\rho) = \min_{\xi: r(\xi)\leq k} C_\xi(\rho) = \min_{\xi: r(\xi)\leq k} \left(\sum_{X\in\xi} S(\rho_X) - S(\rho) \right). \tag{3.90c}$$

Similarly, using (3.88) we can define the *relative entropy of k-partitionable entanglement*, the *relative entropy of k-producible entanglement* and the *relative entropy of k-stretchable entanglement* as

$$E_{k\text{-part}}(\rho) := E_{\boldsymbol{\xi}_{k\text{-part}}}(\rho) = \min_{\sigma\in\mathcal{D}_{k\text{-part sep}}} D(\rho\|\sigma), \tag{3.91a}$$

$$E_{k\text{-prod}}(\rho) := E_{\boldsymbol{\xi}_{k\text{-prod}}}(\rho) = \min_{\sigma\in\mathcal{D}_{k\text{-prod sep}}} D(\rho\|\sigma), \tag{3.91b}$$

$$E_{k\text{-str}}(\rho) := E_{\boldsymbol{\xi}_{k\text{-str}}}(\rho) = \min_{\sigma\in\mathcal{D}_{k\text{-str sep}}} D(\rho\|\sigma). \tag{3.91c}$$

Using (3.64) and (3.89), we can see that these relative entropies also have *multipartite monotonicity* [10, 11],

$$k \geq l \iff C_{k\text{-part}} \geq C_{l\text{-part}}, \tag{3.92a}$$

$$k \leq l \iff C_{k\text{-prod}} \geq C_{l\text{-prod}}, \tag{3.92b}$$

$$k \leq l \iff C_{k\text{-str}} \geq C_{l\text{-str}}, \tag{3.92c}$$

$$k \geq l \iff E_{k\text{-part}} \geq E_{l\text{-part}}, \tag{3.92d}$$

$$k \leq l \iff E_{k\text{-prod}} \geq E_{l\text{-prod}}, \tag{3.92e}$$

$$k \leq l \iff E_{k\text{-str}} \geq E_{l\text{-str}}. \tag{3.92f}$$

4. Symmetric Dicke states

In this thesis we study the *symmetric Dicke states* or *elementary symmetric states*, so in this section we describe the usual representation of the symmetric group on the n-partite Hilbert space, describe these states and study some of their properties in more detail.

4.1. Permutation operators.

Let $L = [n] = \{1, 2, \ldots, n\}$ be the system again, and let the dimensions of the elementary subsystems be the same, that is,

$$\dim(\mathcal{H}_a) = d \qquad \forall a \in L. \tag{4.1}$$

Permutations are functions $\sigma : L \to L$ that are *bijective*. These functions form a group, called the *symmetric group* S_L, which is indeed a group, since S_L is *closed* under multiplication (that is, $\sigma, \tau \in \mathrm{S}_L$ then $\sigma\tau \in \mathrm{S}_L$); which is *associative* (that is, $\sigma, \tau, \omega \in \mathrm{S}_L$, $\sigma(\tau\omega) = (\sigma\tau)\omega$); there exists an *identity permutation* $e \in \mathrm{S}_L$ (that is, $e(a) = a$); and every permutation $\sigma \in \mathrm{S}_L$ has an *inverse* σ^{-1} (that is, $\sigma^{-1}\sigma = e$). S_L describes the permutations of subsystems and we would like to have the representation of S_L on $\mathcal{H}_L$, that permutes the Hilbert spaces of the elementary subsystems.

First, after fixing the local bases $\{|\phi_{a,i}\rangle \in \mathcal{H}_a\}$, let us have for all $a, b \in L$ the operators

$$R_{a,b} = \sum_{i\in[d]} |\phi_{b,i}\rangle\langle\phi_{a,i}| \in \mathrm{Lin}\left(\mathcal{H}_a, \mathcal{H}_b\right). \tag{4.2}$$

Note that $R_{a,b}^\dagger = R_{b,a}$ and $R_{a,a} = I_a \in \mathrm{Lin}(\mathcal{H}_a)$. It is easy to see that these satisfy the multiplication rule

$$R_{b,c}R_{a,b} = R_{a,c}. \tag{4.3}$$

Note that $R_{a,b}^{-1} = R_{b,a}$. Let us see how such an $R_{a,b} \in \mathrm{Lin}\,(\mathcal{H}_a, \mathcal{H}_b)$ operator acts on an arbitrary vector $|\psi\rangle \in \mathcal{H}_a$. For this, let us write the basis expansion of $|\psi\rangle$ as

$$|\psi\rangle = \sum_{j\in[d]} \psi_j |\phi_{a,j}\rangle, \tag{4.4}$$

then the effect of the operator $R_{a,b}$ on the vector $|\psi\rangle$ is

$$R_{a,b}|\psi\rangle = \sum_{i\in[d]} |\phi_{b,i}\rangle\langle\phi_{a,i}| \sum_{j\in[d]} \psi_j |\phi_{a,j}\rangle = \sum_{j\in[d]} \psi_j \sum_{i\in[d]} |\phi_{b,i}\rangle\delta_{ij} = \sum_{j\in[d]} \psi_j |\phi_{b,j}\rangle, \tag{4.5}$$

which means that the operator $R_{a,b}$ acts on the vector $|\psi\rangle$ by "transferring" it to the Hilbert space $\mathcal{H}_b$ such that the expansion coefficient belonging to the basis vector $|\phi_{b,j}\rangle \in \mathcal{H}_b$ is equal to the expansion coefficient belonging to the basis vector $|\phi_{a,j}\rangle \in \mathcal{H}_a$. Note that although the fixing of the bases was necessary for the construction of these permutation operators, the action of the operators themselves is *basis-independent*. Of course, the vectors of the different spaces $\mathcal{H}_a$ had to be identified, which we did by specifying the basis, but we could have done it differently.

For a permutation $\sigma \in \mathrm{S}_L$, a *permutation operator* can be constructed as

$$\sigma \quad \longmapsto \quad R_\sigma = \bigotimes_{a\in L} R_{a,\sigma(a)}. \tag{4.6}$$

For the multiplication of permutation operators, we have

$$R_{\sigma\tau} = R_\sigma R_\tau, \tag{4.7}$$

that is, the operators R_σ give a *representation* of the group S_L. To prove this, we use the (4.6) form of the permutation operator. Then

$$\begin{aligned} R_\sigma R_\tau &= \bigotimes_{a\in L} R_{a,\sigma(a)} \cdot \bigotimes_{b\in L} R_{b,\tau(b)} = \bigotimes_{a\in L} R_{a,\sigma(a)} R_{\tau^{-1}(a),a} \\ &= \bigotimes_{a\in L} R_{\tau^{-1}(a),\sigma(a)} = \bigotimes_{b\in L} R_{b,\sigma(\tau(b))} = R_{\sigma\tau}, \end{aligned} \tag{4.8}$$

where we have used the multiplication rule (4.3), and index relabeling. For the adjoint of the permutation operators, we have

$$R_\sigma^\dagger = R_\sigma^{-1} = R_{\sigma^{-1}}, \tag{4.9}$$

that is, the representation is *unitary*. This easily follows from the definition (4.6) of R_σ and the corresponding properties of $R_{a,b}$.

Let us introduce the operator

$$P := \frac{1}{n!} \sum_{\sigma\in \mathrm{S}_L} R_\sigma, \tag{4.10}$$

which is a *projection* (A.59), meaning it is self-adjoint and idempotent. First, let us prove that the operator P is self-adjoint. Based on the definition (4.10) of the operator P, and using the property (4.9) of the permutation operator,

$$P^\dagger = \frac{1}{n!} \sum_{\sigma\in \mathrm{S}_L} R_\sigma^\dagger = \frac{1}{n!} \sum_{\sigma\in \mathrm{S}_L} R_{\sigma^{-1}} = \frac{1}{n!} \sum_{\tau^{-1}\in \mathrm{S}_L} R_\tau = P. \tag{4.11}$$

Now let us prove that the operator P is idempotent. Here we also use the form (4.10) of the operator P, and use the property (4.7) of the permutation operator,

$$\begin{aligned} P^2 &= \frac{1}{n!}\sum_{\sigma\in \mathrm{S}_L} R_\sigma \frac{1}{n!}\sum_{\tau\in \mathrm{S}_L} R_\tau = \frac{1}{(n!)^2}\sum_{\sigma\in \mathrm{S}_L}\sum_{\tau\in \mathrm{S}_L} R_\sigma R_\tau = \frac{1}{(n!)^2}\sum_{\sigma\in \mathrm{S}_L}\sum_{\tau\in \mathrm{S}_L} R_{\sigma\tau} \\ &= \frac{1}{(n!)^2}\sum_{\sigma\in \mathrm{S}_L}\sum_{\sigma^{-1}\omega\in \mathrm{S}_L} R_\omega = \frac{1}{(n!)^2} n! \sum_{\omega\in\sigma(\mathrm{S}_L)} R_\omega = \frac{1}{n!}\sum_{\omega\in \mathrm{S}_L} R_\omega = P, \end{aligned} \tag{4.12}$$

where we used the relabeling $\sigma\tau = \omega$. With the help of the projection P introduced earlier, we can define the *symmetric subspace of the system's Hilbert space*, which is the image of the projection P,

$$\mathrm{Ran}\,(P) =: \mathcal{H}_L^{\mathrm{Symm}} \subseteq \mathcal{H}_L. \tag{4.13}$$

It is easily shown that vectors in the symmetric Hilbert subspace are those for which acting with any permutation operator leaves invariant,

$$\mathcal{H}_L^{\mathrm{Symm}} = \big\{|\psi\rangle \in \mathcal{H}_L |\ R_\sigma|\psi\rangle = |\psi\rangle,\ \forall \sigma \in \mathrm{S}_L\big\}. \tag{4.14}$$

4.2. **Qubit Dicke states.** In this section, we consider multiqubit systems, that is let the dimension of all elementary subsystems be $d_a = 2$. First, let us consider the state

$$|\,\mathrm{D}_k^0\rangle = \bigotimes_{a\in[n-k]} |\phi_{a,1}\rangle \otimes \bigotimes_{b\in L\setminus[n-k]} |\phi_{b,2}\rangle, \tag{4.15}$$

In the following, we will use the simplified notation

$$|\phi_{a,i}\rangle = |i\rangle \tag{4.16}$$

for the basis vectors, and also omit the tensor product symbol as

$$|i_1\rangle \otimes |i_2\rangle = |i_1 i_2\rangle, \tag{4.17}$$

which is usual in the quantum computation community. (Note that sometimes also the numbering of the basis vectors goes from 0 to $d-1$, which we do not adopt here.) By the use of this notation we have $|\,\mathrm{D}_k^0\rangle = |\underbrace{11\ldots1}_{\text{n-k}}\underbrace{22\ldots2}_{\text{k}}\rangle$.

Let us introduce the *(nonnormalized) qubit Dicke vector* of index k as

$$|\tilde{\mathrm{D}}_k\rangle := \Big(|\underbrace{11\ldots1}_{\text{n-k}}\underbrace{22\ldots2}_{\text{k}}\rangle + \text{“all permutations”}\Big). \tag{4.18}$$

The norm of this vector is $\tilde{N}_k = \sqrt{\langle\tilde{\mathrm{D}}_k|\tilde{\mathrm{D}}_k\rangle}$. To calculate this, note that the terms in the sum are orthonormal, because if a term is multiplied by itself, the result will be the product of the numbers $\langle 1|1\rangle$ and $\langle 2|2\rangle$, which is 1 due to (3.2). And if two different terms are multiplied, in addition to the terms $\langle 1|1\rangle$ and $\langle 2|2\rangle$, the terms $\langle 2|1\rangle$ and $\langle 1|2\rangle$ also appear in the product, whose value is 0 according to (3.2), so the result is also 0. Thus, the value of the scalar product $\langle\tilde{\mathrm{D}}_k|\tilde{\mathrm{D}}_k\rangle$ is equal to the number of terms to be summed, which is the number of ways to arrange k ones and $n-k$ twos in a sequence, whose value is the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Thus

$$\tilde{N}_k = \sqrt{\binom{n}{k}} = \binom{n}{k}^{\frac{1}{2}}. \tag{4.19}$$

Based on this, the *qubit Dicke state with index k* is

$$|\,\mathrm{D}_k\rangle := \frac{1}{\tilde{N}_k}|\tilde{\mathrm{D}}_k\rangle = \binom{n}{k}^{-\frac{1}{2}}\Big(|\underbrace{11\ldots1}_{\text{n-k}}\underbrace{22\ldots2}_{\text{k}}\rangle + \text{“all permutations”}\Big). \tag{4.20}$$

Such a state vector is symmetric and can be considered the first-quantized (particle picture) state of an n-photon system, where k photons are in the excited mode, and $n-k$ photons are in the ground mode. These states are *orthonormal*, that is,

$$\langle \mathrm{D}_k \,|\, \mathrm{D}_l \rangle = \delta_{kl}. \tag{4.21}$$

The normalization holds by construction, so only orthogonality is needed. For this, we proceed as in the calculation of $\tilde{N}_k$, and calculate the product $\langle \mathrm{D}_k \,|\, \mathrm{D}_l \rangle$ in the case where $k \neq l$. We saw earlier that the scalar product of the terms in the form (4.20) of the Dicke states is non-zero if and only if they are exactly the same. However, if two states with unequal indices are multiplied, there will be no matching terms in those states, since the terms of $|\,\mathrm{D}_k\rangle$ contain k vectors $|2\rangle$, and the terms of $|\,\mathrm{D}_l\rangle$ contain l vectors $|2\rangle$, so the value of this product is zero, $\langle \mathrm{D}_k \,|\, \mathrm{D}_l \rangle = 0$, if $k \neq l$, that is, Dicke states with different indices are orthogonal.

We would like to obtain a more easily usable closed formula for Dicke states instead of (4.20), which does not contain "text". For this, let us use the permutation operators (4.6), as it is suspected that Dicke states can also be produced using these, since in the original definition, we summed the permutations of the vector (4.15). For this, let us introduce the vector

$$|\,\tilde{\tilde{\mathrm{D}}}_k\rangle := \sum_{\sigma \in \mathrm{S}_L} R_\sigma |\,\mathrm{D}_k^0\rangle. \tag{4.22}$$

Recall that $|\tilde{\mathrm{D}}_k\rangle$ contains all the $\binom{n}{k}$ different terms. $|\,\tilde{\tilde{\mathrm{D}}}_k\rangle$ contains the same terms with the same constant prefactors C_k summing up to $|\mathrm{S}_L| = n!$, that is $|\,\tilde{\tilde{\mathrm{D}}}_k\rangle = C_k |\tilde{\mathrm{D}}_k\rangle$, where $C_k \binom{n}{k} = n!$, since there are permutations leaving $|\,\mathrm{D}_k^0\rangle$ invariant. Then we have

$$\tilde{\tilde{N}}_k = \sqrt{\langle \tilde{\tilde{\mathrm{D}}}_k \,|\, \tilde{\tilde{\mathrm{D}}}_k \rangle} = \frac{n!}{\binom{n}{k}} \tilde{N}_k = n! \binom{n}{k}^{-\frac{1}{2}}. \tag{4.23}$$

Based on this, the qubit Dicke state with index k reads also as

$$|\,\mathrm{D}_k\rangle = \frac{1}{\tilde{\tilde{N}}_k} |\,\tilde{\tilde{\mathrm{D}}}_k\rangle = \binom{n}{k}^{\frac{1}{2}} \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_L} R_\sigma |\,\mathrm{D}_k^0\rangle = \binom{n}{k}^{\frac{1}{2}} P |\,\mathrm{D}_k^0\rangle, \tag{4.24}$$

where we have recognized the projection P introduced earlier in (4.10). From this, it can be seen that a Dicke state $|\,\mathrm{D}_k\rangle$ can be obtained by acting with the projector P on a vector that is the tensor product of k vectors $|2\rangle$ and $n-k$ vectors $|1\rangle$, which is a basis vector in the Hilbert space $\mathcal{H}_L$ of the composite system based on (3.4). Thus, if we act with the projector P on a basis vector, we get a Dicke state up to normalization. On the other hand, every Dicke state can be obtained in this way. Since the image of P is the space $\mathcal{H}_L^{\mathrm{Symm}}$, the Dicke states span the image of the projector P. Since the Dicke states $|\,\mathrm{D}_k\rangle$ are orthonormal, hence linearly independent, they form a basis in the image of the projector P, which is the symmetric Hilbert space according to (4.13). Based on this, the dimension of $\mathcal{H}_L^{\mathrm{Symm}}$ is equal to the number of Dicke states, that is,

$$\dim(\mathcal{H}_L^{\mathrm{Symm}}) = n+1. \tag{4.25}$$

We will perform this proof more precisely, using group-theoretical methods, in the arbitrary dimensional case in the subsequent section.

In the following, we use the multi-index notation for Dicke states, which will also be used in the arbitrary dimensional, qu*d*it case. The Dicke state $|\,\mathrm{D}_k\rangle$, previously labeled by the index k, is labeled by the multi-index $\boldsymbol{n}$ as $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$, also known as the *excitation vector* or *occupation number vector*. In the 2-dimensional, qubit case, the multi-index takes the form

$$\boldsymbol{n} = (n_1, n_2) = (n-k, k) \in \mathbb{N}_0^2, \tag{4.26}$$

the first component, n_1, gives the number of times the vector $|1\rangle$ appears in the terms of the Dicke state (4.20), that is, $n_1 = n-k$, and the second component, n_2, gives the number of times the vector $|2\rangle$ appears in the terms, that is, $n_2 = k$. Based on these, the sum of the components of the multi-index $\boldsymbol{n}$ is equal to the number of elementary subsystems, $\|\boldsymbol{n}\|_1 = n_1 + n_2 = n$. The previous formulae obtained by the k labeling can simply be translated to the multiindex labeling by (4.26), in particular, $\binom{n}{k} = \binom{n}{n_2} = \frac{n!}{n_2!(n-n_2)!} = \binom{n}{n_1}$. We will see later that the multi-index notation is better than the original index notation if the number of subsystems is not fixed. The qubit Dicke states for 1,2,3 and 4 elementary subsystems in the multi-index notation are

$$
\begin{aligned}
n=1: \quad & |\,\mathrm{D}_{(1,0)}\rangle = |1\rangle, \quad |\,\mathrm{D}_{(0,1)}\rangle = |2\rangle, \\
n=2: \quad & |\,\mathrm{D}_{(2,0)}\rangle = |11\rangle, \quad |\,\mathrm{D}_{(1,1)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |\,\mathrm{D}_{(0,2)}\rangle = |22\rangle, \\
n=3: \quad & |\,\mathrm{D}_{(3,0)}\rangle = |11\rangle, \quad |\,\mathrm{D}_{(2,1)}\rangle = \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) = |\mathrm{W}\rangle, \\
& |\,\mathrm{D}_{(1,2)}\rangle = \frac{1}{\sqrt{3}}(|122\rangle + |212\rangle + |221\rangle), \quad |\,\mathrm{D}_{(0,3)}\rangle = |222\rangle, \\
n=4: \quad & |\,\mathrm{D}_{(4,0)}\rangle = |1111\rangle, \quad |\,\mathrm{D}_{(3,1)}\rangle = \frac{1}{\sqrt{4}}(|1112\rangle + |1121\rangle + |1211\rangle + |2111\rangle), \\
& |\,\mathrm{D}_{(2,2)}\rangle = \frac{1}{\sqrt{6}}(|1122\rangle + |1212\rangle + |1221\rangle + |2112\rangle + |2121\rangle + |2211\rangle), \\
& |\,\mathrm{D}_{(1,3)}\rangle = \frac{1}{\sqrt{4}}(|1222\rangle + |2122\rangle + |2212\rangle + |2221\rangle), \quad |\,\mathrm{D}_{(0,4)}\rangle = |2222\rangle.
\end{aligned}
\tag{4.27}
$$

It is somewhat surprising that, as we will see, a Dicke state labeled by $\boldsymbol{n} = (n_1, n_2)$ consisting of n elementary subsystems can be constructed from Dicke states consisting of fewer elementary subsystems using the tensor product. First, let us consider the case where we construct the Dicke vector consisting of n elementary subsystems from vectors consisting of 1 and $n-1$ elementary subsystems, $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = |1\rangle \otimes |\xi_1\rangle + |2\rangle \otimes |\xi_2\rangle$. This can always be done, however, it turns out that the vectors $|\xi_1\rangle$ and $|\xi_2\rangle$ are also Dicke vectors,

$$
|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = |1\rangle \otimes |\tilde{\mathrm{D}}_{(n_1-1,n_2)}\rangle + |2\rangle \otimes |\tilde{\mathrm{D}}_{(n_1,n_2-1)}\rangle.
\tag{4.28}
$$

This is remarkable, since we have two nonsymmetric terms containing symmetric factors, adding up to a symmetric vector. This is indeed correct, which can be seen by noting that the two terms in the sum obviously have different terms (basis elements), all of these has n_1 $|1\rangle$ and n_2 $|2\rangle$, and the total number of terms is

$$
\begin{aligned}
& \binom{n-1}{n_2} + \binom{n-1}{n_1} = \frac{(n-1)!}{n_2!(n_1-1)!} + \frac{(n-1)!}{n_1!(n_2-1)!} = \\
& = (n-1)!\left(\frac{1}{(n-n_1)!(n_1-1)!} + \frac{1}{n_1!(n-n_1-1)!}\right) = \\
& = (n-1)!\left(\frac{n_1}{(n-n_1)!n_1!} + \frac{n-n_1}{n_1!(n-n_1)!}\right) = \frac{n!}{n_1!n_2!} = \binom{n}{n_2},
\end{aligned}
\tag{4.29}
$$

which is the number of different permutations of the vector $|\underbrace{11\ldots1}_{n_1}\underbrace{22\ldots2}_{n_2}\rangle$, that is, we indeed obtained the Dicke vector $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$. Later, we will give a more general proof for the arbitrary dimensional, qu*d*it case. If we use the forms from (4.27) of the Dicke states consisting of 1 elementary subsystem, $|1\rangle = |\,\mathrm{D}_{(1,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0)}\rangle$ and $|2\rangle = |\,\mathrm{D}_{(0,1)}\rangle = |\tilde{\mathrm{D}}_{(0,1)}\rangle$, then we get that

$$
|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = |\tilde{\mathrm{D}}_{(1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(n_1-1,n_2)}\rangle + |\tilde{\mathrm{D}}_{(0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(n_1,n_2-1)}\rangle.
\tag{4.30}
$$

Thus, a Dicke state consisting of n elementary subsystems is essentially divided into Dicke states consisting of 1 and $n-1$ elementary subsystems. Similarly, a Dicke state consisting of n elementary subsystems can be decomposed into sums of products of Dicke states consisting of m and $n-m$ elementary subsystems. This can be done by taking a Dicke state consisting of m elementary subsystems, in which there are $m_2 \leq n_2$ vectors $|2\rangle$, and multiplying it by a Dicke state consisting of $n-m$ elementary subsystems, in which there are $n_2 - m_2$ vectors $|2\rangle$, that is,

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \sum_{m_2=\max\{0,m_2-(n-m)\}}^{\min\{m,m_2\}} |\tilde{\mathrm{D}}_{(m-m_2,m_2)}\rangle \otimes |\tilde{\mathrm{D}}_{(n_1-m+m_2,n_2-m_2)}\rangle. \tag{4.31}$$

If we also use the multi-index for the Dicke state consisting of m elementary subsystems, with the multi-index $\boldsymbol{m} = (m_1, m_2)$, where $m_1 + m_2 = m$, then the formula takes the very convenient form

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \sum_{\substack{\boldsymbol{m}=(0,0)\\ m_1+m_2=m}}^{\boldsymbol{n}} |\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle, \tag{4.32}$$

or with normalization

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \binom{n}{n_2}^{-\frac{1}{2}} \sum_{\substack{\boldsymbol{m}=(0,0)\\ m_1+m_2=m}}^{\boldsymbol{n}} \binom{m}{m_2}^{\frac{1}{2}} |\,\mathrm{D}_{\boldsymbol{m}}\rangle \otimes \binom{n-m}{n_2-m_2}^{\frac{1}{2}} |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}}\rangle, \tag{4.33}$$

where the limits of the sum are understood as summing over those multi-indices $\boldsymbol{m}$ whose components sum up to m (that is, $\sum_i m_i = m$) and every component is not larger than the corresponding component of $\boldsymbol{n}$ (that is, $m_i \leq n_i$). We will prove later that (4.33) is indeed a Dicke state, as a special case of the arbitrary dimensional, qu*d*it case.

4.3. **Qudit Dicke states.** Dicke states can be generalized to arbitrary d-dimensional elementary subsystems, qu*d*its using the multi-indexing seen earlier. In this case, the generalization of (4.15) for arbitrary d-dimensional elementary subsystems is $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle = |\underbrace{11\ldots1}_{n_1}\underbrace{22\ldots2}_{n_2}\ldots\underbrace{dd\ldots d}_{n_d}\rangle$. Using this, the generalization of (4.18) is

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \Big(|\underbrace{11\ldots1}_{n_1}\underbrace{22\ldots2}_{n_2}\ldots\underbrace{dd\ldots d}_{n_d}\rangle + \text{“all permutations”}\Big), \tag{4.34}$$

where the component n_i of the multi-index $\boldsymbol{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$ gives the number of times the basis vector $|i\rangle$ appears in the vector to be permuted, and the sum of its components gives the number of elementary subsystems, $\|\boldsymbol{n}\|_1 = \sum_i n_i = n$. The set of possible occupations for an n-partite (n elementary subsystems) qu*d*it (d dimensional elementary subsystems) Dicke states is denoted by

$$I_n^d := \Big\{\boldsymbol{n} \in \mathbb{N}_0^d \;\Big|\; \|\boldsymbol{n}\|_1 = \sum_i n_i = n\Big\}. \tag{4.35}$$

The calculation of the normalization factor is completely analogous to the calculation of $\tilde{N}_k$ in the 2-dimensional case, (4.19), so the value of the product $\langle \mathrm{D}_{\boldsymbol{n}} \,|\, \mathrm{D}_{\boldsymbol{n}}\rangle$ is equal to the number of terms in the formula (4.34), which is the number of ways to arrange n_1 ones, n_2 twos, …, in a sequence, which is given by the multinomial coefficient

$$\tilde{N}_{\boldsymbol{n}} = \sqrt{\langle \mathrm{D}_{\boldsymbol{n}} \,|\, \mathrm{D}_{\boldsymbol{n}}\rangle} = \sqrt{\frac{(\sum_i n_i)!}{\prod_i n_i!}} = \sqrt{\frac{n!}{\prod_i n_i!}} = \binom{n}{\boldsymbol{n}}^{\frac{1}{2}}. \tag{4.36}$$

Based on this, the $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ Dicke state with index $\boldsymbol{n}$, is given by

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \frac{1}{\tilde{N}_{\boldsymbol{n}}}|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \binom{n}{\boldsymbol{n}}^{-\frac{1}{2}}\Big(|\underbrace{11\ldots1}_{n_1}\underbrace{22\ldots2}_{n_2}\ldots\underbrace{dd\ldots d}_{n_d}\rangle + \text{“all permutations”}\Big), \tag{4.37}$$

which is the generalization of the formula (4.20) for d-dimensional elementary, qu$d$it subsystems. These states are also *orthonormal*, that is, $\langle \mathrm{D}_{\boldsymbol{n}}\,|\,\mathrm{D}_{\boldsymbol{n}'}\rangle = \delta_{\boldsymbol{n}\boldsymbol{n}'}$, which can be seen similarly to the qubit case.

A closed formula for qu$d$it Dicke states can also be found using the permutation operators as in the qubit case. For this, let us introduce the vector

$$|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle = \sum_{\sigma\in \mathrm{S}_L} R_\sigma|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle \tag{4.38}$$

To calculate the normalization factor $\tilde{\tilde{N}}_{\boldsymbol{n}} = \sqrt{\langle\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\,|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle}$, we rewrite the vector $|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle$ as

$$|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle = \sum_{C\in \mathrm{S}_L/S_{L,\boldsymbol{n}}}\sum_{\sigma\in C} R_\sigma|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle, \tag{4.39}$$

where $S_{L,\boldsymbol{n}}$ is the *stabilizer* of the vector $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$, that is, the set of permutations $\omega\in \mathrm{S}_L$ for which $R_\omega|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle = |\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$. Thus, the ω permutations are permutations that can be decomposed into a product of permutations ω_i, where each of these only permutes those subsystems $a\in X_i$ where the components in the vector $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$ are the same, $|i\rangle$,

$$S_{L,\boldsymbol{n}} = \Big\{\omega = \prod_{i=1}^{d}\omega_i \;\Big|\; \omega_i\in \mathrm{S}_{X_i},\ X_i = \{a_i+1, a_i+2,\ldots,a_i+n_i\},\ a_i = \sum_{j=1}^{i-1} n_j\Big\}. \tag{4.40}$$

Based on this, the size of the stabilizer can be calculated,

$$|S_{L,\boldsymbol{n}}| = \prod_{i=1}^{d} n_i!. \tag{4.41}$$

$\mathrm{S}_L/S_{L,\boldsymbol{n}}$ is the *set of left cosets of the stabilizer*, that is,

$$\mathrm{S}_L/S_{L,\boldsymbol{n}} = \big\{\tau S_{L,\boldsymbol{n}} \;\big|\; \tau\in \mathrm{S}_L\big\}. \tag{4.42}$$

(We note that G/N is often used to denote the factor group by the normal subgroup N, but $G/H, H\setminus G$ is also used more generally for left/right cosets.) It is obvious that two cosets are either identical or disjoint, and their size is the same. Two cosets are identical if the permutations τ and τ' in the definition (4.42) give the same result when acting on $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$. Thus, the cosets are *equivalence classes*. Based on this, continuing (4.39)

$$|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle = \sum_{C\in \mathrm{S}_L/S_{L,\boldsymbol{n}}}\sum_{\sigma\in C} R_\sigma|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle = |S_{L,\boldsymbol{n}}|\sum_{C\in \mathrm{S}_L/S_{L,\boldsymbol{n}}} R_{\sigma(C)}|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle = |S_{L,\boldsymbol{n}}|\cdot|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle, \tag{4.43}$$

where $\sigma(C)$ selects an arbitrary element from the coset C, and we used that every $\sigma\in C$ is of the form $\sigma=\tau\omega$ where $\omega\in S_{L,\boldsymbol{n}}$, so every $\sigma\in C$ in a fixed C acts on $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$ in the same way. Thus, in (4.43), the last summation sums all different permutations of the vector $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$, which is exactly $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$ given in (4.34). Then the norm $\tilde{\tilde{N}}_{\boldsymbol{n}}$ can be calculated,

$$\tilde{\tilde{N}}_{\boldsymbol{n}} = \sqrt{\langle\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\,|\,\tilde{\tilde{\mathrm{D}}}_{\boldsymbol{n}}\rangle} = |S_{L,\boldsymbol{n}}|\sqrt{\langle\tilde{\mathrm{D}}_{\boldsymbol{n}}|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle} = n!\binom{n}{\boldsymbol{n}}^{-\frac{1}{2}}. \tag{4.44}$$

Based on this, the qu*d*it Dicke state of occupation $\boldsymbol{n}$ reads also as

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \binom{n}{\boldsymbol{n}}^{\frac{1}{2}} \frac{1}{n!} \sum_{\sigma\in \mathrm{S}_L} R_\sigma |\,\mathrm{D}^0_{\boldsymbol{n}}\rangle = \binom{n}{\boldsymbol{n}}^{\frac{1}{2}} P|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle, \tag{4.45}$$

where we also recognized the projector P introduced in (4.10), with which we can obtain the arbitrary dimensional form of the relation (4.24). With similar considerations to the two-dimensional case, we get that the vectors $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ form a basis in the image of the projector P, the symmetric Hilbert space, based on which the dimension of $\mathcal{H}_L^{\mathrm{Symm}}$ for arbitrary dimension d is equal to the number of possible multi-indices $\boldsymbol{n}$, which is equal to the number of different vectors $|\,\mathrm{D}^0_{\boldsymbol{n}}\rangle$ that can be written. This is equal to the number of ways of placing n balls into d bins, which is equivalent to the number of ways of placing $d-1$ dividers between n balls (stars and bars problem [60]), that is, to choose the $d-1$ positions of dividers from the $n+d-1$ possible positions of the balls and dividers together,

$$\dim(\mathcal{H}_L^{\mathrm{Symm}}) = \binom{n+d-1}{d-1}. \tag{4.46}$$

For the qubit case, $d=2$, this gives back the result (4.25), since

$$\binom{n+2-1}{2-1} = \binom{n+1}{1} = n+1. \tag{4.47}$$

As an example, the qutrit ($d=3$) Dicke states for 1,2 and 3 elementary subsystems are

(4.48)

$$\begin{aligned}
n=1:\quad & |\,\mathrm{D}_{(1,0,0)}\rangle = |1\rangle, \quad |\,\mathrm{D}_{(0,1,0)}\rangle = |2\rangle, \quad |\,\mathrm{D}_{(0,0,1)}\rangle = |3\rangle,\\
n=2:\quad & |\,\mathrm{D}_{(2,0,0)}\rangle = |11\rangle, \quad |\,\mathrm{D}_{(0,2,0)}\rangle = |22\rangle, \quad |\,\mathrm{D}_{(0,0,2)}\rangle = |33\rangle,\\
& |\,\mathrm{D}_{(1,1,0)}\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \quad |\,\mathrm{D}_{(1,0,1)}\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |31\rangle),\\
& |\,\mathrm{D}_{(0,1,1)}\rangle = \frac{1}{\sqrt{2}}(|23\rangle + |32\rangle),\\
n=3:\quad & |\,\mathrm{D}_{(3,0,0)}\rangle = |111\rangle, \quad |\,\mathrm{D}_{(0,3,0)}\rangle = |222\rangle, \quad |\,\mathrm{D}_{(0,0,3)}\rangle = |333\rangle,\\
& |\,\mathrm{D}_{(2,1,0)}\rangle = \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) = |\mathrm{W}\rangle, \quad |\,\mathrm{D}_{(1,2,0)}\rangle = \frac{1}{\sqrt{3}}(|122\rangle + |212\rangle + |221\rangle),\\
& |\,\mathrm{D}_{(2,0,1)}\rangle = \frac{1}{\sqrt{3}}(|113\rangle + |131\rangle + |311\rangle), \quad |\,\mathrm{D}_{(1,0,2)}\rangle = \frac{1}{\sqrt{3}}(|133\rangle + |313\rangle + |331\rangle),\\
& |\,\mathrm{D}_{(0,2,1)}\rangle = \frac{1}{\sqrt{3}}(|223\rangle + |232\rangle + |322\rangle), \quad |\,\mathrm{D}_{(0,1,2)}\rangle = \frac{1}{\sqrt{3}}(|233\rangle + |323\rangle + |332\rangle),\\
& |\,\mathrm{D}_{(1,1,1)}\rangle = \frac{1}{\sqrt{6}}(|123\rangle + |132\rangle + |213\rangle + |231\rangle + |312\rangle + |321\rangle),
\end{aligned}$$

Qu*d*it Dicke states of n elementary subsystems can also be decomposed in terms of tensor products of Dicke states consisting of m and $n-m$ elementary subsystems as

$$|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} |\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle, \tag{4.49}$$

or with normalization

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \binom{n}{\boldsymbol{n}}^{-\frac{1}{2}} \left[\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \binom{m}{\boldsymbol{m}}^{\frac{1}{2}} |\,\mathrm{D}_{\boldsymbol{m}}\rangle \otimes \binom{n-m}{\boldsymbol{n}-\boldsymbol{m}}^{\frac{1}{2}} |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}}\rangle \right], \tag{4.50}$$

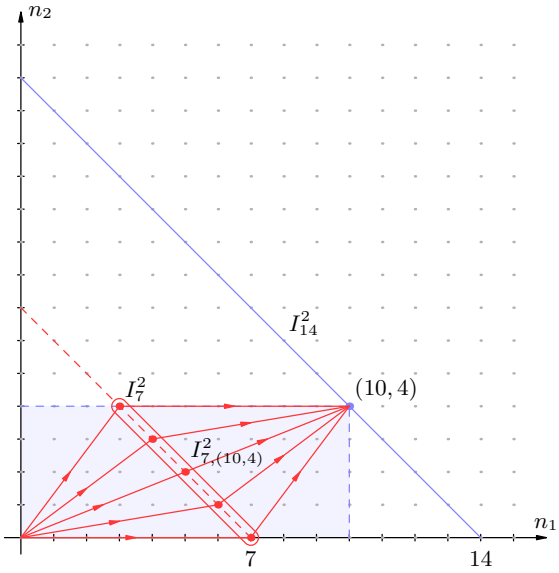


FIGURE 8. The set $I_{m,\boldsymbol{n}}^d$, for the 14-qubit Dicke state $|\mathrm{D}_{\boldsymbol{n}}\rangle$ with excitation $\boldsymbol{n} = (10, 4)$ and subsystem size $m = 7$.

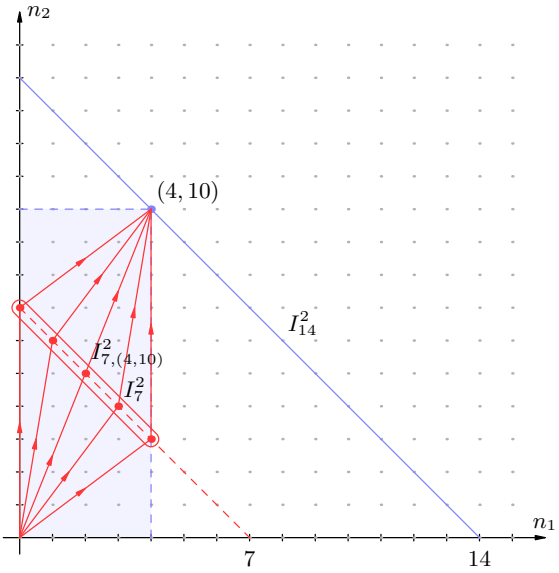


FIGURE 9. The set $I_{m,\boldsymbol{n}}^d$, for the 14-qubit Dicke state $|\mathrm{D}_{\boldsymbol{n}}\rangle$ with excitation $\boldsymbol{n} = (4, 10)$ and subsystem size $m = 7$.

where we introduced the multi-index set $I_{m,\boldsymbol{n}}^d$, which contains the multi-indices corresponding to the decomposition of the Dicke state of occupation $\boldsymbol{n}$ in dimension d with respect to the $m|(n-m)$ bipartition,

$$I_{m,\boldsymbol{n}}^d = \left\{ \boldsymbol{m} \in \mathbb{N}_0^d \;\middle|\; \|\boldsymbol{m}\|_1 = \sum_i m_i = m,\ m_i \leq n_i, \forall i \in \{1, 2, \ldots, d\} \right\} \subseteq I_m^d. \tag{4.51}$$

In the qu*d*it case, determining the set $I_{m,\boldsymbol{n}}^d$ in a closed form is difficult, since even in the $d = 2$ qubit case, the limits of the summation were not fixed. This set can be visualized as the intersection of the hyper-rectangle $\mathbf{0} \leq \boldsymbol{m} \leq \boldsymbol{n}$ with the hyperplane $|\boldsymbol{m}| = m$, which can not be parameterized nicely in a closed form for $d \geq 2$. We can see two examples for the (4.51) sets in the qubit case in Figures 8 and 9.

To prove (4.49), we want to see, that on the left-hand side and the right-hand side of (4.49) has the same terms in the basis expansions. To see this, we have that (i) for an arbitrary occupation $\boldsymbol{m} \in I_{m,\boldsymbol{n}}^d$, all the terms in the basis expansion of the vector $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ are the tensor product of basis vectors of the form $|i_1 \ldots i_m i_{m+1}, \ldots, i_n\rangle = |i_1 \ldots i_m\rangle \otimes |i_{m+1} \ldots i_n\rangle$ with coefficients $+1$; (ii) for different occupations $\boldsymbol{m}, \boldsymbol{m}' \in I_{m,\boldsymbol{n}}^d$, the terms in the basis expansions of the vectors $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ and $|\tilde{\mathrm{D}}_{\boldsymbol{m}'}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}'}\rangle$ contain different basis vectors; (iii) introducing the notation $m_j = |\{a \in L \mid i_a = j, 1 \leq a \leq m\}|$ and $n_j - m_j = |\{a \in L \mid i_a = j, m < a \leq n\}|$, and the multi-indices $\boldsymbol{n} = (n_1, n_2, \ldots n_d)$ and $\boldsymbol{m} = (m_1, m_2 \ldots, m_d)$, we get that every term $|i_1 \ldots i_m i_{m+1}, \ldots, i_n\rangle = |i_1 \ldots i_m\rangle \otimes |i_{m+1} \ldots i_n\rangle$ appearing in the (4.34) form of $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$ can be written as $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ for an arbitrary $\boldsymbol{m} \in I_{m,\boldsymbol{n}}^d$; (iv) for an arbitrary occupation $\boldsymbol{m} \in I_{m,\boldsymbol{n}}^d$, every term in $|\tilde{\mathrm{D}}_{\boldsymbol{m}}\rangle \otimes |\tilde{\mathrm{D}}_{\boldsymbol{n}-\boldsymbol{m}}\rangle$ is contained in $|\tilde{\mathrm{D}}_{\boldsymbol{n}}\rangle$.

Using (4.49) we can write the nonnormalized form of the (4.48) qutrit Dicke states as

$$
\begin{aligned}
n=2:\quad & |\tilde{\mathrm{D}}_{(2,0,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,0)}\rangle, \quad |\tilde{\mathrm{D}}_{(0,2,0)}\rangle = |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(0,0,2)}\rangle = |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,1)}\rangle,\\
& |\tilde{\mathrm{D}}_{(1,1,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,0)}\rangle + |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(1,0,1)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,1)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(0,1,1)}\rangle = |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,1)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,0)}\rangle,\\
n=3:\quad & |\tilde{\mathrm{D}}_{(3,0,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(2,0,0)}\rangle, \quad |\tilde{\mathrm{D}}_{(0,3,0)}\rangle = |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,2,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(0,0,3)}\rangle = |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,2)}\rangle,\\
& |\tilde{\mathrm{D}}_{(2,1,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,1,0)}\rangle + |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(2,0,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(1,2,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,2,0)}\rangle + |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,1,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(2,0,1)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,1)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(2,0,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(1,0,2)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,2)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,1)}\rangle,\\
& |\tilde{\mathrm{D}}_{(0,2,1)}\rangle = |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,1)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,2,0)}\rangle,\\
& |\tilde{\mathrm{D}}_{(0,1,2)}\rangle = |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,0,2)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,1)}\rangle,\\
& |\tilde{\mathrm{D}}_{(1,1,1)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(0,1,1)}\rangle + |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,0,1)}\rangle + |\tilde{\mathrm{D}}_{(0,0,1)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,1,0)}\rangle.
\end{aligned}
\tag{4.52}
$$

It is easy to see, using (4.48) that these are indeed correct. Note that the (3.72a) Schmidt decomposition of the W state is just the normalized form of $|\tilde{\mathrm{D}}_{(2,1,0)}\rangle = |\tilde{\mathrm{D}}_{(1,0,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(1,1,0)}\rangle + |\tilde{\mathrm{D}}_{(0,1,0)}\rangle \otimes |\tilde{\mathrm{D}}_{(2,0,0)}\rangle$ from (4.52).

Since the Dicke states form orthonormal bases, the formula (4.50) is the

$$
|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}}}\,|\,\mathrm{D}_{\boldsymbol{m}}\rangle \otimes |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}}\rangle \tag{4.53}
$$

Schmidt decomposition (3.9) of the $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ Dicke state with Schmidt coefficients following the multinomial generalization of the hypergeometric distribution

$$
\eta^{\boldsymbol{n}}_{\boldsymbol{m}} = \frac{\binom{m}{\boldsymbol{m}}\binom{n-m}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}} = \frac{1}{\binom{n}{m}}\prod_{i=1}^{d}\binom{n_i}{m_i} = \eta^{\boldsymbol{n}}_{\boldsymbol{n}-\boldsymbol{m}}, \tag{4.54}
$$

where $d=2$ gives the usual (binomial) hypergeometric distribution. Using this decomposition and (3.10a), the *m-partite reduced Dicke states* are

$$
\rho_{\boldsymbol{n},m} := \mathrm{Tr}_{n-m}\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big) = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\,|\,\mathrm{D}_{\boldsymbol{m}}\rangle\langle\mathrm{D}_{\boldsymbol{m}}\,|. \tag{4.55}
$$

Note that the reduced states of subsystems of size m has the same form because of the symmetry of the Dicke states. Tr_{n-m} simply denotes the partial trace over an arbitrary subsystem of size $n-m$.

4.4. **Combinatorial identities from Dicke states.** Using the (4.55) form of reduced Dicke states, and the fact that every density operator is of trace one, we get that for all $\boldsymbol{n}\in I^d_n$ and $m\le n$,

$$
\mathrm{Tr}(\rho_{\boldsymbol{n},m}) = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} = 1, \tag{4.56}
$$

which, using the two forms (4.54) of the Schmidt coefficients, boils down to

$$\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \prod_{i=1}^{d} \binom{n_i}{m_i} = \binom{n}{m}, \tag{4.57a}$$

$$\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \binom{m}{\boldsymbol{m}} \binom{n-m}{\boldsymbol{n}-\boldsymbol{m}} = \binom{n}{\boldsymbol{n}}. \tag{4.57b}$$

These two identities are equivalent, and both of these are the generalizations of the Chu-Vandermonde identity [60, 61], $\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$. The (4.57a) form is already known [62], and (4.57b) is given by multinomial coefficients in a more elegant form. Note that these identities came simply form quantum probability theory (4.56) of Dicke states, while the direct proof would be more lengthy.

Using (3.8), we get that for all $\boldsymbol{n} \in I^d_n$ and $k \leq m \leq n$,

$$\begin{aligned}
\mathrm{Tr}_{m-k}\big(\mathrm{Tr}_{n-m}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle \mathrm{D}_{\boldsymbol{n}}\,|)\big) &= \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \sum_{\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\eta^{\boldsymbol{m}}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,| \\
= \sum_{\boldsymbol{k}\in I^d_k} \sum_{\substack{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}} \\ \text{s.t.}\,\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\eta^{\boldsymbol{m}}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,| &= \sum_{\boldsymbol{k}\in I^d_{k,\boldsymbol{n}}} \sum_{\substack{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}} \\ \text{s.t.}\,\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\eta^{\boldsymbol{m}}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,| \\
&= \mathrm{Tr}_{n-k}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle \mathrm{D}_{\boldsymbol{n}}\,|) = \rho_{\boldsymbol{m},k} = \sum_{\boldsymbol{k}\in I^d_{k,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,|,
\end{aligned} \tag{4.58}$$

where we used that if $\boldsymbol{k} \notin I^d_{k,\boldsymbol{n}}$, then $\boldsymbol{k} \notin I^d_{k,\boldsymbol{m}}$, so in the sum for $\boldsymbol{k}$ we can have $\boldsymbol{k} \in I^d_{k,\boldsymbol{n}}$ instead of $\boldsymbol{k} \in I^d_{k,\boldsymbol{n}}$. From (4.58), we get for all $\boldsymbol{n} \in I^d_n$ and $\boldsymbol{k} \in I^d_{k,\boldsymbol{n}}$

$$\sum_{\substack{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}} \\ \text{s.t.}\,\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\eta^{\boldsymbol{m}}_{\boldsymbol{k}} = \eta^{\boldsymbol{n}}_{\boldsymbol{k}}. \tag{4.59}$$

It turns out that this boils down to the (4.57a), however, this is given directly for the generalized hypergeometric distribution. Indeed, using the (4.54) form of the Schmidt coefficients, we get

$$\sum_{\substack{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}} \\ \text{s.t.}\,\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}}} \frac{1}{\binom{n}{m}} \frac{1}{\binom{m}{k}} \prod_{i=1}^{d} \binom{n_i}{m_i}\binom{m_i}{k_i} = \frac{1}{\binom{n}{k}} \prod_{i=1}^{d} \binom{n_i}{k_i}. \tag{4.60}$$

After using the $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ identity and simplifications, (4.60) boils down to

$$1 = \sum_{\substack{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}} \\ \text{s.t.}\,\boldsymbol{k}\in I^d_{k,\boldsymbol{m}}}} \frac{1}{\binom{n-k}{m-k}} \prod_{i=1}^{d} \binom{n_i-k_i}{m_i-k_i} = \sum_{\substack{\boldsymbol{m}:\boldsymbol{k}\leq\boldsymbol{m}\leq\boldsymbol{n} \\ \|\boldsymbol{m}\|_1=m}} \frac{1}{\binom{n-k}{m-k}} \prod_{i=1}^{d} \binom{n_i-k_i}{m_i-k_i}, \tag{4.61}$$

which is equivalent to (4.57a) with the $\boldsymbol{n} \mapsto \boldsymbol{n}-\boldsymbol{k}$ and $\boldsymbol{m} \mapsto \boldsymbol{m}-\boldsymbol{k}$ substitutions.

5. Detecting correlation and entanglement of Dicke states

In this section, we show that the pure and reduced Dicke states are entangled and therefore correlated, using entanglement detection methods from Section 3.8.

5.1. **Pure states.** To detect entanglement in the pure $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ Dicke states, we use the (4.53) Schmidt decomposition. For states with at least two occupations ($\boldsymbol{n}$ has at least two nonzero elements) the (4.51) $I^d_{m,\boldsymbol{n}}$ set contains at least two occupations, meaning that these states have a Schmidt rank at least two for an $m|(n-m)$ bipartition, so these states are entangled for this bipartition. In the case of Dicke states with only one occupation, $\boldsymbol{n}=(0,0,\ldots,n,\ldots,0)$, the pure Dicke states are $|\,\mathrm{D}_{\boldsymbol{n}}\rangle=|ii\ldots i\rangle$, which are obviously not correlated and entangled.

Since Dicke states are entangled, we can formulate entanglement witnesses detecting entanglement in the vicinity of them. For an $X|\overline{X}$ bipartition the (3.70) entanglement witness by Dicke state $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ is

$$W_{X|\overline{X}}=\max_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}I-|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|=\max_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}I-|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|. \tag{5.1}$$

The (3.71) GME witness by Dicke state $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ is

$$W_{\mathrm{bisep}}=\max_{1\leq m\leq n/2}\max_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}I-|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|=\max_{1\leq m\leq n/2}\max_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}}\frac{\binom{m}{\boldsymbol{m}}\binom{n-m}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}I-|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|. \tag{5.2}$$

For qubit Dicke states these GME witnesses were calculated [13, 16], namely, the witness detecting GME in the vicinity of the n-qubit Dicke state with occupation $\boldsymbol{n}=(n/2,n/2)$ is

$$W=\frac{1}{2}\frac{n}{n-1}I-|\,\mathrm{D}_{(n/2,n/2)}\rangle\langle\mathrm{D}_{(n/2,n/2)}\,|, \tag{5.3}$$

and the witness detecting GME in the vicinity of the n-qubit Dicke state with occupation $\boldsymbol{n}=(n-k,k)$, with $1<k<n/2$, is

$$W=\frac{n-k}{n}I-|\,\mathrm{D}_{(n-k,k)}\rangle\langle\mathrm{D}_{(n-k,k)}\,|. \tag{5.4}$$

5.2. **Reduced states.** To detect entanglement in the reduced Dicke states, we use the (3.66) PPT criterion. Our calculation will follow Ref [44].

First of all, note that in (4.45) we have seen that Dicke states are symmetric and we can formulate a stronger version of the (3.66) PPT criterion for symmetric states. For this, we use the fact that every symmetric state is either fully separable or GME [63], since every pure decomposition of symmetric states consists of pure symmetric states, which are known to be uncorrelated or GME. Therefore if a symmetric state is entangled for one bipartition, it is entangled for all bipartitions and can not be mixed from bipartite separable states (GME), and if a symmetric state is separable for a bipartition, it is separable for all partitions (fully separable). Using this, the (3.66) criterion takes the form

$$\text{symmetric state } \rho \text{ is fully separable} \quad\Longrightarrow\quad \forall X\in L:\rho^{\mathrm{T}_X}\geq 0, \tag{5.5a}$$

$$\text{symmetric state } \rho \text{ is GME} \quad\Longleftarrow\quad \exists X\in L:\rho^{\mathrm{T}_X}\not\geq 0. \tag{5.5b}$$

To detect bipartite entanglement in the $\rho_{\boldsymbol{n},m}$ m-partite reduced Dicke states (4.55) with respect to a bipartition into k ($1\leq k\leq m-1$) and $m-k$ subsystems, writing a Schmidt decomposition in the vectors in (4.55) is beneficial. This way the reduced state takes the form

$$\rho_{\boldsymbol{n},m}=\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}}|\,\mathrm{D}_{\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{k}'}\,|\otimes|\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|. \tag{5.6}$$

Note that the partial transposed operator of the state $\rho_{\boldsymbol{n},m}$ takes the same form for all subsystems with size k due to the symmetry of the Dicke states. T_k denotes the partial transpose over an

arbitrary subsystem with size k,

$$\rho^{\mathrm{T}_k}_{\boldsymbol{n},m} = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}} \sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}} |\,\mathrm{D}_{\boldsymbol{k}'}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,| \otimes |\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|. \tag{5.7}$$

To detect entanglement, we need to show that the $\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}$ partial transpose is non positive, which is true if we find a vector $|\psi\rangle$, such that $\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle \leq 0$. For this $|\psi\rangle$, let us use the following ansatz

$$|\psi\rangle := \alpha|\,\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle \otimes |\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle + \beta|\,\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle \otimes |\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle, \tag{5.8}$$

with the $\hat{\boldsymbol{m}} \in I^d_m$, $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}' \in I^d_k$ and $\alpha, \beta \in \mathbb{C}$ free parameters. With this vector

$$\begin{aligned}\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle = &\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}} \\ &\Big[\overline{\alpha}\alpha\langle \mathrm{D}_{\hat{\boldsymbol{k}}}\,|\,\mathrm{D}_{\boldsymbol{k}'}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle\langle \mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\,|\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle \\ &+ \overline{\alpha}\beta\langle \mathrm{D}_{\hat{\boldsymbol{k}}}\,|\,\mathrm{D}_{\boldsymbol{k}'}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle\langle \mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\,|\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle \\ &+ \overline{\beta}\alpha\langle \mathrm{D}_{\hat{\boldsymbol{k}}'}\,|\,\mathrm{D}_{\boldsymbol{k}'}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\hat{\boldsymbol{k}}}\rangle\langle \mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\,|\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\rangle \\ &+ \overline{\beta}\beta\langle \mathrm{D}_{\hat{\boldsymbol{k}}'}\,|\,\mathrm{D}_{\boldsymbol{k}'}\rangle\langle \mathrm{D}_{\boldsymbol{k}}\,|\,\mathrm{D}_{\hat{\boldsymbol{k}}'}\rangle\langle \mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\,|\,\mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}}\rangle\langle \mathrm{D}_{\boldsymbol{m}-\boldsymbol{k}'}\,|\,\mathrm{D}_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\rangle\Big].\end{aligned} \tag{5.9}$$

Using the $\langle \mathrm{D}_{\boldsymbol{n}}\,|\,\mathrm{D}_{\boldsymbol{n}'}\rangle = \delta_{\boldsymbol{n},\boldsymbol{n}'}$ orthonormality of the Dicke states, we get

$$\begin{aligned}\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle = &\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \sum_{\boldsymbol{k},\boldsymbol{k}'\in I^d_{k,\boldsymbol{m}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\boldsymbol{k}'}} \\ &\Big[\overline{\alpha}\alpha\delta_{\hat{\boldsymbol{k}},\boldsymbol{k}'}\delta_{\boldsymbol{k},\hat{\boldsymbol{k}}}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\boldsymbol{k}}\delta_{\boldsymbol{m}-\boldsymbol{k}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'} + \overline{\alpha}\beta\delta_{\hat{\boldsymbol{k}},\boldsymbol{k}'}\delta_{\boldsymbol{k},\hat{\boldsymbol{k}}'}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\boldsymbol{k}}\delta_{\boldsymbol{m}-\boldsymbol{k}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}} \\ &\overline{\beta}\alpha\delta_{\hat{\boldsymbol{k}}',\boldsymbol{k}'}\delta_{\boldsymbol{k},\hat{\boldsymbol{k}}}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}},\boldsymbol{m}-\boldsymbol{k}}\delta_{\boldsymbol{m}-\boldsymbol{k}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'} + \overline{\beta}\beta\delta_{\hat{\boldsymbol{k}}',\boldsymbol{k}'}\delta_{\boldsymbol{k},\hat{\boldsymbol{k}}'}\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}},\boldsymbol{m}-\boldsymbol{k}}\delta_{\boldsymbol{m}-\boldsymbol{k}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\Big].\end{aligned} \tag{5.10}$$

After summing for $\boldsymbol{k}$ and for $\boldsymbol{k}'$, we get

$$\begin{aligned}\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle = &\sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\Big[\overline{\alpha}\alpha\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\hat{\boldsymbol{k}}}\delta_{\boldsymbol{m}-\hat{\boldsymbol{k}},\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}\delta\big(\hat{\boldsymbol{k}}\in I^d_{k,\boldsymbol{m}}\big) \\ &+ \overline{\alpha}\beta\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}',\boldsymbol{m}-\hat{\boldsymbol{k}}'}\delta_{\boldsymbol{m}-\hat{\boldsymbol{k}},\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{k}}\in I^d_{k,\boldsymbol{m}}\big)\delta\big(\hat{\boldsymbol{k}}'\in I^d_{k,\boldsymbol{m}}\big) \\ &+ \overline{\beta}\alpha\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}},\boldsymbol{m}-\hat{\boldsymbol{k}}}\delta_{\boldsymbol{m}-\hat{\boldsymbol{k}}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}'}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{k}}\in I^d_{k,\boldsymbol{m}}\big)\delta\big(\hat{\boldsymbol{k}}'\in I^d_{k,\boldsymbol{m}}\big) \\ &+ \overline{\beta}\beta\delta_{\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}},\boldsymbol{m}-\hat{\boldsymbol{k}}'}\delta_{\boldsymbol{m}-\hat{\boldsymbol{k}}',\hat{\boldsymbol{m}}-\hat{\boldsymbol{k}}}\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{k}}'\in I^d_{k,\boldsymbol{m}}\big)\Big],\end{aligned} \tag{5.11}$$

where $\delta\big(\boldsymbol{a} \in A\big)$ gives 1 if $\boldsymbol{a} \in A$ and 0 otherwise. After using the properties of the Kronecker-delta, namely $\delta_{\boldsymbol{a},\boldsymbol{a}'} = \delta^2_{\boldsymbol{a},\boldsymbol{a}'} = \delta_{\boldsymbol{a}',\boldsymbol{a}}$ and $\delta_{\boldsymbol{a}-\boldsymbol{b},\boldsymbol{a}'-\boldsymbol{b}'} = \delta_{\boldsymbol{a},\boldsymbol{a}'-\boldsymbol{b}'+\boldsymbol{b}}$, we get

$$\begin{aligned}\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle = \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \Bigg[& \overline{\alpha}\alpha\delta_{\boldsymbol{m},\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\boldsymbol{m}}\big) \\ & + \overline{\alpha}\beta\delta_{\boldsymbol{m},\hat{\boldsymbol{m}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\boldsymbol{m}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\boldsymbol{m}}\big) \\ & + \overline{\beta}\alpha\delta_{\boldsymbol{m},\hat{\boldsymbol{m}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\boldsymbol{m}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\boldsymbol{m}}\big) \\ & + \overline{\beta}\beta\delta_{\boldsymbol{m},\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\eta^{\boldsymbol{m}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\boldsymbol{m}}\big)\Bigg],\end{aligned} \tag{5.12}$$

where $\boldsymbol{\Delta} = \hat{\boldsymbol{k}}' - \hat{\boldsymbol{k}}$. Now, after summing for $\boldsymbol{m}$, we get

$$\begin{aligned}\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle = & \,\overline{\alpha}\alpha\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\eta^{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}}\delta\big(\hat{\boldsymbol{m}}-\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\big) \\ & + \overline{\alpha}\beta\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big) \\ & + \overline{\beta}\alpha\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}}\sqrt{\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big) \\ & + \overline{\beta}\beta\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\eta^{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}}+\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\big)\end{aligned} \tag{5.13}$$

To detect entanglement, we need $\langle\psi|\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}|\psi\rangle < 0$. But since (5.13) is a hermitian form of nonnegative coefficients of the complex numbers α and β, which can only take negative values if its discriminant,

(5.14)
$$\begin{aligned}&\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\eta^{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}}-\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}}+\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\big) \\ &- \big(\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\big)^2\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}\big)\delta\big(\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}\big)\end{aligned}$$

is negative. Notice that the second term can not vanish if we want the discriminant to be negative, so the constraints in the deltas need to hold, namely $\hat{\boldsymbol{m}} \in I^d_{m,\boldsymbol{n}}$, $\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}$ and $\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$. We can see that if these constraints hold, $\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}$ and $\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}$ should hold as well. To see this, let us go back to the definition (4.51) and let us use that if $\hat{\boldsymbol{k}} \in I^d_{k,\hat{\boldsymbol{m}}}$, then $\hat{\boldsymbol{k}} \leq \hat{\boldsymbol{m}}$, so $\boldsymbol{0} \leq \hat{\boldsymbol{m}} - \hat{\boldsymbol{k}}$. Adding $\hat{\boldsymbol{k}}'$ to both sides, we get $\hat{\boldsymbol{k}}' \leq \hat{\boldsymbol{m}} + \hat{\boldsymbol{k}}' - \hat{\boldsymbol{k}} = \hat{\boldsymbol{m}} + \boldsymbol{\Delta}$ together with $\hat{\boldsymbol{k}}' \in I^d_k$, we get that $\hat{\boldsymbol{k}}' \in I^d_{k,\hat{m}} + \boldsymbol{\Delta}$. (We can see that $\boldsymbol{k} \in I^d_{k,\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}$, using the constrain $\hat{\boldsymbol{k}}' \in I^d_{k,\hat{\boldsymbol{m}}}$.) With the help of these, the negativity of the discriminant is equivalent to

$$\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\eta^{\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}_{\hat{\boldsymbol{k}}'}\delta\big(\hat{\boldsymbol{m}}-\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}}+\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big) < \big(\eta^{\boldsymbol{n}}_{\hat{\boldsymbol{m}}}\big)^2\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}}\eta^{\hat{\boldsymbol{m}}}_{\hat{\boldsymbol{k}}'}. \tag{5.15}$$

After substituting the (4.54) form of the Schmidt coefficients, we get

$$\binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}+\boldsymbol{\Delta}}\binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}-\boldsymbol{\Delta}}\delta\big(\hat{\boldsymbol{m}}-\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big)\delta\big(\hat{\boldsymbol{m}}+\boldsymbol{\Delta} \in I^d_{m,\boldsymbol{n}}\big) < \binom{n-m}{\boldsymbol{n}-\hat{\boldsymbol{m}}}^2. \tag{5.16}$$

If one of the constraints on the left hand size is violated, then the relation holds, however this is not always the case. If there are at least two nonzero coefficients in $\boldsymbol{n}$, for example $n_i, n_j \neq 0$, we can always choose $\boldsymbol{m} \in I^d_{m,\boldsymbol{n}}$, such that $m_i, m_j \neq 0$ and $\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}'$ such that they only differ in two positions, meaning $\Delta_i = 1$, $\Delta_j = -1$ and zero elsewhere. In that case, after substituting the

(4.36) form of the multinomial coefficients, the (5.16) inequality reads as

(5.17)
$$\frac{(n-m)!}{(n_i-\hat{m}_i+1)!(n_j-\hat{m}_j-1)!}\frac{(n-m)!}{(n_i-\hat{m}_i-1)!(n_j-\hat{m}_j+1)!}<\left(\frac{(n-m)!}{(n_i-\hat{m}_i)!(n_j-\hat{m}_j)!}\right)^2,$$

which boils down to

(5.18)
$$\frac{n_i-\hat{m}_i}{n_i-\hat{m}_i+1}\frac{n_j-\hat{m}_j}{n_j-\hat{m}_j+1}<1,$$

which holds for every $\hat{\boldsymbol{m}}\in I^d_{m,\boldsymbol{n}}$.

Summing up, in this section we showed that for at least two occupations ($\boldsymbol{n}$ has at least two nonzero elements) the m-partite reduced Dicke states, $\rho_{\boldsymbol{n},m}=\mathrm{Tr}_{n-m}\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big)$ for all $2\leq m\leq n$ are NPT for all bipartitions, $\rho^{\mathrm{T}_k}_{\boldsymbol{n},m}\not\geq 0$ for all $1\leq k\leq m$, which together with the (5.5) Peres-Horodecki criterion for symmetric states tells us that these states NPT-GME. Note that in the case $m=n$ we get that the pure Dicke states $|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|$ are entangled, which we have seen before.

In the case of states with only one occupation, $\boldsymbol{n}=(0,\ldots,n,\ldots,0)$, the Dicke states and all reduced states reads as $|ii\ldots i\rangle\langle ii\ldots i|$ uncorrelated states, which are obviously not entangled.

Our result was also justified later, following from the general theory of the entanglement of mixtures of qudit Dicke states in Ref [33].

6. Measuring correlation and entanglement of Dicke states

In this section we examine the (3.77) relative entropy of ξ-correlation and (3.85) relative entropy of $X|\overline{X}$-entanglement of the pure Dicke states $|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|$ and the m-partite reduced Dicke states $\rho_{\boldsymbol{n},m}$ in different cases.

6.1. **Pure states.** Based on the (4.53) Schmidt decomposition of Dicke states and (3.84), the realtive entropy of $X|\overline{X}$-correlation of the $|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|$ pure Dicke states is

(6.1)
$$\begin{aligned}C_{X|\overline{X}}\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big)&=-2\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\\&=-2\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right).\end{aligned}$$

We can calculate the relative entropy of ξ-correlation of the $|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|$ pure Dicke states for an arbitrary partition $\xi\in P_{\mathrm{I}}$, based on (3.77), using that $S(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|)=0$, since it is a pure state. The relative entropy of ξ-correlation is $C_\xi(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|)=\sum_{X\in\xi}S_X(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|)$. For an arbitrary ξ partition the relative entropy of ξ-correlation of the pure Dicke states is

(6.2)
$$\begin{aligned}C_\xi\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big)&=-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\right)\\&=-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)\right),\end{aligned}$$

which for bipartitions $\xi=X|\overline{X}$ gives back the formula (6.1).

Examples of these correlation measures can be seen in Figures 10. and 11. Note that in the figures only the size of the X_i parts of the partitions are taken into consideration, because of the symmetry of the Dicke states, only the sizes of the parts matter in the (6.2) relative entropy of

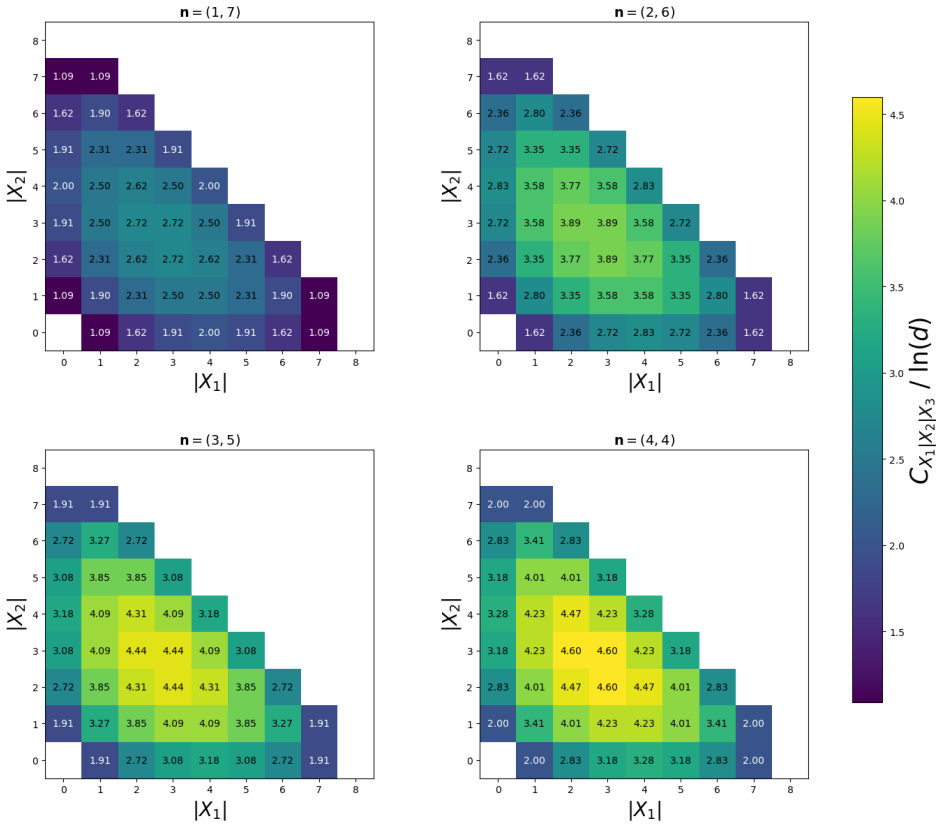


FIGURE 10. The C_ξ correlation (6.2) of $|\,\mathrm{D}_{(n_1,n_2)}\rangle$ pure n-qubit Dicke states consisting of $n = 8$ elementary subsystems for a partition $\xi = X_1|X_2|X_3$, in terms of $|X_1|$ and $|X_2|$, for different values of n_2.

correlation. From these figures one can see that the correlation increases as the partitions get more and more uniform. Note that this is independent of the (3.79) multipartite monotonicity of correlation, since the different tripartitions are not related by refinement. The correlation also increases for more and more uniform occupations. It can also be seen in Figure 11 that correlation is invariant under the exchange of the n_i elements of the occupations. This is because such a change corresponds to an exchange of the basis vectors in the elementary subsystems, which is a local unitary transformation, under which the correlation is invariant. In Figure 10, one can see that the correlation is also invariant under the exchange of the sizes of the partitions, which is the consequence of the symmetry of the Dicke states.

Unfortunately, entanglement can only be calculated for bipartitions because the minimization in the (3.80) formula of the relative entropy of ξ-entanglement could not be given in a closed form for an arbitrary partition or a mixed state, unlike in the relative entropy of correlation (3.77). Based on the (6.1) relative entropy of $X|\overline{X}$-correlation of pure Dicke states and (3.85), the relative entropy of $X|\overline{X}$-entanglement of the pure Dicke states is

$$E_{X|\overline{X}}\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big) = \frac{1}{2}C_{X|\overline{X}}\big(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|\big) = -\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}} \frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}} \ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right). \tag{6.3}$$

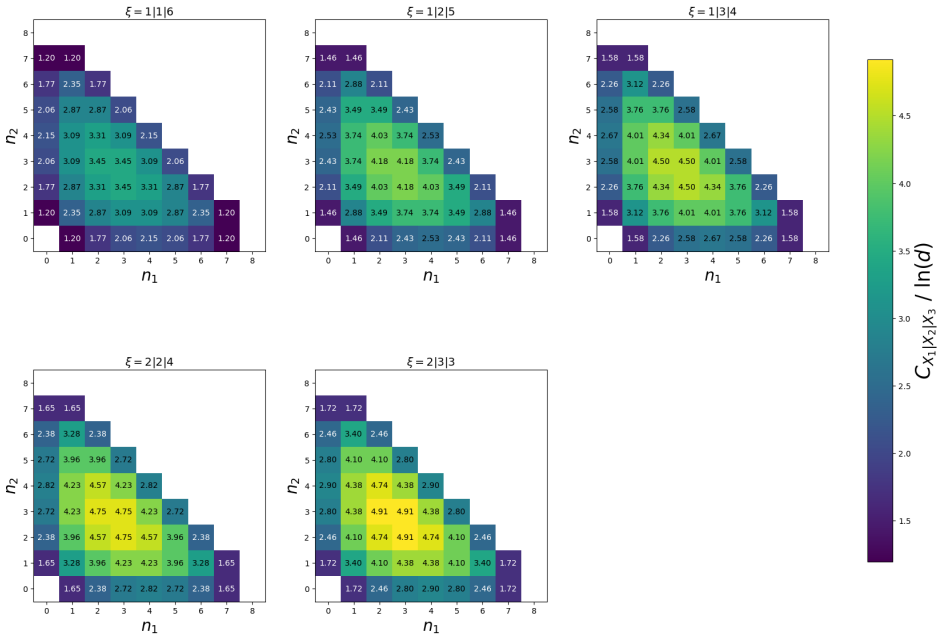


FIGURE 11. The C_ξ correlation (6.2) of $|\,\mathrm{D}_{(n_1,n_2,n_3)}\rangle$ pure n-qutrit Dicke states consisting of $n = 8$ elementary subsystems for different partitions, in terms of of n_1 and n_2 values.

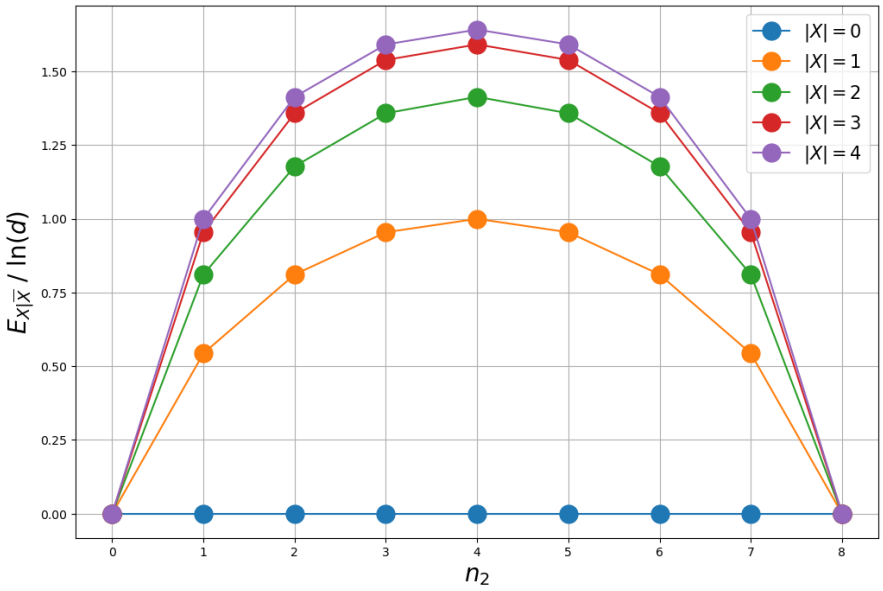


FIGURE 12. The $E_{X|\overline{X}}$ entanglement (6.3) of $|\,\mathrm{D}_{(n_1,n_2)}\rangle$ pure n-qubit Dicke states consisting of $n = 8$ elementary subsystems, for different $X|\overline{X}$ bipartitions, in terms of n_2.

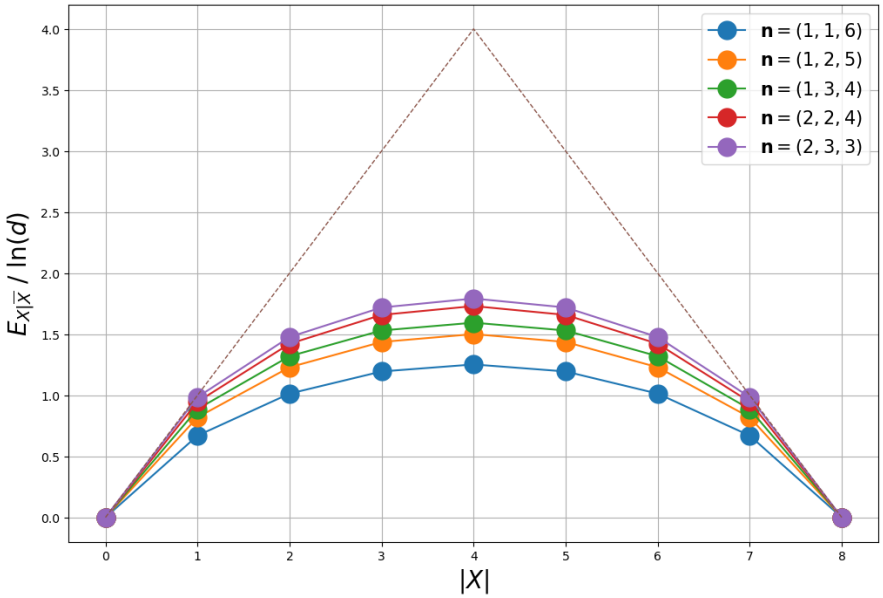


FIGURE 13. The $E_{X|\overline{X}}$ entanglement of $|\,\mathrm{D}_{(n_1,n_2,n_3)}\rangle$ pure n-qutrit Dicke states consisting of $n = 8$ elementary subsystems of dimension $d = 3$, for an $X|\overline{X}$ bipartition, for different occupation number vectors $\boldsymbol{n}$ in terms of $|X|$. The dotted line denotes the upper bound [9] $E_{X|\overline{X}} \leq \min\{|X|, |\overline{X}|\} \ln(d)$.

Examples of these entanglement measures can be seen in Figures 12., 13. and 14. Note that in the figures only the size of the X_i parts of the partitions are taken into consideration, because of the symmetry of the Dicke states, only the sizes of the parts matter in the (6.3) relative entropy of entanglement. From these figures one can see that the entanglement increases as the partitions get more and more uniform. Note that this is independent of the (3.82) multipartite monotonicity of entanglement, since the different bipartitions are not related by refinement. The entanglement also increases for more uniform occupations. This suggest that as we get more and more uniform occupations, we get farther from the Dicke states with occupation one, which are uncorrelated and separable. From Figure 13 one can see that the entanglement is invariant under the exchange of the sizes of the partitions, which is the manifestation of the symmetry of the Dicke states, but as the partitions get more and more uniform, entanglement gets farther from its theoretical maximum. It can also be seen in Figures 12 and 14 that entanglement is invariant under the exchange of the n_i elements of the occupations. This is because such a change corresponds to an exchange of the basis vectors in the elementary subsystems, which is a local unitary transformation, under which the entanglement is invariant.

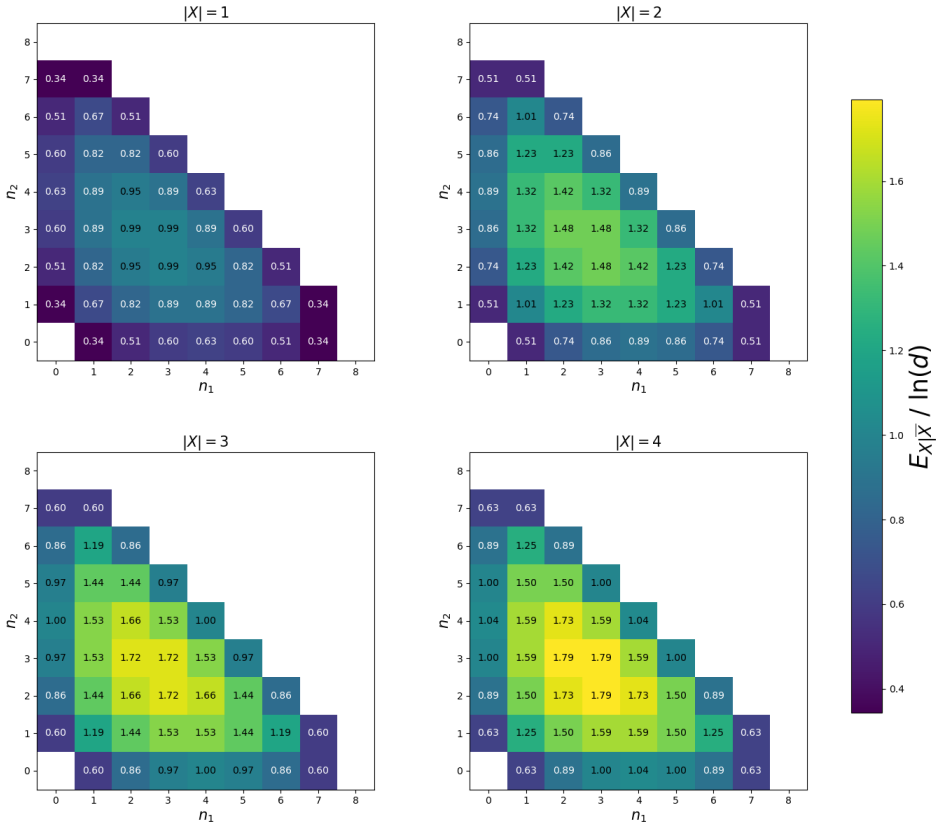


FIGURE 14. The $E_{X|\overline{X}}$ entanglement of $|\,\mathrm{D}_{(n_1,n_2,n_3)}\rangle$ pure n-qutrit Dicke states consisting of $n = 8$ elementary subsystems of dimension $d = 3$, for different $X|\overline{X}$ bipartitions, in terms of the n_1 and n_2 components of the occupation number vector $\boldsymbol{n}$.

6.2. **Reduced states.** To calculate the relative entropy of ξ-correlation of the $\rho_{\boldsymbol{n},m}$ m-partite reduced Dicke states (4.55), we should use that according to (3.8), the k-partite reduced states of these states are the $\rho_{\boldsymbol{n},k}$ k-partite reduced Dicke states. Using the (4.55) form of $\rho_{\boldsymbol{n},m}$, the (3.77) relative entropy of ξ correlation of the m-partite reduced Dicke states for an arbitrary partition is

(6.4)

$$C_\xi(\rho_{\boldsymbol{n},m}) = \sum_{X\in\xi} S(\rho_{\boldsymbol{n},|X|}) - S(\rho_{\boldsymbol{n},m}) = -\sum_{X\in\xi}\left[\sum_{\boldsymbol{k}\in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{k}})\right] + \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})$$

$$= -\sum_{|X|\in\xi}\left(\sum_{\boldsymbol{k}\in I^d_{|X|,\boldsymbol{n}}} \frac{\binom{|X|}{\boldsymbol{k}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{k}}}{\binom{n}{\boldsymbol{n}}} \ln\left(\frac{\binom{|X|}{\boldsymbol{k}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{k}}}{\binom{n}{\boldsymbol{n}}}\right)\right) + \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \frac{\binom{m}{\boldsymbol{m}}\binom{n-m}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}} \ln\left(\frac{\binom{n}{\boldsymbol{m}}\binom{n-m}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)$$

These relative entropies are illustrated in Figure 15. From this figure, it is easy to see that the correlation increases as the size of the subsystem increases. It can also be seen, that for uniform partitions (where every part has the same size, $\xi = \frac{m}{|\xi|}\big|\frac{m}{|\xi|}\big|\ldots\big|\frac{m}{|\xi|}$), the correlation is larger if the number of parts is larger although these partitions are generally not related by the

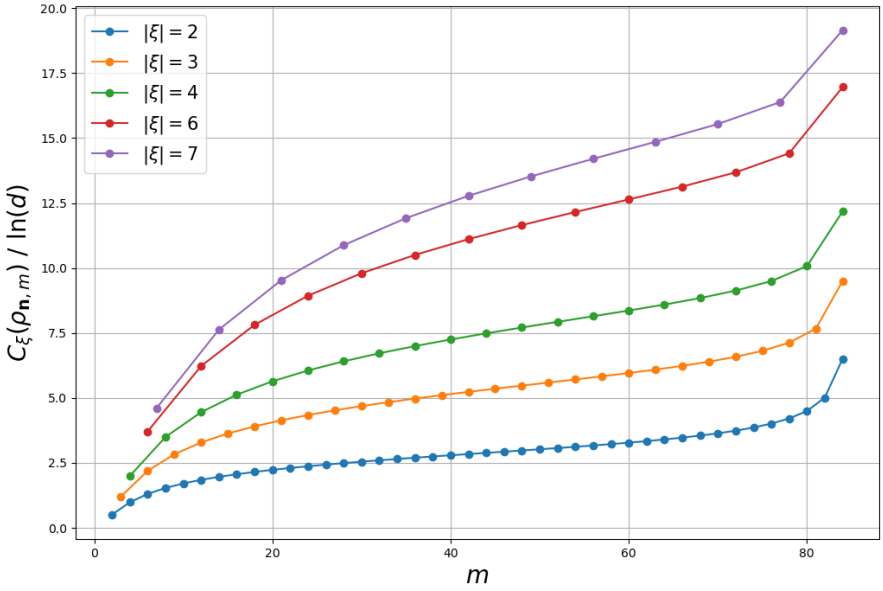


FIGURE 15. The C_ξ relative entropy of correlation (6.4) of the m-qubit reduced states $\rho_{(42,42),m}$ of the 84-qubit Dicke state $|\,\mathrm{D}_{(42,42)}\rangle$, for different ξ partitions, where the parts of the partitions have the same size, $\xi = \frac{m}{|\xi|}\big|\frac{m}{|\xi|}\big|\ldots\big|\frac{m}{|\xi|}$.

partial ordering of the partitions. For example, the partitions containing two parts (blue graph) are coarser than the partitions containing four parts (green graph), $\frac{m}{4}|\frac{m}{4}|\frac{m}{4}|\frac{m}{4} \preceq \frac{m}{2}|\frac{m}{2}$, but not related to the partitions containing three parts (orange graph), $\frac{m}{3}|\frac{m}{3}|\frac{m}{3} \not\preceq \frac{m}{2}|\frac{m}{2}$.

Unfortunately, entanglement can only be calculated for pure states and bipartitions because the minimization in the (3.80) formula of the relative entropy of ξ-entanglement could not be given in a closed form for an arbitrary partition or a mixed state, unlike in the relative entropy of correlation (3.77).

7. Partitionability, producibility and stretchability of Dicke states

In this section we calculate the (3.90a) relative entropy of k-partitionability correlation, the (3.90b) relative entropy of k-producibility correlation and the (3.90c) relative entropy of k-stretchability correlation of the $|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|$ pure Dicke states and the $\rho_{\boldsymbol{n},m}$ m-partite reduced Dicke states using (6.2) and (6.4). Unfortunately these calculations are not possible for entanglement, since we could only evaluate the measures of entanglement for pure states and bipartitions.

7.1. **Pure states.** For an arbitrary $\boldsymbol{\xi}$ partition ideal, the relative correlation of $\boldsymbol{\xi}$-correlation of pure Dicke states can be easily calculated using (3.87) and (6.2),

$$
\begin{aligned}
C_{\boldsymbol{\xi}}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) &= \min_{\xi\in\boldsymbol{\xi}} C_\xi(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) = \min_{\xi\in\boldsymbol{\xi}}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\right)\right] \\
&= \min_{\xi\in\boldsymbol{\xi}}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}} \frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)\right)\right].
\end{aligned}
\tag{7.1}
$$

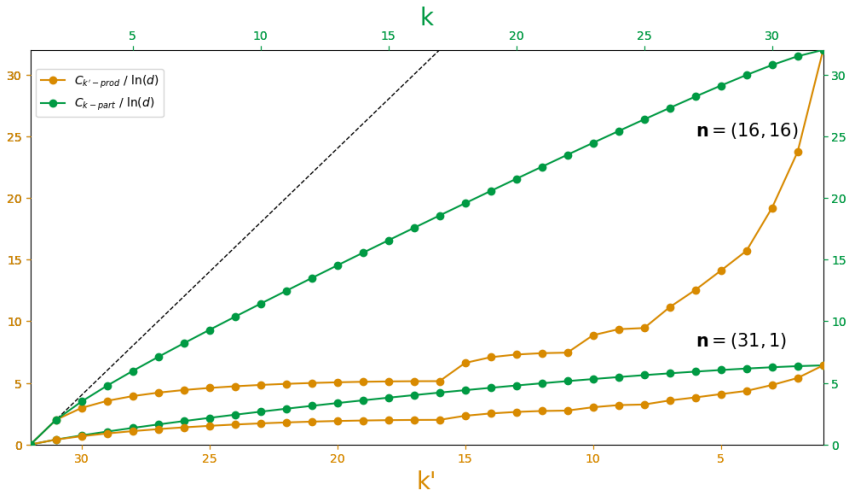


FIGURE 16. The $C_{k\text{-part}}$ relative entropy of k-partitionability correlation (7.2a) and the $C_{k'\text{-prod}}$ relative entropy of k'-producibility correlation (7.2b) of two 32-qubit Dicke states in terms of k and k'. The dotted line denotes the theoretical maximum of the k-partitionable correlation.

Using (3.90), we can easily calculate the relative entropy of k-partitionability, k-producbility and k-stretchability correlation of pure Dicke states are

$$\begin{aligned} C_{k\text{-part}}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) &= \min_{\xi:h(\xi)\geq k} C_\xi(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) = \min_{\xi:h(\xi)\geq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\right)\right] \\ &= \min_{\xi:h(\xi)\geq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)\right)\right], \end{aligned} \tag{7.2a}$$

$$\begin{aligned} C_{k\text{-prod}}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) &= \min_{\xi:w(\xi)\leq k} C_\xi(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) = \min_{\xi:w(\xi)\leq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\right)\right] \\ &= \min_{\xi:w(\xi)\leq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)\right)\right], \end{aligned} \tag{7.2b}$$

$$\begin{aligned} C_{k\text{-str}}(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) &= \min_{\xi:r(\xi)\leq k} C_\xi(|\,\mathrm{D}_{\boldsymbol{n}}\rangle\langle\mathrm{D}_{\boldsymbol{n}}\,|) = \min_{\xi:r(\xi)\leq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\eta^{\boldsymbol{n}}_{\boldsymbol{m}}\ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}})\right)\right] \\ &= \min_{\xi:r(\xi)\leq k}\left[-\sum_{X\in\xi}\left(\sum_{\boldsymbol{m}\in I^d_{|X|,\boldsymbol{n}}}\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\ln\left(\frac{\binom{|X|}{\boldsymbol{m}}\binom{n-|X|}{\boldsymbol{n}-\boldsymbol{m}}}{\binom{n}{\boldsymbol{n}}}\right)\right)\right]. \end{aligned} \tag{7.2c}$$

Although the minimizations in (7.1) and (7.2) can not be calculated analytically, it is easy numerically.

Examples of the (7.2) relative entropies can be seen in Figures 16 and 17. From these figures, we can see that the k-partitionability correlation is monotonically increasing, while the k-producibility and k-stretchability correlations are monotonically decreasing in k, which is the

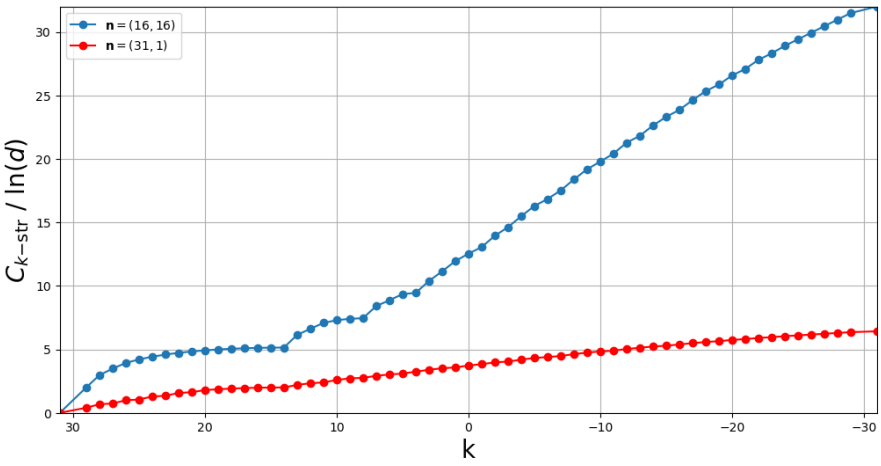


FIGURE 17. The $C_{k\text{-str}}$ relative entropy of k-stretchability correlation (7.2c) of two 32-qubit Dicke states in terms of k.

(3.92) multipartite monotonicity of these correlations. One can see that the more uniform the occupation, the larger the correlation is, which we have seen before. It can also be seen from Figure 16, that while the k-partitionability correlation (green graphs) forms a (discrete) convex curve, the k-producibility correlation (orange graphs) does not, so the k-partitionability correlation seems a more natural quantity for Dicke states. However, note that these are not general features of these measures, the k-partitionability correlation is not necessary convex, see [9]. In Figure 17 we can see that the k-stretchability correlation inherits the non-convexity from the k-producibility correlation. The exact cause of the convexity or non-convexity of these measures is unknown, it represents an interesting direction of further research, related to the (relative entropy based) geometry of the state spaces of k-partitionably, k-producibly and k-stretchably uncorrelated states.

7.2. **Reduced states.** Using (3.87) and (6.4), the relative entropy of $\boldsymbol{\xi}$-correlation of $\rho_{\boldsymbol{n},m}$ m-partite reduced Dicke states is

$$C_{\boldsymbol{\xi}}(\rho_{\boldsymbol{n},m}) = \min_{\xi\in\boldsymbol{\xi}} C_{\xi}(\rho_{\boldsymbol{n},m}) = \min_{\xi\in\boldsymbol{\xi}} \left[-\sum_{X\in\xi} \left(\sum_{\boldsymbol{k}\in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{k}}) \right) + \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}}) \right]. \tag{7.3}$$

Using (3.90), we can easily calculate the relative entropy of k-partitionability, k-producbility and k-stretchability correlation of m-partite reduced Dicke states are

$$\begin{aligned} C_{k\text{-part}}(\rho_{\boldsymbol{n},m}) &= \min_{\xi:h(\xi)\geq k} C_{\xi}(\rho_{\boldsymbol{n},m}) \\ &= \min_{\xi:h(\xi)\geq k} \left[-\sum_{X\in\xi} \left(\sum_{\boldsymbol{k}\in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{k}}) \right) + \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}}) \right] \end{aligned} \tag{7.4a}$$

$$\begin{aligned} C_{k\text{-prod}}(\rho_{\boldsymbol{n},m}) &= \min_{\xi:w(\xi)\leq k} C_{\xi}(\rho_{\boldsymbol{n},m}) \\ &= \min_{\xi:w(\xi)\leq k} \left[-\sum_{X\in\xi} \left(\sum_{\boldsymbol{k}\in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{k}}) \right) + \sum_{\boldsymbol{m}\in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}}) \right] \end{aligned} \tag{7.4b}$$

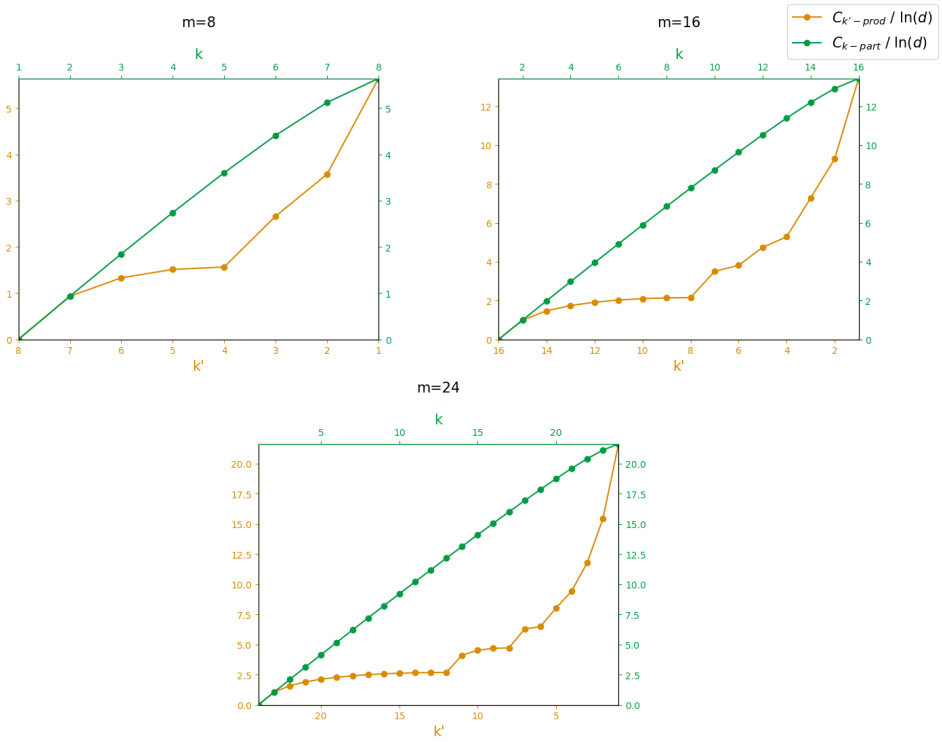


FIGURE 18. The $C_{k\text{-part}}$ relative entropy of k-partitionability correlation (7.4a) and the $C_{k'\text{-prod}}$ relative entropy of k'-producibility correlation (7.4b) of the m-partite reduced states of the 32-qubit Dicke state with occupation $\boldsymbol{n} = (16, 16)$ for different m in terms of k and k'.

$$
\begin{aligned}
C_{k\text{-str}}(\rho_{\boldsymbol{n},m}) &= \min_{\xi : r(\xi) \leq k} C_\xi(\rho_{\boldsymbol{n},m}) \\
&= \min_{\xi : r(\xi) \leq k} \left[- \sum_{X \in \xi} \left(\sum_{\boldsymbol{k} \in I^d_{|X|,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{k}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{k}}) \right) + \sum_{\boldsymbol{m} \in I^d_{m,\boldsymbol{n}}} \eta^{\boldsymbol{n}}_{\boldsymbol{m}} \ln(\eta^{\boldsymbol{n}}_{\boldsymbol{m}}) \right].
\end{aligned} \tag{7.4c}
$$

Although the minimizations in in (7.3) and (7.4) can not be calculated analytically, it is easy numerically.

The (7.4) relative entropies for some m-partite reduced Dicke states can be seen in Figures 18 and 19. From these figures we can see that the k-partitionability, k-producibility and k-stretchability correlation shows the same behavior as for the pure states. We can also see that the larger the size of the subsystem is, the larger the correlation, which we have seen before.

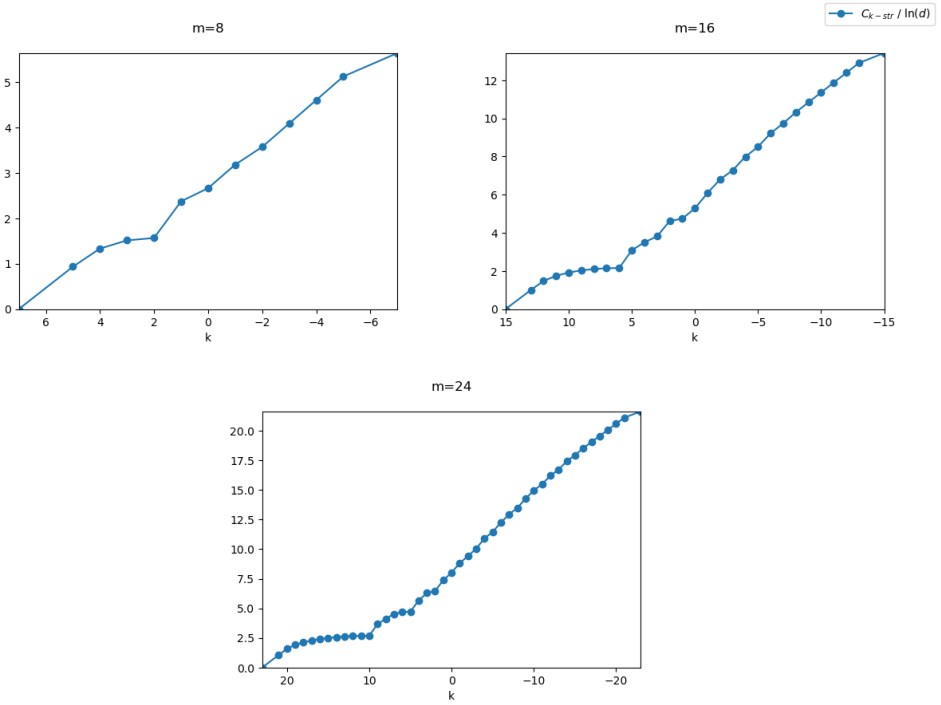


FIGURE 19. The $C_{k\text{-str}}$ relative entropy of k-stretchability correlation (7.4c) of the m-partite reduced states of the 32-qubit Dicke state with occupation $\boldsymbol{n} = (16, 16)$ for different m in terms of k.

8. Matrix product state form of Dicke states

Writing down the MPS form (3.20) of an arbitrary Dicke state $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ is an easy task, since the Schmidt vectors in the Schmidt decomposition (4.53) of Dicke states are also Dicke states and the basis vectors of the elementary subsystems $\mathcal{H}_a$ and the vectors $|\tau\rangle$ appearing in (3.13) during the derivation of the matrix product state are also Dicke states. First, let us take the Schmidt decomposition of the vector $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ for a $\{1\}|\{2,3,\dots,n\}$ bipartition

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{\boldsymbol{m}_1 \in I^d_{1,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}_1}}\,|\,\mathrm{D}_{\boldsymbol{m}_1}\rangle \otimes |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle = \sum_{i_1} \sum_{\boldsymbol{m}_1 \in I^d_{1,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}_1}}\langle i_1|\,\mathrm{D}_{\boldsymbol{m}_1}\rangle |i_1\rangle \otimes |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle. \tag{8.1}$$

Let us introduce the $\boldsymbol{e}_i = (0,\dots,1,\dots,0)$, $(\boldsymbol{e}_i)_j = \delta_{i,j}$ single occupations. Using these, the value of the scalar product appearing in (8.1) is $\langle i_1|\,\mathrm{D}_{\boldsymbol{m}_1}\rangle = \delta(\boldsymbol{e}_{i_1}, \boldsymbol{m}_1)$, since $|i\rangle = |\,\mathrm{D}_{\boldsymbol{e}_i}\rangle$ is also a Dicke state, and these states are orthonormal. Substituting this back into (8.1) we get that

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{i_1} \sum_{\boldsymbol{m}_1 \in I^d_{1,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}_1}}\,\delta(\boldsymbol{e}_{i_1}, \boldsymbol{m}_1) |i_1\rangle \otimes |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle, \tag{8.2}$$

based on this, the first Γ matrix is

$$\Gamma^{(1)i_1}_{\boldsymbol{m}_1} = \delta(\boldsymbol{e}_{i_1}, \boldsymbol{m}_1). \tag{8.3}$$

Now we perform the Schmidt decomposition of the vector $|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle$ for $m=1$, as

$$\begin{aligned}|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle &= \sum_{\boldsymbol{t}\in I^d_{1,\boldsymbol{n}-\boldsymbol{m}_1}} \sqrt{\eta_{\boldsymbol{t}}^{\boldsymbol{n}-\boldsymbol{m}_1}}\,|\,\mathrm{D}_{\boldsymbol{t}}\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1-\boldsymbol{t}}\rangle \\ &= \sum_{i_2}\sum_{\boldsymbol{t}\in I^d_{1,\boldsymbol{n}-\boldsymbol{m}_1}} \sqrt{\eta_{\boldsymbol{t}}^{\boldsymbol{n}-\boldsymbol{m}_1}}\langle i_2|\,\mathrm{D}_{\boldsymbol{t}}\rangle|i_2\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1-\boldsymbol{t}}\rangle,\end{aligned} \tag{8.4}$$

where the value of the scalar product is $\langle i_2|\,\mathrm{D}_{\boldsymbol{t}}\rangle=\delta(\boldsymbol{e}_{i_2},\boldsymbol{t})$. We write the vector $|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1-\boldsymbol{t}}\rangle$ as

$$|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1-\boldsymbol{t}}\rangle = \sum_{\boldsymbol{m}_2\in I_2^d}\langle \mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\,|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1-\boldsymbol{t}}\rangle|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\rangle = \sum_{\boldsymbol{m}_2\in I_2^d}\delta(\boldsymbol{m}_2,\boldsymbol{m}_1+\boldsymbol{t})|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\rangle, \tag{8.5}$$

where we used that Dicke states form a basis. Substituting this back into (8.4), we get that

$$|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle = \sum_{i_2}\sum_{\boldsymbol{t}\in I^d_{1,\boldsymbol{n}-\boldsymbol{m}_1}}\sum_{\boldsymbol{m}_2\in I_2^d}\sqrt{\eta_{\boldsymbol{t}}^{\boldsymbol{n}-\boldsymbol{m}_1}}\delta(\boldsymbol{e}_{i_2},\boldsymbol{t})\delta(\boldsymbol{m}_2,\boldsymbol{m}_1+\boldsymbol{t})|i_2\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\rangle. \tag{8.6}$$

Performing the summation over $\boldsymbol{t}$, we get

$$|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_1}\rangle = \sum_{i_2}\sum_{\boldsymbol{m}_2\in I^d_{2,\boldsymbol{n}}}\sqrt{\eta_{\boldsymbol{m}_2-\boldsymbol{m}_1}^{\boldsymbol{n}-\boldsymbol{m}_1}}\delta(\boldsymbol{e}_{i_2},\boldsymbol{m}_2-\boldsymbol{m}_1)|i_2\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\rangle. \tag{8.7}$$

Note, that in (8.5) and in (8.6) we had $\boldsymbol{m}_2\in I_2^d$ but in (8.7) we have $\boldsymbol{m}_2\in I^d_{2,\boldsymbol{n}}$ in the sums for $\boldsymbol{m}_2$. This is because in (8.7) we have a $\sqrt{\eta_{\boldsymbol{m}_2-\boldsymbol{m}_1}^{\boldsymbol{n}-\boldsymbol{m}_1}}$ factor, which gives zero if $\boldsymbol{m}_2\in I_2^d\setminus I^d_{2,\boldsymbol{n}}$. Substituting this back into (8.2), we get that

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{i_1,i_2}\sum_{\boldsymbol{m}_1\in I^d_{1,\boldsymbol{n}}}\sum_{\boldsymbol{m}_2\in I^d_{2,\boldsymbol{n}}}\sqrt{\eta_{\boldsymbol{m}_1}^{\boldsymbol{n}}}\delta(\boldsymbol{e}_{i_1},\boldsymbol{m}_1)\sqrt{\eta_{\boldsymbol{m}_2-\boldsymbol{m}_1}^{\boldsymbol{n}-\boldsymbol{m}_1}}\delta(\boldsymbol{e}_{i_2},\boldsymbol{m}_2-\boldsymbol{m}_1)|i_1i_2\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_2}\rangle. \tag{8.8}$$

Based on this, the next Γ matrix is

$$\Gamma^{(2)i_2}_{\boldsymbol{m}_1,\boldsymbol{m}_2} = \frac{\sqrt{\eta_{\boldsymbol{m}_2-\boldsymbol{m}_1}^{\boldsymbol{n}-\boldsymbol{m}_1}}}{\sqrt{\eta_{\boldsymbol{m}_2}^{\boldsymbol{n}}}}\delta(\boldsymbol{e}_{i_2},\boldsymbol{m}_2-\boldsymbol{m}_1). \tag{8.9}$$

Now assume that similar steps have been performed a times. Then we need to decompose the vector $|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a}\rangle$. Simimar to (8.4), we write the Schmidt decomposition of the vector for $m=1$, as

$$\begin{aligned}|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a}\rangle &= \sum_{\boldsymbol{t}\in I^d_{1,\boldsymbol{n}-\boldsymbol{m}_a}} \sqrt{\eta_{\boldsymbol{t}}^{\boldsymbol{n}-\boldsymbol{m}_a}}\,|\,\mathrm{D}_{\boldsymbol{t}}\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a-\boldsymbol{t}}\rangle \\ &= \sum_{i_{a+1}}\sum_{\boldsymbol{t}\in I^d_{1,\boldsymbol{n}-\boldsymbol{m}_a}} \sqrt{\eta_{\boldsymbol{t}}^{\boldsymbol{n}-\boldsymbol{m}_a}}\langle i_{a+1}|\,\mathrm{D}_{\boldsymbol{t}}\rangle|i_{a+1}\rangle\otimes|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a-\boldsymbol{t}}\rangle,\end{aligned} \tag{8.10}$$

where the value of the scalar product is $\langle i_{a+1}|\,\mathrm{D}_{\boldsymbol{t}}\rangle=\delta(\boldsymbol{e}_{i_{a+1}},\boldsymbol{t})$. Similarly to (8.5), we write the vector $|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a-\boldsymbol{t}}\rangle$ as

$$\begin{aligned}|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a-\boldsymbol{t}}\rangle &= \sum_{\boldsymbol{m}_{a+1}\in I^d_{a+1}}\langle \mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_{a+1}}\,|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a-\boldsymbol{t}}\rangle|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_{a+1}}\rangle \\ &= \sum_{\boldsymbol{m}_{a+1}\in I^d_{a+1}}\delta(\boldsymbol{m}_{a+1},\boldsymbol{m}_a+\boldsymbol{t})|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_{a+1}}\rangle,\end{aligned} \tag{8.11}$$

using that Dicke states form a basis. Substituting this back into (8.10) and performing the summation over $\boldsymbol{t}$, we get that

$$|\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_a}\rangle = \sum_{i_{a+1}} \sum_{\boldsymbol{m}_{a+1}\in I^d_{a+1,\boldsymbol{n}}} \sqrt{\eta^{\boldsymbol{n}-\boldsymbol{m}_a}_{\boldsymbol{m}_{a+1}-\boldsymbol{m}_a}}\,\delta(\boldsymbol{e}_{i_{a+1}}, \boldsymbol{m}_{a+1}-\boldsymbol{m}_a)|i_{a+1}\rangle \otimes |\,\mathrm{D}_{\boldsymbol{n}-\boldsymbol{m}_{a+1}}\rangle, \tag{8.12}$$

based on this, the Γ matrix is

$$\Gamma^{(a+1)i_{a+1}}_{\boldsymbol{m}_a,\boldsymbol{m}_{a+1}} = \frac{\sqrt{\eta^{\boldsymbol{n}-\boldsymbol{m}_a}_{\boldsymbol{m}_{a+1}-\boldsymbol{m}_a}}}{\sqrt{\eta^{\boldsymbol{n}}_{\boldsymbol{m}_{a+1}}}}\,\delta(\boldsymbol{e}_{i_{a+1}}, \boldsymbol{m}_{a+1}-\boldsymbol{m}_a). \tag{8.13}$$

We can simplify this form if we notice that $\boldsymbol{m}_{a+1}-\boldsymbol{m}_a = \boldsymbol{e}_{i_{a+1}}$ is a single excitation index, and for such indices, the Schmidt coefficient can be easily calculated in a closed form. For arbitrary $\boldsymbol{n}'$ and single excitation indices $\boldsymbol{m}' = \boldsymbol{e}_j$ (if $n'_j > 0$), we get that

$$\eta^{\boldsymbol{n}'}_{\boldsymbol{m}'} = \frac{\binom{1}{\boldsymbol{m}'}\binom{|\boldsymbol{n}'|-1}{\boldsymbol{n}'-\boldsymbol{m}'}}{\binom{|\boldsymbol{n}'|}{\boldsymbol{n}'}} = \frac{1\frac{(|\boldsymbol{n}'|-1)!}{\prod_{i=1}^d(\boldsymbol{n}'-\boldsymbol{m}')_i!}}{\binom{|\boldsymbol{n}'|}{\boldsymbol{n}'}} = \frac{\frac{|\boldsymbol{n}'|!/|\boldsymbol{n}'|}{(\prod_{i=1}^d n'_i!)/n'_j}}{\binom{|\boldsymbol{n}'|}{\boldsymbol{n}'}} = \frac{n'_j}{|\boldsymbol{n}'|}. \tag{8.14}$$

With this form of the Schmidt coefficient, the Γ matrices are

$$\Gamma^{(a+1)i_{a+1}}_{\boldsymbol{m}_a,\boldsymbol{m}_{a+1}} = \frac{\sqrt{(\boldsymbol{n}-\boldsymbol{m}_a)_{i_{a+1}}}}{\sqrt{(n-a)\eta^{\boldsymbol{n}}_{\boldsymbol{m}_{a+1}}}}\,\delta(\boldsymbol{e}_{i_{a+1}}, \boldsymbol{m}_{a+1}-\boldsymbol{m}_a). \tag{8.15}$$

Note that the B matrices in the *right canonical MPS* form (3.23) of Dicke states take a simpler form than the Γ matrices, namely

$$B^{(a)i_a}_{\boldsymbol{m}_{a-1},\boldsymbol{m}_a} = \sqrt{\eta^{\boldsymbol{n}-\boldsymbol{m}_{a-1}}_{\boldsymbol{m}_a-\boldsymbol{m}_{a-1}}}\,\delta(\boldsymbol{e}_{i_a}, \boldsymbol{m}_a-\boldsymbol{m}_{a-1}) = \sqrt{\frac{(\boldsymbol{n})_{i_a}-(\boldsymbol{m}_{a-1})_{i_a}}{n-a+1}}\,\delta(\boldsymbol{e}_{i_a}, \boldsymbol{m}_a-\boldsymbol{m}_{a-1}), \tag{8.16}$$

With the B matrices seen in (8.16), the right canonical MPS form of the vector $|\,\mathrm{D}_{\boldsymbol{n}}\rangle$ is

$$|\,\mathrm{D}_{\boldsymbol{n}}\rangle = \sum_{i_1,i_2,\ldots,i_n} \sum_{\boldsymbol{m}_1\in I^d_{1,\boldsymbol{n}}} \sum_{\boldsymbol{m}_2\in I^d_{2,\boldsymbol{n}}} \cdots \sum_{\boldsymbol{m}_{n-1}\in I^d_{n-1,\boldsymbol{n}}} B^{(1)i_1}_{\boldsymbol{m}_1} B^{(2)i_2}_{\boldsymbol{m}_1,\boldsymbol{m}_2} \ldots B^{(n)i_n}_{\boldsymbol{m}_{n-1}} \bigotimes_{a=1}^{n} |i_a\rangle. \tag{8.17}$$

This calculation first appeared in the TDK essay of the Author in November, 2023, in Hungarian. Later, before publishing it in English, it was published independently by researchers from the USA [43].

Since there are Kronecker deltas in the form of the (8.15) and (8.17) matrices, these matrices are sparse and easy to write out. For example, the (8.17) B matrices in the right canonical MPS form of the $|\,\mathrm{D}_{n_1,n_2}\rangle$ qubit Dicke states, where without the loss of generality we assume that $n_1 \geq n_2$, are for $a \leq n_2$, the $a \times (a+1)$ size matrices,

(8.18a)

$$B^{(a)1} = \begin{pmatrix} 0 & \sqrt{\frac{n_1}{n-a+1}} & & \\ & 0 & \sqrt{\frac{n_1-1}{n-a+1}} & \\ & & 0 & \ldots \\ & & & 0 \quad \ldots \end{pmatrix}, \quad B^{(a)2} = \begin{pmatrix} \sqrt{\frac{n_2-a+1}{n-a+1}} & & & \\ & \sqrt{\frac{n_2-a+2}{n-a+1}} & & \\ & & \ldots & \\ & & & \ldots \end{pmatrix},$$

for $n_2 \leq a \leq n_1$, the $(n_2+1)\times(n_2+1)$ size matrices,

$$B^{(a)1} = \begin{pmatrix} 1 & & & \\ & \sqrt{\frac{n-a}{n-a+1}} & & \\ & & \dots & \\ & & & \dots \end{pmatrix}, \; B^{(a)2} = \begin{pmatrix} 0 & & & \\ \sqrt{\frac{1}{n-a+1}} & 0 & & \\ & \sqrt{\frac{2}{n-a+1}} & 0 & \\ & & \dots & \end{pmatrix}, \tag{8.18b}$$

for $n_1 \leq a \leq n$, the $(n-a+2)\times(n-a+1)$ size matrices,

$$B^{(a)1} = \begin{pmatrix} 1 & & & \\ & \sqrt{\frac{n-a}{n-a+1}} & & \\ & & \dots & \\ & & & \dots \end{pmatrix}, \; B^{(a)2} = \begin{pmatrix} 0 & & & \\ \sqrt{\frac{1}{n-a+1}} & 0 & & \\ & \sqrt{\frac{2}{n-a+1}} & 0 & \\ & & \dots & \end{pmatrix}. \tag{8.18c}$$

Note that the size of the (8.17) right canonical MPS matrices are increasing, until the a index of the matrices reaches the n_1 number of excited elementary subsystems, where it hits a plateau, until the a index of the matrices reaches the n_1 number of non-excited elementary subsystems, where the size of the matrices starts to decrease. Since the (4.51) $I^d_{m,\boldsymbol{n}}$ set can not be parametrized nicely for $d > 2$, it is a more complicated task to analytically derive the (8.17) matrices for arbitrary occupations with $d > 2$. Note that even in this case, the form of the matrices would be similar to the matrices in (8.18).

9. Summary

In this thesis we examined various aspects of the multipartite correlation and entanglement of qudit Dicke states. Dicke states serve as a great example for the illustration of multipartite entanglement because these are symmetric states that can be described easily with few parameters, simplifying the derivations a lot, making possible to obtain results in closed form. This is clearly visible already in our first step, the Schmidt decomposition (4.53), on which all our results are built upon. We note that these results may be considered to be somewhat implicit for general qu*d*its due to the summation limits, but for qubits, they become more manageable, explicit formulas.

First, we conjectured and proved that arbitrary reduced (mixed) states of arbitrary pure qu*d*it Dicke states are entangled. This is noteworthy, since entanglement detection in general mixed states cannot be carried out without matrix diagonalization or other optimization methods. This means that the entanglement in the pure Dicke states, as well as in any of its reduced states, is *robust*, that is, it does not disappear if subsystems are lost, but the remaining subsystems also remain entangled. This was known, for example, for the 3-qubit single-excited Dicke state, also called W state. Note that the multi-qu*d*it GHZ state $\frac{1}{\sqrt{d}}\sum_i |i,i,\dots,i\rangle$ is maximally entangled, so more entangled than the Dicke states, however, its entanglement is *fragile*, that is, it disappears if subsystems are lost.

Second, we evaluated the relative entropies of the different multipartite correlation and entanglement properties of pure qu*d*it Dicke states and Dicke subsystems. This is noteworthy, since these measures cannot be evaluated for general states, without matrix diagonalization or other optimization methods. We could also illustrate these measures for concrete cases, which leads to some interesting further research directions. We saw that correlation and entanglement are stronger with respect to a more uniform partition, and also if the excitation is more uniform, that is, the number of different excitations is similar.

Third, we derived the matrix product state form of the pure qu*d*it Dicke states. This is noteworthy, since calculating the matrix product state form in a closed, analytical form cannot be done in general for an arbitrary vector, only few examples exist in the literature.

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Appendix A. Hilbert spaces, operators and maps

In quantum theory *measurable quantities* or *observables* are represented as linear operators acting on a Hilbert spaces. In this section, we will familiarize ourselves in more detail with Hilbert spaces [64, 65].

A.1. **Hilbert spaces.** An *inner product* or *scalar product* on a complex vector space $\mathcal{H}$ is a map

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \quad \longrightarrow \quad \mathbb{C} \tag{A.1}$$

which is *non-negative* (that is, $\langle \psi, \psi \rangle \geq 0$); *non-degenerate* (that is, $\langle \psi, \psi \rangle = 0 \Leftrightarrow \psi = 0$); *conjugate symmetric* or *Hermitian* (that is, $\langle \psi, \varphi \rangle = \overline{\langle \varphi, \psi \rangle}$); and *linear in the second argument* (that is, $\langle \psi, \sum_i \lambda_i \varphi_i \rangle = \sum_i \lambda_i \langle \psi, \varphi_i \rangle$ for a finite sum). From this it follows that it is *conjugate linear* in the first argument (that is, $\langle \sum_i \lambda_i \psi_i, \varphi \rangle = \sum_i \overline{\lambda_i} \langle \psi_i, \varphi \rangle$ for a finite sum). For an inner product it can be shown that the *Cauchy-Bunyakovski-Schwarz inequality* holds, that is, for all $\psi, \varphi \in \mathcal{H}$,

$$|\langle \psi, \varphi \rangle|^2 \leq \langle \psi, \psi \rangle \cdot \langle \varphi, \varphi \rangle. \tag{A.2}$$

An inner product leads to a *norm*, which can be defined as

$$\|\psi\| = \sqrt{\langle \psi, \psi \rangle}, \tag{A.3}$$

by which the Cauchy-Bunyakovski-Schwarz inequality takes the form

$$|\langle \psi, \varphi \rangle| \leq \|\psi\| \cdot \|\varphi\| \tag{A.4}$$

for all $\psi, \varphi \in \mathcal{H}$. A complex vector space $\mathcal{H}$ with an inner product is a *Hilbert space*, if it is a complete normed (Banach) space with respect to the norm induced by the inner product, that is, every Cauchy sequence in $\mathcal{H}$ converges.

In vector spaces the defining structure is the linear combination, so the maps preserving it are naturally important. A map $A : \mathcal{H} \longrightarrow \mathcal{K}$ is called *linear* if for all vectors $\psi_i \in \mathcal{H}$ and all scalars $c_i \in \mathbb{C}$,

$$A\Big(\sum_i c_i \psi_i\Big) = \sum_i c_i A\psi_i. \tag{A.5}$$

Such maps are called *(linear) operators*. The space of these operators are denoted by $\mathrm{Lin}(\mathcal{H}, \mathcal{K})$. It can be shown that linear operators also form a vector space. The *adjoint* $A^\dagger \in \mathrm{Lin}(\mathcal{K}, \mathcal{H})$ of an operator $A \in \mathrm{Lin}(\mathcal{H}, \mathcal{K})$ is defined such that for all $\psi \in \mathcal{H}$ and $\varphi \in \mathcal{K}$

$$\langle \varphi, A\psi \rangle = \langle A^\dagger \varphi, \psi \rangle. \tag{A.6}$$

The $\mathrm{Lin}(\mathcal{H}, \mathbb{C})$ space of linear functionals, mapping from $\mathcal{H}$ to the base field $\mathbb{C}$, is called the *dual* of the $\mathcal{H}$ Hilbert space. This space is denoted by $\mathcal{H}^*$. A conjugate linear bijection can be naturally introduced between the elements of the Hilbert space and the elements of its dual,

$$\begin{array}{rcl} \dagger : \mathcal{H} & \longrightarrow & \mathcal{H}^*, \\ \psi & \longmapsto & \psi^\dagger, \end{array} \tag{A.7}$$

where $\psi^\dagger$ is a functional given as

$$\begin{array}{rcll} \psi^\dagger : \mathcal{H} & \longrightarrow & \mathbb{C}, & \\ \varphi & \longmapsto & \psi^\dagger(\varphi) = \langle \psi, \varphi \rangle & \forall \varphi \in \mathcal{H}. \end{array} \tag{A.8}$$

Based on this bijection we can introduce the *Dirac bra-ket notation*, according to which the elements of the Hilbert space are denoted as

$$|\psi\rangle \in \mathcal{H}, \tag{A.9}$$

called *"ket vectors"*, and the elements of the dual space are denoted as

$$(|\psi\rangle)^\dagger =: \langle \psi| \in \mathcal{H}^*, \tag{A.10}$$

called *"bra vectors"*. In this notation, $\psi^\dagger(\varphi) = (|\psi\rangle)^\dagger(|\varphi\rangle) = \langle\psi|(|\varphi\rangle)$, which is conveniently written as $\langle\psi|\varphi\rangle$, so the scalar product of vectors $|\psi\rangle \in \mathcal{H}$ and $|\varphi\rangle \in \mathcal{H}$ is written in the form

$$\langle \psi, \varphi \rangle = \langle \psi | \varphi \rangle. \tag{A.11}$$

The *transpose* $A^{\mathrm{T}} \in \mathrm{Lin}(\mathcal{K}^*, \mathcal{H}^*)$ of an operator $A \in \mathrm{Lin}(\mathcal{H}, \mathcal{K})$ is defined such that for all $\psi \in \mathcal{H}$ and $\varphi \in \mathcal{K}$

$$\langle \varphi | A | \psi \rangle = \big(A^{\mathrm{T}} \langle \varphi |\big) |\psi\rangle. \tag{A.12}$$

Note that the transposition (A.12), contrary to the adjoint (A.6), can be formulated in vector spaces, without the inner product, as it is given in terms of functionals acting on vectors. We do not do this, as we do not consider vector spaces which are not Hilbert spaces.

A.2. **Bases and linear decomposition.** A set of vectors in a vector space $\mathcal{H}$ is called a *basis* if they are linearly independent and span the space. From this it follows that *any* vector in the space can be written *uniquely* as a linear combination of these basis vectors. The number of basis vectors, which is equal to the maximum number of linearly independent vectors, is the *dimension* of the Hilbert space, $\dim(\mathcal{H})$, which are finite throughout this thesis and from now on in this appendix. Using the notation $[d] := \{1, 2, \ldots, d\} \in \mathbb{N}$, a basis is written as

$$\Big\{ |\phi_j\rangle \in \mathcal{H} \;\Big|\; j \in [\dim(\mathcal{H})] \Big\}. \tag{A.13}$$

In Hilbert spaces, a basis written above is called *orthonormal*, if

$$\langle\phi_i|\phi_j\rangle = \delta_{ij}. \tag{A.14}$$

In this thesis we always use orthonormal bases, unless otherwise stated. An arbitrary vector $|\psi\rangle \in \mathcal{H}$ can be written in the form

$$|\psi\rangle = \sum_i c_i|\phi_i\rangle. \tag{A.15}$$

The coefficients c_i can be obtained by multiplying this expression by $|\phi_j\rangle$ from the left. Then

$$\langle\phi_j|\psi\rangle = \langle\phi_j|\sum_i c_i\phi_i\rangle = \sum_i c_i\langle\phi_j|\phi_i\rangle = \sum_i c_i\delta_{ji} = c_j. \tag{A.16}$$

A dual basis $\{\langle\phi_i|\}$ can also be defined on the space $\mathcal{H}^*$ based on the basis vectors $\{|\phi_i\rangle\}$ of the space $\mathcal{H}$ using the bijection (A.7). The operator $I \in \mathrm{Lin}(\mathcal{H},\mathcal{H}) = \mathrm{Lin}(\mathcal{H})$ is called the *identity operator* if it maps every vector to itself, that is, $I|\psi\rangle = |\psi\rangle$ for every $|\psi\rangle \in \mathcal{H}$. It can be shown in any basis the identity operator can be written in the form

$$I = \sum_i |\phi_i\rangle\langle\phi_i|. \tag{A.17}$$

The action of this gives the linear decomposition of an arbitrary vector $|\psi\rangle \in \mathcal{H}$,

$$|\psi\rangle = I|\psi\rangle = \sum_i \Big(|\phi_i\rangle\langle\phi_i|\Big)|\psi\rangle = \sum_i c_i|\phi_i\rangle. \tag{A.18}$$

Similarly, we can obtain the linear decomposition of an arbitrary operator $A \in \mathrm{Lin}(\mathcal{H})$, since $IAI|\psi\rangle = IA|\psi\rangle = A|\psi\rangle$, so

$$\begin{aligned} A = IAI = \sum_i |\phi_i\rangle\langle\phi_i|A\sum_j |\phi_j\rangle\langle\phi_j| = \sum_i\sum_j |\phi_i\rangle\langle\phi_i|A|\phi_j\rangle\langle\phi_j| = \\ = \sum_i\sum_j \langle\phi_i|A|\phi_j\rangle|\phi_i\rangle\langle\phi_j| = \sum_i\sum_j A_{ij}|\phi_i\rangle\langle\phi_j|. \end{aligned} \tag{A.19}$$

Thus, the *dyads* $|\phi_i\rangle\langle\phi_j|$ form a basis in the space $\mathrm{Lin}(\mathcal{H})$, and the coefficient of an arbitrary operator $A \in \mathrm{Lin}(\mathcal{H})$ corresponding to the basis element $|\phi_i\rangle\langle\phi_j|$ is $A_{ij} = \langle\phi_i|A|\phi_j\rangle$, which is also called the "matrix element" of the operator.

The *trace* of an operator $A \in \mathrm{Lin}(\mathcal{H})$ with respect to a given basis $\{|\phi_i\rangle\}$ is the map given as

$$\begin{aligned} \mathrm{Tr}(A) = \sum_i \langle\phi_i|A\phi_i\rangle = \sum_i \langle\phi_i|\Big(\sum_j\sum_k A_{jk}|\phi_j\rangle\langle\phi_k|\Big)|\phi_i\rangle = \\ = \sum_i\sum_j\sum_k A_{jk}\langle\phi_i|\phi_j\rangle\langle\phi_k|\phi_i\rangle = \sum_i\sum_j\sum_k A_{jk}\delta_{ij}\delta_{ki} = \sum_j\sum_k A_{jk}\delta_{jk} = \sum_k A_{kk}, \end{aligned} \tag{A.20}$$

that is, the sum of the elements on the main diagonal of the matrix of the operator. The value of this, as we will see later, is *basis-independent*.

A.3. **Tensor product of Hilbert spaces.** The *tensor product* of the Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ is a Hilbert space $\mathcal{K}\otimes\mathcal{H}$, with a bilinear map

$$\begin{aligned} \otimes : \mathcal{K}\times\mathcal{H} &\longrightarrow \mathcal{K}\otimes\mathcal{H}, \\ |\varphi\rangle, |\psi\rangle &\longmapsto |\varphi\rangle\otimes|\psi\rangle, \end{aligned} \tag{A.21}$$

that is,

$$\Big(\sum_i c_i|\varphi_i\rangle\Big)\otimes|\psi\rangle = \sum_i c_i\Big(|\varphi_i\rangle\otimes|\psi\rangle\Big), \qquad |\varphi\rangle\otimes\Big(\sum_i c_i|\psi_i\rangle\Big) = \sum_i c_i\Big(|\varphi\rangle\otimes|\psi_i\rangle\Big), \tag{A.22}$$

where the inner product is given as

$$\langle \varphi_1 \otimes \psi_1 | \varphi_2 \otimes \psi_2 \rangle = \langle \varphi_1 | \varphi_2 \rangle_{\mathcal{K}} \langle \psi_1 | \psi_2 \rangle_{\mathcal{H}}, \tag{A.23}$$

where $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ is the inner product defined on the Hilbert space $\mathcal{H}$, and $\langle \cdot | \cdot \rangle_{\mathcal{K}}$ is the inner product defined on the Hilbert space $\mathcal{K}$. Elements of the form $|\varphi\rangle \otimes |\psi\rangle$ are called *elementary tensors*, and $\mathcal{K} \otimes \mathcal{H}$ is spanned by these,

$$\mathcal{K} \otimes \mathcal{H} = \operatorname{Span}\big\{ |\varphi\rangle \otimes |\psi\rangle \;\big|\; |\varphi\rangle \in \mathcal{K}, |\psi\rangle \in \mathcal{H} \big\}. \tag{A.24}$$

We can also define the tensor product of operators, for arbitrary $A \in \operatorname{Lin}(\mathcal{H}_1, \mathcal{H}'_1)$ and $B \in \operatorname{Lin}(\mathcal{H}_2, \mathcal{H}'_2)$ and $|\psi_1\rangle \in \mathcal{H}_1$, $|\psi_2\rangle \in \mathcal{H}_2$

$$\big(A \otimes B\big)\big(|\psi_1\rangle \otimes |\psi_2\rangle\big) = \big(A|\psi_1\rangle\big) \otimes \big(B|\psi_2\rangle\big). \tag{A.25}$$

Let $\{|\phi_i\rangle_{\mathcal{H}}\}$ be a basis in the space $\mathcal{H}$, and $\{|\phi_i\rangle_{\mathcal{K}}\}$ be a basis in the space $\mathcal{K}$. Then

$$\begin{aligned} |\varphi\rangle \otimes |\psi\rangle &= \Big(\sum_i \varphi_i |\phi_i\rangle_{\mathcal{K}}\Big) \otimes \Big(\sum_j \psi_j |\phi_j\rangle_{\mathcal{H}}\Big) = \\ &= \sum_i \sum_i \varphi_i \psi_j \big(|\phi_i\rangle_{\mathcal{K}} \otimes |\phi_j\rangle_{\mathcal{H}}\big) = \sum_i \sum_i \big(|\varphi\rangle \otimes |\psi\rangle\big)_{ij} \big(|\phi_i\rangle_{\mathcal{K}} \otimes |\phi_j\rangle_{\mathcal{H}}\big), \end{aligned} \tag{A.26}$$

so the elementary tensors $|\phi_i\rangle_{\mathcal{K}} \otimes |\phi_j\rangle_{\mathcal{H}}$ form a basis in the space $\mathcal{K} \otimes \mathcal{H}$, and the coefficient corresponding to these for an arbitrary elementary tensor $|\varphi\rangle \otimes |\psi\rangle \in \mathcal{K} \otimes \mathcal{H}$ is $\varphi_i \psi_j$.

Let us now study the space

$$\mathcal{K} \otimes \mathcal{H}^* = \operatorname{Span}\big\{ |\xi\rangle \otimes \langle \psi| \;\big|\; |\xi\rangle \in \mathcal{K}, |\psi\rangle \in \mathcal{H} \big\}. \tag{A.27}$$

(The tensor product sign is often omitted in such cases, $|\xi\rangle\langle\psi| := |\xi\rangle \otimes \langle\psi|$.) This space is isomorphic to the space $\operatorname{Lin}(\mathcal{H}, \mathcal{K})$ by using the following definition of action. Let an arbitrary element $A = |\xi\rangle\langle\psi| \in \mathcal{K} \otimes \mathcal{H}^*$ act linearly on an arbitrary vector $|\chi\rangle \in \mathcal{H}$ as

$$A|\chi\rangle = \big(|\xi\rangle\langle\psi|\big)|\chi\rangle = |\xi\rangle\langle\psi|\chi\rangle \in \mathcal{K}, \tag{A.28}$$

extending linearly to $\mathcal{K} \otimes \mathcal{H}^*$. So we use the identification $\mathcal{K} \otimes \mathcal{H}^* = \operatorname{Lin}(\mathcal{H}, \mathcal{K})$. In particular $\mathcal{H} \otimes \mathcal{H}^* = \operatorname{Lin}(\mathcal{H})$.

We previously introduced the adjoint of linear operators in (A.6), and now we give a more general definition using the tensor product, giving back (A.6) for operators. The adjoint is a *conjugate linear* operation such that

$$\begin{aligned} \dagger : \mathcal{K} \otimes \mathcal{H} &\longrightarrow \mathcal{H}^* \otimes \mathcal{K}^* \\ |\varphi\rangle \otimes |\psi\rangle &\longmapsto \langle\varphi| \otimes \langle\psi|. \end{aligned} \tag{A.29}$$

In the special case of operators, this is

$$\begin{aligned} \dagger : \operatorname{Lin}(\mathcal{H}, \mathcal{K}) = \mathcal{K} \otimes \mathcal{H}^* &\longrightarrow \mathcal{H} \otimes \mathcal{K}^* = \operatorname{Lin}(\mathcal{K}, \mathcal{H}), \\ |\xi\rangle\langle\chi| &\longmapsto |\chi\rangle\langle\xi|, \end{aligned} \tag{A.30}$$

since in finite dimension $(\mathcal{K}^*)^* = \mathcal{K}$. It is easy to see that this is indeed equivalent to (A.6), as for elementary tensors $A = |\xi\rangle\langle\chi| \in \mathcal{K} \otimes \mathcal{H}^* = \operatorname{Lin}(\mathcal{H}, \mathcal{K})$ we have $A^\dagger|\varphi\rangle = |\chi\rangle\langle\xi|\varphi\rangle$, then the left-hand side of (A.6) is $\langle \varphi, A\psi\rangle = \langle\varphi|A|\psi\rangle = \langle\varphi|\xi\rangle\langle\chi|\psi\rangle$, and the right-hand side is $\langle A^\dagger\phi, \psi\rangle = \langle\langle\xi|\phi\rangle|\chi\rangle, \psi\rangle = \overline{\langle\xi|\phi\rangle}\langle\chi|\psi\rangle = \langle\phi|\xi\rangle\langle\chi|\psi\rangle$. The adjoint of an arbitrary operator $A \in \operatorname{Lin}(\mathcal{H}, \mathcal{K})$, decomposed in the basis is

$$\begin{aligned} A^\dagger &= \Big(\sum_i \sum_j A_{ij} |\phi_i\rangle_{\mathcal{K}} \langle\phi_j|_{\mathcal{H}}\Big)^\dagger = \sum_i \sum_j \overline{A_{ij}} \big(|\phi_i\rangle_{\mathcal{K}} \langle\phi_j|_{\mathcal{H}}\big)^\dagger \\ &= \sum_i \sum_j \overline{A_{ij}} |\phi_j\rangle_{\mathcal{H}} \langle\phi_i|_{\mathcal{K}} = \sum_i \sum_j (A^\dagger)_{ji} |\phi_j\rangle_{\mathcal{H}} \langle\phi_i|_{\mathcal{K}}, \end{aligned} \tag{A.31}$$

where the last form is the linear decomposition of the operator $A^\dagger \in \mathrm{Lin}(\mathcal{K},\mathcal{H})$. Based on this we can deduce that

$$(A^\dagger)_{ij} = \overline{A_{ji}}, \tag{A.32}$$

that is, the matrix of the adjoint operator is given by flipping the conjugate of the matrix of the operator over its main diagonal, which is the *conjugate transpose* of the operator's matrix.

We previously introduced the transpose of linear operators in (A.12), and now we give a more general definition using the tensor product, giving back (A.12) for operators. The transpose is a *linear* operation such that

$$\begin{aligned} \mathrm{T} : \mathcal{K}\otimes\mathcal{H} &\longrightarrow \mathcal{H}\otimes\mathcal{K} \\ |\varphi\rangle\otimes|\psi\rangle &\longmapsto |\psi\rangle\otimes|\varphi\rangle. \end{aligned} \tag{A.33}$$

In the special case of operators, this is

$$\begin{aligned} \mathrm{T} : \mathrm{Lin}(\mathcal{H},\mathcal{K}) = \mathcal{K}\otimes\mathcal{H}^* &\longrightarrow \mathcal{H}^*\otimes\mathcal{K} = \mathrm{Lin}(\mathcal{K}^*,\mathcal{H}^*), \\ |\xi\rangle\otimes\langle\chi| &\longmapsto \langle\chi|\otimes|\xi\rangle, \end{aligned} \tag{A.34}$$

since in finite dimension $(\mathcal{K}^*)^* = \mathcal{K}$. It is easy to see that this is indeed equivalent to (A.12), as for elementary tensors $A = |\xi\rangle\langle\chi| \in \mathcal{K}\otimes\mathcal{H}^* = \mathrm{Lin}(\mathcal{H},\mathcal{K})$ we have $A^{\mathrm{T}}\langle\varphi| = \big(\langle\chi|\otimes|\xi\rangle\big)\langle\varphi| = \langle\chi|\langle\varphi|\xi\rangle = \langle\varphi|\xi\rangle\langle\chi|$ (note that $|\xi\rangle \in (\mathcal{K}^*)^* = \mathcal{K}$ is considered here, acting as $|\xi\rangle(\langle\varphi|) = \langle\varphi|\xi\rangle$), then the left-hand side of (A.12) is $\langle\varphi|A|\psi\rangle = \langle\varphi|\xi\rangle\langle\chi|\psi\rangle$, and the right-hand side is $(A^{\mathrm{T}}\langle\varphi|)|\psi\rangle = \langle\varphi|\xi\rangle\langle\chi|\psi\rangle$. The transpose of an arbitrary operator $A \in \mathrm{Lin}(\mathcal{H},\mathcal{K}) = \mathcal{K}\otimes\mathcal{H}^*$, decomposed in the basis is

$$\begin{aligned} A^{\mathrm{T}} &= \Big(\sum_i\sum_j A_{ij}|\phi_i\rangle_{\mathcal{K}}\langle\phi_j|_{\mathcal{H}}\Big)^{\mathrm{T}} = \sum_i\sum_j A_{ij}\big(|\phi_i\rangle_{\mathcal{K}}\langle\phi_j|_{\mathcal{H}}\big)^{\mathrm{T}} \\ &= \sum_i\sum_j A_{ij}\langle\phi_j|_{\mathcal{H}}\otimes|\phi_i\rangle_{\mathcal{K}} = \sum_i\sum_j (A^{\mathrm{T}})_{ji}\langle\phi_j|_{\mathcal{H}}\otimes|\phi_i\rangle_{\mathcal{K}}, \end{aligned} \tag{A.35}$$

where the last form is the linear decomposition of the operator $A^{\mathrm{T}} \in \mathrm{Lin}(\mathcal{K},\mathcal{H})$. Based on this we can deduce that

$$(A^{\mathrm{T}})_{ij} = A_{ji}, \tag{A.36}$$

that is, the matrix of the transpose operator is given by flipping the matrix of the operator over its main diagonal, which is the *transpose* of the operator's matrix. Note that (A.12) is the basis independent definition of the transpose. However, usually a different, basis dependent definition is used, which is a *linear* map, such that

$$\begin{aligned} \mathrm{T} : \mathrm{Lin}(\mathcal{H},\mathcal{K}) = \mathcal{K}\otimes\mathcal{H}^* &\longrightarrow \mathcal{H}\otimes\mathcal{K}^* = \mathrm{Lin}(\mathcal{K},\mathcal{H}), \\ |\phi_i\rangle_{\mathcal{K}}\otimes\langle\phi_j|_{\mathcal{H}} &\longmapsto |\phi_j\rangle_{\mathcal{H}}\otimes\langle\phi_i|_{\mathcal{K}}. \end{aligned} \tag{A.37}$$

It is easy to check that for an operator expressed in this basis, we have the same correspondence as (A.36). Note that the same does not hold in a different basis.

We previously introduced the trace by a basis in (A.20), now we can formulate the general definition of this using the tensor product. Let the Trace be a linear map such that for all $|\psi\rangle \in \mathcal{H}$ and $\langle\varphi| \in \mathcal{H}^*$

$$\begin{aligned} \mathrm{Tr} : \mathrm{Lin}(\mathcal{H}) = \mathcal{H}\otimes\mathcal{H}^* &\longrightarrow \mathbb{C} \\ |\psi\rangle\langle\varphi| &\longmapsto \langle\varphi|\psi\rangle. \end{aligned} \tag{A.38}$$

It can be shown that this is equivalent to the previous basis-dependent definition (A.20). Indeed, the trace of an arbitrary operator $A \in \mathrm{Lin}(\mathcal{H})$ is

$$\begin{aligned}\mathrm{Tr}\Big(\sum_i \sum_j A_{ij}|\phi_i\rangle\langle\phi_j|\Big) &= \sum_i \sum_j A_{ij}\,\mathrm{Tr}\big(|\phi_i\rangle\langle\phi_j|\big) \\ &= \sum_i \sum_j A_{ij}\langle\phi_j|\phi_i\rangle = \sum_i \sum_j A_{ij}\delta_{ij} = \sum_i A_{ii}.\end{aligned} \tag{A.39}$$

So the trace of an operator is the sum of the elements in the main diagonal of the operator's matrix, but this is the same in any basis, as (A.38) is given without respect to a particular basis.

Let us also have the *partial trace*, which is the linear map given for elementary tensor operators as

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}} : \mathrm{Lin}(\mathcal{K}) \otimes \mathrm{Lin}(\mathcal{H}) &\longrightarrow \mathrm{Lin}(\mathcal{H}), \\ A \otimes B &\longmapsto \mathrm{Tr}(A)B,\end{aligned} \tag{A.40a}$$

and

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}} : \mathrm{Lin}(\mathcal{K}) \otimes \mathrm{Lin}(\mathcal{H}) &\longrightarrow \mathrm{Lin}(\mathcal{K}), \\ A \otimes B &\longmapsto A\,\mathrm{Tr}(B).\end{aligned} \tag{A.40b}$$

Using the trace, the so-called *Hilbert-Schmidt inner product* can be introduced for the space $\mathrm{Lin}(\mathcal{H})$ as

$$\begin{aligned}(\cdot|\cdot) : \mathrm{Lin}(\mathcal{H}) \times \mathrm{Lin}(\mathcal{H}) &\longrightarrow \mathbb{C} \\ A \times B &\longmapsto (A|B) := \mathrm{Tr}\big(A^\dagger B\big).\end{aligned} \tag{A.41}$$

It can be shown that this is indeed an inner product, that is, it is non-negative, non-degenerate, conjugate symmetric, and linear in the second argument. With this inner product $\mathrm{Lin}(\mathcal{H})$ is a Hilbert space, called *Hilbert-Schmidt space*.

A.4. **Schmidt decomposition.** For an arbitrary vector $|\psi\rangle \in \mathcal{K} \otimes \mathcal{H}$, let us form the positive semidefinite operator

$$\mathrm{Tr}_{\mathcal{H}}\big(|\psi\rangle\langle\psi|\big) = \sum_{i=1}^{\dim(\mathcal{K})} \eta_i |\phi_i\rangle_{\mathcal{K}}\langle\phi_i|_{\mathcal{K}}, \tag{A.42}$$

where the η_i eigenvalues of the operator are in descending order, that is, $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{\dim(\mathcal{K})} \geq 0$. Let $r = \mathrm{rk}\big(\mathrm{Tr}_{\mathcal{H}}(|\psi\rangle\langle\psi|)\big)$, then for the η_i eigenvalues we have

$$1 \leq i \leq r : \eta_i \geq 0 \qquad \text{and} \qquad r < i : \eta_i = 0. \tag{A.43}$$

Using the eigenvectors $\big\{|\phi_i\rangle_{\mathcal{K}} \in \mathcal{K}\big\}$ as a basis in $\mathcal{K}$ and an arbitrary $\big\{|\phi_j\rangle_{\mathcal{H}} \in \mathcal{H}\big\}$ basis in $\mathcal{H}$, $|\psi\rangle$ can be decomposed as

$$|\psi\rangle = \sum_i \sum_j \psi_{ij}|\phi_i\rangle_{\mathcal{K}} \otimes |\phi_j\rangle_{\mathcal{H}} = \sum_i |\phi_i\rangle_{\mathcal{K}} \otimes |\chi_i\rangle_{\mathcal{H}}, \tag{A.44}$$

where we introduced the vectors $|\chi_i\rangle_{\mathcal{H}} := \sum_{j=1} \psi_{ij}|\phi_j\rangle_{\mathcal{H}}$. Now let us calculate the partial trace of the operator $|\psi\rangle\langle\psi|$ the vector (A.44), and compare it with the spectral decomposition (A.42),

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}}(|\psi\rangle\langle\psi|) &= \mathrm{Tr}_{\mathcal{H}}\left[\sum_i \Big(|\phi_i\rangle_{\mathcal{K}} \otimes |\chi_i\rangle_{\mathcal{H}}\Big) \otimes \sum_j \Big(\langle\phi_j|_{\mathcal{K}} \otimes \langle\chi_j|_{\mathcal{H}}\Big)\right] = \\ &= \mathrm{Tr}_{\mathcal{H}}\left[\sum_i \sum_j |\phi_i\rangle\langle\phi_j|_{\mathcal{K}} \otimes |\chi_i\rangle\langle\chi_j|_{\mathcal{H}}\right] = \sum_i \sum_j |\phi_i\rangle\langle\phi_j|_{\mathcal{K}}\, \mathrm{Tr}\Big[|\chi_i\rangle\langle\chi_j|_{\mathcal{H}}\Big] = \\ &= \sum_i \sum_j \langle\chi_i|\chi_j\rangle_{\mathcal{H}} |\phi_i\rangle\langle\phi_j|_{\mathcal{K}} = \sum_i \eta_i |\phi_i\rangle\langle\phi_i|_{\mathcal{K}}. \end{aligned} \tag{A.45}$$

This equality holds if $\langle\chi_i|\chi_j\rangle_{\mathcal{H}} = \eta_i\delta_{ij}$. After introducing the vectors

$$|\varphi_i\rangle_{\mathcal{K}} := |\phi_i\rangle_{\mathcal{K}}, \qquad |\varphi_i\rangle_{\mathcal{H}} := \frac{1}{\sqrt{\eta_i}}|\chi_i\rangle_{\mathcal{H}}, \tag{A.46}$$

we obtain the *Schmidt canonical form* or *Schmidt decomposition* of the vector $|\psi\rangle$ as

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\eta_i}|\varphi_i\rangle_{\mathcal{K}} \otimes |\varphi_i\rangle_{\mathcal{H}}, \tag{A.47}$$

where the $\{|\varphi_i\rangle_{\mathcal{K}}\}$ and $\{|\varphi_j\rangle_{\mathcal{H}}\}$ orthonormal bases are the so-called *Schmidt bases*, the scalars $\sqrt{\eta_i}$ are the *Schmidt coefficients*, and r is the *Schmidt rank* of the vector.

A.5. **Linear operators.** In this section we consider particular classes of linear operators and we also mention the structures.

An arbitrary linear operator $A \in \mathrm{Lin}(\mathcal{H}, \mathcal{K})$ can be written in the form

$$A = \sum_i^{d_{\min}} s_i |\alpha_i\rangle\langle\beta_i|, \tag{A.48}$$

using the notation $d_{\min} = \min\{\dim(\mathcal{H}), \dim(\mathcal{K})\}$. This is just the Schmidt decomposition (A.47) for $\mathrm{Lin}(\mathcal{H}, \mathcal{K}) = \mathcal{K} \otimes \mathcal{H}^*$ instead of $\mathcal{K} \otimes \mathcal{H}$. The (A.48) form is called the *singular value decomposition* (SVD) of the operator A, where the orthonormal basis vectors $\{|\alpha_i\rangle \in \mathcal{K}\}$ and $\{|\beta_i\rangle \in \mathcal{H}\}$ are called the *left and right singular vectors* of the operator, and the scalars $0 \leq s_i \in \mathbb{R}$ are called the *singular values* of the operator [6,66]. The *rank* of an operator is the number of its non-zero singular values, denoted by $\mathrm{rk}(A)$. Note that A transforms the left and right singular vectors into each other with rescaling by the singular values as

$$A|\beta_i\rangle = s_i|\alpha_i\rangle, \qquad A^\dagger|\alpha_i\rangle = s_i|\beta_i\rangle. \tag{A.49}$$

The set of the linear operators $\mathrm{Lin}(\mathcal{H}, \mathcal{K})$ forms a Hilbert space, which is the *Hilbert-Schmidt space* mentioned above.

An operator $A \in \mathrm{Lin}(\mathcal{H})$ is called *normal* if it commutes with its adjoint, that is,

$$[A, A^\dagger] = AA^\dagger - A^\dagger A = 0. \tag{A.50}$$

It can be shown that such operators can be written in the form

$$A = \sum_i a_i |\alpha_i\rangle\langle\alpha_i|, \tag{A.51}$$

called the *spectral decomposition* of the operator A, where the scalars $a_i \in \mathbb{C}$ are called the *eigenvalues* of the operator A, and the orthonormal basis vectors $|\alpha_i\rangle \in \mathcal{H}$ are the *eigenvectors* of the operator. Note that A rescales the eigenvectors by the eigenvalues,

$$A|\alpha_i\rangle = a_i|\alpha_i\rangle. \tag{A.52}$$

An operator $A \in \mathrm{Lin}(\mathcal{H})$ is called *self-adjoint* or *Hermitian* (these are the same in finite dimension) if

$$A^\dagger = A. \tag{A.53}$$

These operators are obviously normal, and in their spectral decomposition (A.51), all their eigenvalues are real, that is,

$$a_i \in \mathbb{R} \qquad \forall i = 1, 2, \ldots, d. \tag{A.54}$$

This can be seen by writing the equation (A.53) using the spectral decomposition (A.51), $A = \sum_i a_i |\alpha_i\rangle\langle\alpha_i| = A^\dagger = \sum_i \overline{a_i} |\alpha_i\rangle\langle\alpha_i|$. This is satisfied if $a_i = \overline{a_i}$, meaning that all eigenvalues are real. The set of self-adjoint linear operators is closed under real linear combinations (linear combination with real coefficients). This is a *real Hilbert space (Euclidean space)* with the Hilbert-Schmidt inner product.

An operator $A \in \mathrm{Lin}(\mathcal{H})$ is called *unitary* if

$$A^\dagger = A^{-1}. \tag{A.55}$$

These operators are obviously normal, and in their spectral decomposition (A.51), the absolute value of all eigenvalues is 1, that is,

$$|a_i| = 1 \qquad \forall i = 1, 2, \ldots, d. \tag{A.56}$$

This can be seen by writing the equation (A.55) using the spectral decomposition (A.51) and using the (A.17) form of the identity operator $AA^\dagger = \sum_i \sum_j a_i \overline{a_j} |\alpha_i\rangle\langle\alpha_i||\alpha_j\rangle\langle\alpha_j| = \sum_i |a_i|^2 |\alpha_i\rangle\langle\alpha_i| = I = \sum_i |\alpha_i\rangle\langle\alpha_i|$. This is satisfied if $|a_i| = 1$. The set of unitary operators $\mathrm{U}(\mathcal{H})$ is closed under multiplication and forms a *Lie group*, which is a compact smooth manifold [4, 67].

An operator $A \in \mathrm{Lin}(\mathcal{H})$ is called *positive semi-definite*, $A \geq 0$, if for all $|\psi\rangle \in \mathcal{H}$

$$\langle\psi|A\psi\rangle \geq 0. \tag{A.57}$$

All eigenvalues of these operators are greater than or equal to 0, that is,

$$a_i \geq 0 \qquad \forall i = 1, 2, \ldots, d. \tag{A.58}$$

This can be seen by writing the equation (A.57) using the spectral decomposition (A.51), $\langle\psi|A\psi\rangle = \langle\psi|\big(\sum_i a_i |\alpha_i\rangle\langle\alpha_i|\big)|\psi\rangle = \sum_i a_i \langle\psi|\alpha_i\rangle\langle\alpha_i|\psi\rangle = \sum_i a_i |\langle\psi|\alpha_i\rangle|^2 \geq 0$. This is satisfied if $a_i \geq 0$. The set of positive sem-definite operators is closed under conical combination (linear combination with non-negative coefficients) and forms a *pointed cone* in the space of the self-adjoint operators [4, 46].

An operator $P \in \mathrm{Lin}(\mathcal{H})$ is called a *projection* if

$$P^2 = P, \qquad P^\dagger = P, \tag{A.59}$$

meaning that they are idempotent and self-adjoint. These operators are obviously normal, and in their spectral decomposition (A.51), all their eigenvalues are 0 or 1, that is,

$$a_i \in \{0, 1\} \qquad \forall i = 1, 2, \ldots, d. \tag{A.60}$$

This can be seen by writing the equation (A.59) using the spectral decomposition (A.51), $P^2 = \sum_i \sum_j a_i a_j |\alpha_i\rangle\langle\alpha_i||\alpha_j\rangle\langle\alpha_j| = \sum_i a_i^2 |\alpha_i\rangle\langle\alpha_i| = \sum_i a_i |\alpha_i\rangle\langle\alpha_i| = P$. This is satisfied if $a_i^2 = a_i$, so if every eigenvalue is either 0 or 1. The set of projections of fixed rank k, $\mathrm{Gr}_k(\mathcal{H})$ is a compact smooth manifold called *Grassmannian* [4, 67].

A.6. **Linear maps of operators.** The linear maps of operators play a very important role in quantum information and entanglement theory. Here we recall some definitions and properties of these maps used in the main text. We mainly follow Refs. [4,6].

Let us consider the

$$\Phi:\ \mathrm{Lin}(\mathcal{H}) \quad \longrightarrow \quad \mathrm{Lin}(\mathcal{K}) \tag{A.61}$$

linear maps, which are also called *superoperators*. These maps are called *positive* if they preserve the positive semidefiniteness of an operator, that is

$$\forall A \geq 0:\ \Phi(A) \geq 0. \tag{A.62}$$

Φ is called *n-positive*, if for all $\mathcal{H}'$ of dimension $\dim(\mathcal{H}') = n$,

$$\mathrm{Lin}(\mathcal{H}') \otimes \mathrm{Lin}(\mathcal{H}) \ni A \geq 0:\ (\mathcal{I}_{\mathcal{H}'} \otimes \Phi)(A) \geq 0, \tag{A.63}$$

where $\mathcal{I}_{\mathcal{H}'} : \mathrm{Lin}(\mathcal{H}') \to \mathrm{Lin}(\mathcal{H}')$ is the identity superoperator. It can be shown, that if Φ is n-positive, then it is k-positive for all $k \leq n$. The maps which are n-positive for every $n \in \mathbb{N}$ are called *completely positive* (CP). *Choi's theorem* states that a Φ superoperator is completely positive if and only if for any particular basis $\{|\phi_i\rangle\} \in \mathcal{H}$ the so-called *Choi operator of* Φ is positive,

$$C_\Phi = (\mathcal{I}_{\mathcal{H}} \otimes \Phi)\Big(\sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes |\phi_i\rangle\langle\phi_j|\Big) = \sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes \Phi\big(|\phi_i\rangle\langle\phi_j|\big) \geq 0. \tag{A.64}$$

This theorem is useful because the completely positiveness of a superoperator is traced back to the positivity of a usual operator, for which we have many tools to determine. *Unital* superoperators are those, which preserve the identity,

$$\Phi(I_{\mathcal{H}}) = I_{\mathcal{K}}. \tag{A.65}$$

A superoperator is *noise preserving* if

$$\Phi\Big(\frac{1}{d} I_{\mathcal{H}}\Big) = \frac{1}{d'} I_{\mathcal{K}}, \tag{A.66}$$

where $\dim(\mathcal{H}) = d$, $\dim(\mathcal{K}) = d'$.

A $\Phi : \mathrm{Lin}(\mathcal{H}) \to \mathrm{Lin}(\mathcal{K})$ superoperator is completely positive if and only if it has a *Kraus representation*, that is, it can be written in the form

$$\Phi(A) = \sum_i K_i A K_i^\dagger, \tag{A.67}$$

where the $K_i \in \mathrm{Lin}(\mathcal{H}, \mathcal{K})$ operators are called *Kraus operators*. A superoperator Φ is called *trace-preserving* (TP) if it preserves the trace, that is,

$$\forall A:\ \mathrm{Tr}\big(\Phi(A)\big) = \mathrm{Tr}(A). \tag{A.68}$$

The completely positive, trace-preserving (CPTP) superoperators are called *quantum stochastic maps*, which are also called *quantum channels*. A completely positive superoperator is stochastic (trace-preserving) if and only if the Kraus operators satisfy

$$\sum_i K_i^\dagger K_i = I_{\mathcal{H}}, \tag{A.69}$$

it is not trace-increasing if and only if

$$\sum_i K_i^\dagger K_i \leq I. \tag{A.70}$$

(This means that $I - \sum_i K_i^\dagger K_i \geq 0$.) A completely positive superoperator is unital if and only if its Kraus operators satisfy

$$\sum_i K_i K_i^\dagger = I_{\mathcal{K}}. \tag{A.71}$$

A completely positive superoperator is noise preserving if and only if its Kraus operators satisfy

$$\sum_i K_i K_i^\dagger = \frac{d}{d'} I_{\mathcal{K}}. \tag{A.72}$$

Stochastic, noise preserving superoperators are called *bistochastic*. Note that in the case when $d = d'$, unital and noise preserving superoperators coincide.

Note that the Kraus representation of a completely positive superoperator is not unique. The minimal number of Kraus operators is called the *Kraus rank* of the Φ superoperator, $\mathrm{rk}(\Phi)$. The Kraus representation with $\mathrm{rk}(\Phi)$ number of Kraus operators which are Hilbert-Schmidt orthogonal, $(K_i|K_j) = \mathrm{Tr}(K_i^\dagger K_j) = c_i \delta_{ij}$ is called the *canonical Kraus representation*. The stochastic (CPTP) superoperators form a convex set with extreme points of superoperators having linearly independent $K_i^\dagger K_j$ operators formed from the Kraus operators in the canonical Kraus representation.

An example for superoperators is the (A.38) trace. It can easily be seen that it is positive, using that it is the sum of the elements in the main diagonal (A.39), the (A.51) spectral decomposition and the fact that every eigenvalue of a positive operator is non negative (A.58). To show that it is completely positive, let us use (A.64) Choi's theorem,

$$\begin{aligned}(\mathcal{I} \otimes \mathrm{Tr})\Big(\sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes |\phi_i\rangle\langle\phi_j|\Big) &= \sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes \mathrm{Tr}(|\phi_i\rangle\langle\phi_j|) \\ &= \sum_{ij} \delta_{ij} |\phi_i\rangle\langle\phi_j| = \sum_i |\phi_i\rangle\langle\phi_i| = I \geq 0.\end{aligned} \tag{A.73}$$

A Kraus representation of the trace can be given with the Kraus operators

$$K_i = \langle\phi_i|. \tag{A.74}$$

With the help of these Kraus operators, it can easily be seen, that the trace is bistochastic, that is completely positive, trace- and noise-preserving (but not unital) and has Kraus rank $\mathrm{rk}(\mathrm{Tr}) = \dim(\mathcal{H}) = d$.

Another example for superoperators is the (A.37) transpose map. It can easily be seen that it is positive, using (A.35) the (A.51) and the fact that every eigenvalue of a positive operator is non negative (A.58). To show that the transpose is not completely positive, let us use (A.64) Choi's theorem,

$$(\mathcal{I} \otimes \mathrm{T})\Big(\sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes |\phi_i\rangle\langle\phi_j|\Big) = \sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes (|\phi_i\rangle\langle\phi_j|)^{\mathrm{T}} = \sum_{ij} |\phi_i\rangle\langle\phi_j| \otimes |\phi_j\rangle\langle\phi_i|. \tag{A.75}$$

This operator is not positive. To see that, let us multiply it by $|\psi\rangle = |\phi_k\phi_l\rangle - |\phi_l\phi_k\rangle$, $k \neq l$ from both sides.

$$\begin{aligned}
&\langle\psi|\Big(\sum_{ij}|\phi_i\rangle\langle\phi_j|\otimes|\phi_j\rangle\langle\phi_i|\Big)|\psi\rangle \\
&= \sum_{ij}(\delta_{ki}\delta_{jk}\delta_{lj}\delta_{ij} - \delta_{li}\delta_{jk}\delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl}\delta_{lj}\delta_{ik} + \delta_{li}\delta_{jl}\delta_{kj}\delta_{ik}) \\
&= \sum_{j}(\delta_{jk}\delta_{lj}\delta_{kj} - \delta_{jk}\delta_{ll} - \delta_{jl}\delta_{kk} + \delta_{jl}\delta_{kj}\delta_{lk}) \\
&= \delta_{lk} - \delta_{kk}\delta_{ll} - \delta_{ll}\delta_{kk} + \delta_{kl} = -2 < 0.
\end{aligned} \tag{A.76}$$

Appendix B. Entropic functions over the state space

Entropic quantities turn out to be fundamental in the characterization of quantum states. Here we recall the most important ones.

B.1. Von Neumann entropy.

The *von Neumann entropy* [4] of a density operator ρ is

$$\begin{aligned}
S : \mathcal{D}(\mathcal{H}) &\longrightarrow [0, \ln(\dim(\mathcal{H}))] \\
\rho &\longmapsto S(\rho) := -\operatorname{Tr}\big(\rho\ln(\rho)\big).
\end{aligned} \tag{B.1}$$

Using the spectral decomposition (2.11) of the density operator ρ, we get that the von Neumann entropy can be calculated as the classical Shannon entropy of the spectrum

$$S(\rho) = -\sum_i \lambda_i \ln(\lambda_i). \tag{B.2}$$

(The function $f(x) = -x\ln(x)$ can be made continuous from the right at $x = 0$ by choosing $\lim_{x\to 0}(f(x)) = 0$.)

The von Neumann entropy is *concave*,

$$S(\sum_i w_i\rho_i) \geq \sum_i w_i S(\rho_i). \tag{B.3}$$

It can be shown that the von Neumann entropy is zero exactly for pure states,

$$S(\rho) = 0 \quad\Longleftrightarrow\quad \rho \in \mathcal{P}(\mathcal{H}). \tag{B.4}$$

The von Neumann entropy is *Schur concave*,

$$\rho_1 \preceq \rho_2 \quad\Longleftrightarrow\quad S(\rho_1) \geq S(\rho_2), \tag{B.5}$$

where $\rho_1 \preceq \rho_2$ means that ρ_2 *majorizes* ρ_1, which is defined as

$$\rho_1 \preceq \rho_2 : \quad \sum_{i=1}^{k}\lambda_{1i}^{\downarrow} \leq \sum_{i=1}^{k}\lambda_{2i}^{\downarrow} \qquad \forall k, \tag{B.6}$$

where $\lambda_{1i}^{\downarrow}$ and $\lambda_{2i}^{\downarrow}$ are the eigenvalues of the density operators ρ_1 and ρ_2 in descending order [4]. Majorization is the comparison of the mixedness of states, that is, if $\sigma \preceq \rho$ (ρ majorizes σ), then σ is defined to be more mixed than ρ. So the von Neumann entropy and Schur-concave functions in general measure the mixedness of states. Schur concavity is the most important, defining property of entropies. In addition, for unitary operators $U \in \mathrm{U}(\mathcal{H})$

$$S(U\rho U^\dagger) = S(\rho). \tag{B.7}$$

The quantum Hardy-Littlewood-Pólya lemma tells us that for every bistochastic quantum channel $\Phi : \mathcal{D} \to \mathcal{D}$, the mixedness increases, $\Phi(\rho) \preceq \rho$. Based on the Schur concavity (B.5), the von Neumann entropy increases during such processes,

$$S\big(\Phi(\rho)\big) \geq S(\rho). \tag{B.8}$$

The von Neumann entropy is *subadditive*,

$$S(\rho) \leq \sum_{X \in \xi} S(\rho_X), \tag{B.9}$$

moreover, it is *strongly subadditive*,

$$S(\rho_{X\cap Y}) + S(\rho_{X\cup Y}) \leq S(\rho_X) + S(\rho_Y). \tag{B.10}$$

In (B.9) and (B.10), equality holds for uncorrelated systems,

$$S\Big(\bigotimes_{X\in\xi} \rho_X\Big) = \sum_{X\in\xi} S(\rho_X), \tag{B.11}$$

this is called also called *extensivity* of the von Neumann entropy.

B.2. **Umegaki relative entropy.** The *Umegaki relative entropy* [4], also known as the *quantum Kullback-Leibler divergence*, is

$$\begin{array}{rcl} D : \mathcal{D}(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) & \longrightarrow & [0,\infty] \\ \rho, \sigma & \longmapsto & D(\rho\|\sigma) := \mathrm{Tr}\big(\rho[\ln(\rho) - \ln(\sigma)]\big). \end{array} \tag{B.12}$$

Since ρ and σ are not commuting in general, we do not have a simple way of calculating the relative entropy using the eigenvalues only, contrary to the von Neumann entropy.

Since it can be shown that

$$D(\rho\|\sigma) = 0 \quad \Leftrightarrow \quad \rho = \sigma, \tag{B.13}$$

and

$$D(\rho\|\sigma) \geq 0, \tag{B.14}$$

but it is not symmetric, the Umegaki relative entropy is not a *metric*, but a *divergence* [4]. It is known that this relative entropy is the decay rate of the error probability in quantum hypothesis testing, and in this sense, it is a measure of the distinguishability of states. The Umegaki relative entropy is jointly convex in its two variables, that is,

$$D\Big(\sum_i w_i\rho_i \Big\| \sum_i w_i\sigma_i\Big) \leq \sum_i w_i D(\rho_i\|\sigma_i), \tag{B.15}$$

and its value decreases for every quantum channel $\Phi : \mathcal{D} \to \mathcal{D}$, that is,

$$D\big(\Phi(\rho)\big\|\Phi(\sigma)\big) \leq D(\rho\|\sigma), \tag{B.16}$$

meaning that distinguishability can only decrease in physical processes. This monotonicity is the most important, defining property of relative entropies. In addition, for unitary operators $U \in \mathrm{U}(\mathcal{H})$

$$D(U\rho U^\dagger || U\sigma U^\dagger) = D(\rho||\sigma). \tag{B.17}$$

Declaration

I, the undersigned **Péter Ádám Nyári**, declare that in preparing my thesis, I used AI tool(s) as follows.

Subtask	**AI tool used**	**Place of use**	**Notes (optional)**
Figure preparation	Google Gemini 3.1 Pro	Figures 10, 11, 14, 18 and 19	Used in the python code generating the figures
Figure preparation	Google Gemini 3.1 Pro	Figures 2, 3, 4, 5, 6 and 7	Used in the metapost code generating the figures
Grammar check	Google Gemini 3.1 Pro	In some sentences sporadically in the thesis	-

Other than the tasks listed in the itemized list, I did not use any AI tool for any other task when preparing my thesis.

I declare that I have familiarized myself with the content of Rector's Instruction No. 4/2025 (28 Oct.) concerning the use of AI, and that, when preparing my thesis, I complied with the provisions of the Rector's Instruction.

Budapest, May 28, 2026 .....

signature